OPTIONAL PURE / FURTHER CALCULUS

Surface Area of Revolution

Worked solutions 13 questions · 131 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

Question 1	*7 marks*	2
Question 2	*8 marks*	3
Question 3	*7 marks*	5
Question 4	*7 marks*	7
Question 5	*13 marks*	8
Question 6	*14 marks*	11
Question 7	*10 marks*	14
Question 8	*11 marks*	16
Question 9	*8 marks*	18
Question 10	*9 marks*	20
Question 11	*10 marks*	22
Question 12	*12 marks*	25
Question 13	*15 marks*	27

1. A curve C has equation $y = 2\cosh\left(\frac{x}{2}\right)$, for $0 \leq x \leq 2\ln 2$.
The curve C is rotated through 2π radians about the x-axis.
Find the exact area of the surface generated, giving your answer in the form $\pi\,(a + b\ln 2)$, where a and b are rational numbers. **[7]**

Solution

The area of a surface of revolution about the x-axis is

$$S = 2\pi \int_0^{2\ln 2} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Differentiating $y = 2\cosh\left(\frac{x}{2}\right)$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{x}{2}\right)$$

Using $\cosh^2 A - \sinh^2 A = 1$:

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \sinh^2\left(\frac{x}{2}\right) = \cosh^2\left(\frac{x}{2}\right)$$

Since $\cosh\left(\frac{x}{2}\right) > 0$, this gives $\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \cosh\left(\frac{x}{2}\right)$, so

$$S = 2\pi \int_0^{2\ln 2} 2\cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)\mathrm{d}x = 2\pi \int_0^{2\ln 2} 2\cosh^2\left(\frac{x}{2}\right)\mathrm{d}x$$

The double angle formula $\cosh 2A = 2\cosh^2 A - 1$, with $A = \frac{x}{2}$, gives $2\cosh^2\left(\frac{x}{2}\right) = 1 + \cosh x$, so

$$\begin{aligned} S &= 2\pi \int_0^{2\ln 2} (1 + \cosh x)\,\mathrm{d}x \\ &= 2\pi[x + \sinh x]_0^{2\ln 2} \\ &= 2\pi(2\ln 2 + \sinh(2\ln 2)) \end{aligned}$$

Since $e^{2\ln 2} = 4$ and $e^{-2\ln 2} = \frac{1}{4}$:

$$\sinh(2\ln 2) = \frac{e^{2\ln 2} - e^{-2\ln 2}}{2} = \frac{4 - \frac{1}{4}}{2} = \frac{15}{8}$$

Therefore

$$S = 2\pi\left(2\ln 2 + \frac{15}{8}\right) = \pi\left(\frac{15}{4} + 4\ln 2\right)$$

This is in the form $\pi(a + b\ln 2)$ with $a = \frac{15}{4}$, $b = 4$.

2. A curve C has parametric equations $x = 2t$, $y = t^2$, for $0 \leq t \leq \sqrt{3}$, where both the x- and y-units are in cm. The area of the surface generated when C is rotated through 2π radians about the y-axis is $A\,\text{cm}^2$.

(a) Show that $A = \frac{56\pi}{3}$. **[6]**

The reflector of a car headlamp is made in the shape of this surface. The curved surface area of the reflector must be $58\,\text{cm}^2$, accurate to within two percent. The manufacturing process produces reflectors whose curved surface area is accurate to within one half of one percent of $A\,\text{cm}^2$.

(b) Determine whether all reflectors produced will be suitable for use in the headlamp. **[2]**

Solution

(a) For a curve given parametrically and rotated through 2π radians about the y-axis, the surface area is

$$A = 2\pi \int_{t_0}^{t_1} x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Here $x = 2t \geq 0$ on the interval. Differentiating:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$$

so

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2}$$

Substituting into the formula:

$$A = 2\pi \int_0^{\sqrt{3}} 2t \times 2\sqrt{1 + t^2}\,\mathrm{d}t$$

$$= 8\pi \int_0^{\sqrt{3}} t\left(1 + t^2\right)^{\frac{1}{2}}\,\mathrm{d}t$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + t^2\right)^{\frac{3}{2}} = 3t\left(1 + t^2\right)^{\frac{1}{2}}$, the integral is

$$A = 8\pi\left[\tfrac{1}{3}\left(1 + t^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{3}}$$

$$= \frac{8\pi}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{8\pi}{3}(8 - 1)$$

$$= \frac{56\pi}{3}$$

as required.

(b) The area must lie within 2% of $58\,\text{cm}^2$, so the acceptable range is

$$0.98 \times 58 \leq \text{area} \leq 1.02 \times 58 \quad \Longrightarrow \quad 56.84\,\text{cm}^2 \leq \text{area} \leq 59.16\,\text{cm}^2$$

The reflectors produced have area within 0.5% of $\frac{56\pi}{3}$, so their range is

$$0.995 \times \frac{56\pi}{3} \leq \text{area} \leq 1.005 \times \frac{56\pi}{3} \quad \Longrightarrow \quad 58.35\,\text{cm}^2 \leq \text{area} \leq 58.94\,\text{cm}^2$$

to 2 decimal places. Since $56.84 < 58.35$ and $58.94 < 59.16$, the whole range of areas produced lies inside the acceptable range, so all reflectors produced will be suitable for use in the headlamp.

3. The curve C has parametric equations $x = 3t^2$, $y = 3t - t^3$, where $t \geq 0$.
The curve C meets the x-axis at the origin O and at one other point P.

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 9\left(1 + t^2\right)^2 \qquad \textbf{[2]}$$

(b) The arc of C between O and P is rotated through 2π radians about the x-axis to form a surface of revolution.
Find the exact area of this surface. **[5]**

Solution

(a) Differentiating with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3 - 3t^2$$

So

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 36t^2 + \left(9 - 18t^2 + 9t^4\right) \\ &= 9t^4 + 18t^2 + 9 \\ &= 9\left(t^4 + 2t^2 + 1\right) \\ &= 9\left(1 + t^2\right)^2\end{aligned}$$

as required.

(b) The curve meets the x-axis where $y = 0$:

$$3t - t^3 = 0 \quad \Longrightarrow \quad t\left(3 - t^2\right) = 0$$

With $t \geq 0$, this gives $t = 0$ at O and $t = \sqrt{3}$ at P, where $x = 9$. For $0 \leq t \leq \sqrt{3}$, $y = t\left(3 - t^2\right) \geq 0$.
The surface area of revolution about the x-axis is

$$S = 2\pi \int_{t_0}^{t_1} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

By part (a), the square root is $3\big(1+t^2\big)$, since $1+t^2>0$. Hence

$$\begin{aligned}
S &= 2\pi\int_0^{\sqrt{3}}\big(3t-t^3\big)\times 3\big(1+t^2\big)\mathrm{d}t \\
&= 6\pi\int_0^{\sqrt{3}}\big(3t+2t^3-t^5\big)\mathrm{d}t \\
&= 6\pi\left[\frac{3t^2}{2}+\frac{t^4}{2}-\frac{t^6}{6}\right]_0^{\sqrt{3}} \\
&= 6\pi\left(\frac{9}{2}+\frac{9}{2}-\frac{27}{6}\right) \\
&= 6\pi\times\frac{9}{2} \\
&= 27\pi
\end{aligned}$$

4. A curve has equation $y = \cosh x$, for $0 \leq x \leq \ln 2$.
The curve is rotated through 2π radians about the y-axis.
Find the area of the surface generated, giving your answer in the form $\frac{\pi}{2}(p \ln 2 + q)$, where p and q are integers. **[7]**

Solution

For rotation through 2π radians about the y-axis, the surface area is

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$$

Here $\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x$. Using $\cosh^2 x - \sinh^2 x = 1$:

$$\sqrt{1 + \sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$$

since $\cosh x > 0$. Therefore

$$S = 2\pi \int_0^{\ln 2} x \cosh x \, \mathrm{d}x$$

Integration by parts is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x$$

Take $u = x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cosh x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, $v = \sinh x$. Then

$$\begin{aligned}\int_0^{\ln 2} x \cosh x \, \mathrm{d}x &= [x \sinh x]_0^{\ln 2} - \int_0^{\ln 2} \sinh x \, \mathrm{d}x \\ &= [x \sinh x - \cosh x]_0^{\ln 2}\end{aligned}$$

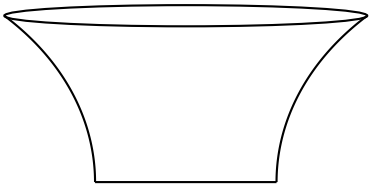
Using the exponential definitions:

$$\sinh(\ln 2) = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}, \qquad \cosh(\ln 2) = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$$

Also $\sinh 0 = 0$ and $\cosh 0 = 1$. So

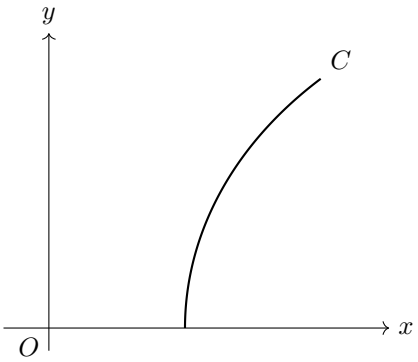
$$\begin{aligned}S &= 2\pi \left[\left(\frac{3}{4} \ln 2 - \frac{5}{4}\right) - (0 - 1)\right] \\ &= 2\pi \left(\frac{3}{4} \ln 2 - \frac{1}{4}\right) \\ &= \frac{\pi}{2}(3 \ln 2 - 1)\end{aligned}$$

This is in the form $\frac{\pi}{2}(p \ln 2 + q)$ with $p = 3$, $q = -1$.



5. The diagram above shows a picture of a ceramic bowl.
The bowl has

- a flat circular base of radius 6 cm
- a height of 11 cm



The diagram above shows a sketch of the curve C with parametric equations

$$x = 6 + 6t^2 \qquad y = 12t - t^3 \qquad 0 \leq t \leq 1$$

The curved inner surface of the bowl is modelled by the surface of revolution formed by rotating C through 2π radians about the y-axis.

(a) Show that, according to the model, the area of the curved inner surface of the bowl is given by

$$36\pi \int_0^1 \left(4 + 5t^2 + t^4\right) \, \mathrm{d}t \qquad \textbf{[5]}$$

(b) Find, according to the model, the total area of the inner surface of the bowl, giving your answer as an exact multiple of π. **[4]**

The inner surface of each bowl is to be glazed. The curved inner surface needs two coats of glaze and the base needs one coat. One pot of glaze covers an area of $2\,\mathrm{m}^2$ with a single coat.

(c) Use your answer to part (b) to find, according to the model, the greatest number of bowls that can be glazed using one pot of glaze. **[3]**

(d) Give a reason why, in practice, one pot of glaze might cover fewer bowls than your answer to part (c). **[1]**

Solution

(a) The area of the surface formed by rotating a curve given parametrically through 2π radians about the y-axis is

$$S = 2\pi \int_{t_0}^{t_1} x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

Differentiating the parametric equations gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 12t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 12 - 3t^2$$

so

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 144t^2 + 144 - 72t^2 + 9t^4 \\ &= 9t^4 + 72t^2 + 144 \\ &= 9\left(t^4 + 8t^2 + 16\right) \\ &= 9\left(4 + t^2\right)^2\end{aligned}$$

Since $4 + t^2 > 0$, the square root is $3\left(4 + t^2\right)$. Substituting into the formula with $x = 6 + 6t^2$:

$$\begin{aligned}S &= 2\pi \int_0^1 \left(6 + 6t^2\right) \times 3\left(4 + t^2\right) \mathrm{d}t \\ &= 2\pi \int_0^1 18\left(1 + t^2\right)\left(4 + t^2\right) \mathrm{d}t \\ &= 36\pi \int_0^1 \left(4 + 5t^2 + t^4\right) \mathrm{d}t\end{aligned}$$

as required.

(b) Evaluating the integral from part (a):

$$\begin{aligned}36\pi \int_0^1 \left(4 + 5t^2 + t^4\right) \mathrm{d}t &= 36\pi \left[4t + \frac{5t^3}{3} + \frac{t^5}{5}\right]_0^1 \\ &= 36\pi \left(4 + \frac{5}{3} + \frac{1}{5}\right) \\ &= 36\pi \times \frac{88}{15} \\ &= \frac{1056\pi}{5}\ \mathrm{cm}^2\end{aligned}$$

The flat base is a circle of radius 6 cm, with area $\pi \times 6^2 = 36\pi\ \mathrm{cm}^2$. The total area of the inner surface is

$$\frac{1056\pi}{5} + 36\pi = \frac{1236\pi}{5}\ \mathrm{cm}^2$$

(c) From part (b), the curved surface has area $\frac{1236\pi}{5} - 36\pi = \frac{1056\pi}{5}\ \mathrm{cm}^2$. The curved surface takes two coats and the base one, so the area glazed per bowl is the total inner area plus the curved area once more:

$$\frac{1236\pi}{5} + \frac{1056\pi}{5} = \frac{2292\pi}{5} \approx 1440.1\ \mathrm{cm}^2$$

One pot covers $2\,\mathrm{m}^2 = 20\,000\,\mathrm{cm}^2$, so the number of bowls is

$$\frac{20\,000}{\frac{2292\pi}{5}} = \frac{100\,000}{2292\pi} \approx 13.89$$

Only complete bowls count, so the greatest number of bowls that can be glazed is 13.

(d) In practice some glaze is wasted when it is applied, for example spilt, left in the pot or absorbed by the clay, so less than $2\,\mathrm{m}^2$ of bowl surface is actually covered.

6. The points $P\left(1, \frac{3}{8}\right)$ and $Q\left(2, \frac{33}{16}\right)$ lie on the curve with equation $y = \frac{1}{8}x^4 + \frac{1}{4x^2}$.
The area of the surface generated when the arc PQ is rotated through 2π radians about the x-axis is denoted by A.

(a) Find the exact value of A, giving your answer as a rational multiple of π. **[6]**

Student X finds an approximation to A by modelling the arc PQ as the straight line segment PQ, then rotating this segment through 2π radians about the x-axis to form the curved surface of a frustum of a cone.

(b) Find the approximation to A obtained by student X, giving your answer in the form $k\pi\sqrt{n}$, where k is rational and n is a positive integer. **[4]**

Student Y finds a second approximation to A by modelling the arc PQ as the line $y = M$, where M is the mean value of $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2}$ over the interval $1 \leq x \leq 2$, then rotating the part of this line between $x = 1$ and $x = 2$ through 2π radians about the x-axis.

(c) Find the percentage error in the approximation to A obtained by student Y, giving your answer to 3 significant figures. **[4]**

Solution

(a) The area of the surface generated when the curve between $x = a$ and $x = b$ is rotated through 2π radians about the x-axis is

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Differentiating $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^3 - \frac{1}{2x^3}$$

so

$$\begin{aligned} 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1 + \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4x^6} \\ &= \frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4x^6} \\ &= \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2 \end{aligned}$$

Since $\frac{1}{2}x^3 + \frac{1}{2x^3} > 0$ for $1 \leq x \leq 2$, the square root is $\frac{1}{2}x^3 + \frac{1}{2x^3}$. Then

$$\begin{aligned} y\left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) &= \left(\frac{1}{8}x^4 + \frac{1}{4x^2}\right)\left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) \\ &= \frac{1}{16}x^7 + \frac{1}{16}x + \frac{1}{8}x + \frac{1}{8x^5} \\ &= \frac{1}{16}x^7 + \frac{3}{16}x + \frac{1}{8}x^{-5} \end{aligned}$$

Substituting into the formula:

$$\begin{aligned} A &= 2\pi \int_1^2 \left(\frac{1}{16}x^7 + \frac{3}{16}x + \frac{1}{8}x^{-5}\right)\mathrm{d}x \\ &= 2\pi\left[\frac{x^8}{128} + \frac{3x^2}{32} - \frac{1}{32x^4}\right]_1^2 \\ &= 2\pi\left[\left(2 + \frac{3}{8} - \frac{1}{512}\right) - \left(\frac{1}{128} + \frac{3}{32} - \frac{1}{32}\right)\right] \\ &= 2\pi\left(\frac{1215}{512} - \frac{36}{512}\right) \\ &= 2\pi \times \frac{1179}{512} \\ &= \frac{1179\pi}{256} \end{aligned}$$

(b) The gradient of PQ is

$$\frac{\frac{33}{16} - \frac{3}{8}}{2 - 1} = \frac{27}{16}$$

so the segment PQ has equation

$$y = \frac{3}{8} + \frac{27}{16}(x - 1), \qquad 1 \le x \le 2$$

On this segment $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{27}{16}$, so

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \frac{729}{256}} = \sqrt{\frac{985}{256}} = \frac{\sqrt{985}}{16}$$

Using the surface area formula from part (a):

$$\begin{aligned} \text{Approximation} &= 2\pi \int_1^2 \left(\frac{3}{8} + \frac{27}{16}(x-1)\right)\frac{\sqrt{985}}{16}\,\mathrm{d}x \\ &= \frac{\pi\sqrt{985}}{8}\left[\frac{3x}{8} + \frac{27}{32}(x-1)^2\right]_1^2 \\ &= \frac{\pi\sqrt{985}}{8}\left[\left(\frac{3}{4} + \frac{27}{32}\right) - \frac{3}{8}\right] \\ &= \frac{\pi\sqrt{985}}{8} \times \frac{39}{32} \\ &= \frac{39}{256}\pi\sqrt{985} \end{aligned}$$

This is in the form $k\pi\sqrt{n}$ with $k = \frac{39}{256}$, $n = 985$.

(c) The mean value of $f(x)$ over $a \le x \le b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

so

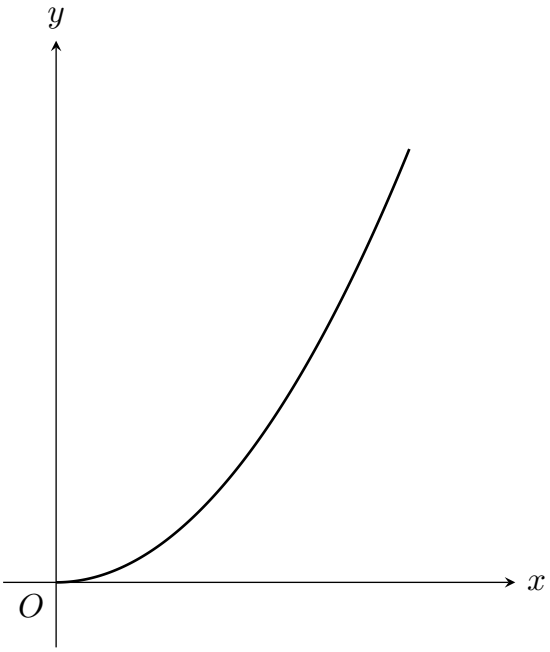
$$\begin{aligned} M &= \frac{1}{2-1}\int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2}\right)\mathrm{d}x \\ &= \left[\frac{x^5}{40} - \frac{1}{4x}\right]_1^2 \\ &= \left(\frac{32}{40} - \frac{1}{8}\right) - \left(\frac{1}{40} - \frac{1}{4}\right) \\ &= \frac{9}{10} \end{aligned}$$

On the line $y = \frac{9}{10}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, so the surface area formula gives

$$\text{Approximation} = 2\pi\int_1^2 \frac{9}{10}\,\mathrm{d}x = \frac{9\pi}{5}$$

The percentage error is

$$\begin{aligned} \frac{\left|\frac{9\pi}{5} - \frac{1179\pi}{256}\right|}{\frac{1179\pi}{256}} \times 100 &= \left(1 - \frac{9}{5}\times\frac{256}{1179}\right)\times 100 \\ &= \frac{3591}{5895}\times 100 \\ &= 60.9\% \qquad (3\text{ s.f.}) \end{aligned}$$



7. The diagram above shows a sketch of the curve with equation

$$y = x^2, \qquad 0 \leq x \leq 1$$

The curve is rotated through 2π radians about the x-axis.

(a) Show that the area of the curved surface generated is given by

$$S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2}\, \mathrm{d}x$$ **[2]**

(b) Hence, using the substitution $2x = \sinh\theta$, show that

$$S = \frac{\pi}{32}\left(a\sqrt{5} - \ln\left(b + \sqrt{5}\right)\right)$$

where a and b are integers to be determined. **[8]**

Solution

(a) The area of the surface generated when the curve is rotated through 2π radians about the x-axis, for $0 \leq x \leq 1$, is

$$S = 2\pi \int_0^1 y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$$

Differentiating $y = x^2$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \quad \Longrightarrow \quad 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + 4x^2$$

so

$$S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2}\, \mathrm{d}x$$

as required.

(b) The substitution $2x = \sinh\theta$ gives

$$x = \frac{1}{2}\sinh\theta, \qquad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{1}{2}\cosh\theta$$

Since $\cosh\theta > 0$, the square root becomes

$$\sqrt{1+4x^2} = \sqrt{1+\sinh^2\theta} = \sqrt{\cosh^2\theta} = \cosh\theta$$

For the limits, $x = 0$ gives $\sinh\theta = 0$, so $\theta = 0$, and $x = 1$ gives $\sinh\theta = 2$, so $\theta = \operatorname{arsinh} 2$. The formula booklet gives

$$\operatorname{arsinh} u = \ln\left(u + \sqrt{u^2+1}\right)$$

so the upper limit is $\theta = \ln\left(2+\sqrt{5}\right)$. Substituting into the result of part (a):

$$S = 2\pi\int_0^{\operatorname{arsinh} 2} \frac{1}{4}\sinh^2\theta \times \cosh\theta \times \frac{1}{2}\cosh\theta\,\mathrm{d}\theta$$

$$= \frac{\pi}{4}\int_0^{\operatorname{arsinh} 2} \sinh^2\theta\cosh^2\theta\,\mathrm{d}\theta$$

Using the double angle formulae $\sinh 2\theta = 2\sinh\theta\cosh\theta$ and $\cosh 4\theta = 1 + 2\sinh^2 2\theta$:

$$\sinh^2\theta\cosh^2\theta = \frac{1}{4}\sinh^2 2\theta = \frac{1}{8}(\cosh 4\theta - 1)$$

so

$$S = \frac{\pi}{32}\int_0^{\operatorname{arsinh} 2} (\cosh 4\theta - 1)\mathrm{d}\theta$$

$$= \frac{\pi}{32}\left[\frac{1}{4}\sinh 4\theta - \theta\right]_0^{\operatorname{arsinh} 2}$$

When $\theta = \operatorname{arsinh} 2$, $\sinh\theta = 2$ and $\cosh\theta = \sqrt{1+\sinh^2\theta} = \sqrt{5}$, so

$$\sinh 2\theta = 2\sinh\theta\cosh\theta = 2\times 2\times\sqrt{5} = 4\sqrt{5}$$

$$\cosh 2\theta = 1 + 2\sinh^2\theta = 1 + 2\times 4 = 9$$

$$\sinh 4\theta = 2\sinh 2\theta\cosh 2\theta = 2\times 4\sqrt{5}\times 9 = 72\sqrt{5}$$

The lower limit contributes $\frac{1}{4}\sinh 0 - 0 = 0$, so

$$S = \frac{\pi}{32}\left(\frac{1}{4}\times 72\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right)$$

$$= \frac{\pi}{32}\left(18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right)$$

This is in the required form with $a = 18$, $b = 2$.

8. **(a)** Show that

$$\int_1^2 \left(t + \frac{1}{t}\right) \ln t \, \mathrm{d}t = a + b \ln 2 + c(\ln 2)^2$$

where a, b and c are rational numbers to be found. **[5]**

(b) A curve C is defined parametrically for $t > 0$ by

$$x = t^2 - 2\ln t \qquad y = 4t$$

The arc of C from $t = 1$ to $t = 2$ is rotated through 2π radians about the y-axis to generate a surface with area S.
Using your answer to part (a), find the exact value of S, giving your answer in the form

$$S = \pi\left(p + q\ln 2 + r(\ln 2)^2\right)$$

where p, q and r are rational numbers to be found. **[6]**

Solution

(a) Split the integral into two parts:

$$\int_1^2 \left(t + \frac{1}{t}\right) \ln t \, \mathrm{d}t = \int_1^2 t \ln t \, \mathrm{d}t + \int_1^2 \frac{\ln t}{t} \, \mathrm{d}t$$

For the first integral, integration by parts is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}t} \, \mathrm{d}t = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}t} \, \mathrm{d}t$$

with $u = \ln t$, $\frac{\mathrm{d}v}{\mathrm{d}t} = t$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{t}$, $v = \frac{t^2}{2}$. Then

$$\begin{aligned}
\int_1^2 t \ln t \, \mathrm{d}t &= \left[\frac{t^2}{2}\ln t\right]_1^2 - \int_1^2 \frac{t^2}{2} \times \frac{1}{t} \, \mathrm{d}t \\
&= 2\ln 2 - \int_1^2 \frac{t}{2} \, \mathrm{d}t \\
&= 2\ln 2 - \left[\frac{t^2}{4}\right]_1^2 \\
&= 2\ln 2 - \left(1 - \frac{1}{4}\right) \\
&= 2\ln 2 - \frac{3}{4}
\end{aligned}$$

For the second integral, $\frac{\mathrm{d}}{\mathrm{d}t}\left((\ln t)^2\right) = \frac{2\ln t}{t}$, so

$$\int_1^2 \frac{\ln t}{t} \, \mathrm{d}t = \left[\frac{1}{2}(\ln t)^2\right]_1^2 = \frac{1}{2}(\ln 2)^2$$

Adding the two results:

$$\int_1^2 \left(t + \frac{1}{t}\right) \ln t \, \mathrm{d}t = -\frac{3}{4} + 2\ln 2 + \frac{1}{2}(\ln 2)^2$$

so the integral is in the form $a + b\ln 2 + c(\ln 2)^2$ with $a = -\frac{3}{4}$, $b = 2$, $c = \frac{1}{2}$.

(b) The surface area generated by rotating a parametric arc through 2π radians about the y-axis is

$$S = 2\pi \int_{t_0}^{t_1} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Differentiating:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t - \frac{2}{t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4$$

so

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 4t^2 - 8 + \frac{4}{t^2} + 16 \\ &= 4t^2 + 8 + \frac{4}{t^2} \\ &= \left(2t + \frac{2}{t}\right)^2\end{aligned}$$

Since $t > 0$, $2t + \frac{2}{t} > 0$, so

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = 2\left(t + \frac{1}{t}\right)$$

On $1 \le t \le 2$, $\frac{\mathrm{d}x}{\mathrm{d}t} \ge 0$ and $x = 1$ when $t = 1$, so $x > 0$ on the arc. Hence

$$\begin{aligned}S &= 2\pi \int_1^2 (t^2 - 2\ln t) \times 2\left(t + \frac{1}{t}\right)\mathrm{d}t \\ &= 4\pi \int_1^2 t^2\left(t + \frac{1}{t}\right)\mathrm{d}t - 8\pi \int_1^2 \left(t + \frac{1}{t}\right)\ln t\,\mathrm{d}t \\ &= 4\pi \int_1^2 (t^3 + t)\mathrm{d}t - 8\pi \int_1^2 \left(t + \frac{1}{t}\right)\ln t\,\mathrm{d}t\end{aligned}$$

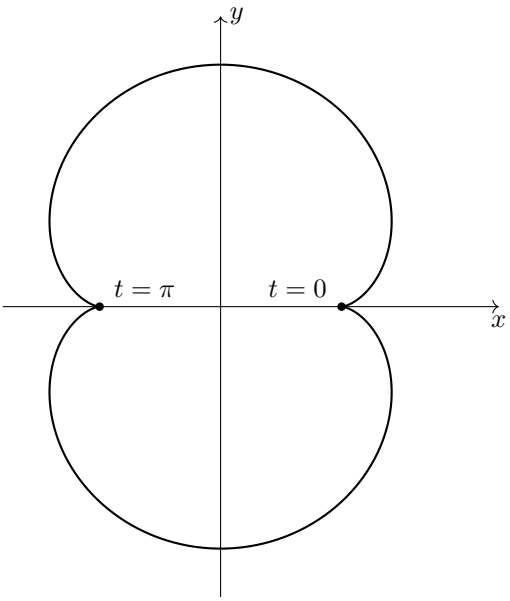
The first integral is

$$\int_1^2 (t^3 + t)\mathrm{d}t = \left[\frac{t^4}{4} + \frac{t^2}{2}\right]_1^2 = (4 + 2) - \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{21}{4}$$

Using part (a) for the second integral:

$$\begin{aligned}S &= 4\pi \times \frac{21}{4} - 8\pi\left(-\frac{3}{4} + 2\ln 2 + \frac{1}{2}(\ln 2)^2\right) \\ &= 21\pi + 6\pi - 16\pi \ln 2 - 4\pi(\ln 2)^2 \\ &= \pi\left(27 - 16\ln 2 - 4(\ln 2)^2\right)\end{aligned}$$

so S is in the form $\pi\left(p + q\ln 2 + r(\ln 2)^2\right)$ with $p = 27$, $q = -16$, $r = -4$.



9. The diagram above shows a curve known as a nephroid.
The curve has parametric equations

$$x = 3\cos t - \cos 3t$$
$$y = 3\sin t - \sin 3t$$
$$(0 \leq t < 2\pi)$$

The section of the curve from $t = 0$ to $t = \pi$ is rotated through 2π radians about the x-axis.
Show that the area of the surface generated is $k\pi^2$, where k is an integer to be found. **[8]**

Solution

The area of the surface generated when a parametric curve is rotated through 2π radians about the x-axis is

$$S = 2\pi \int_{t_0}^{t_1} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Differentiating:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3\sin t + 3\sin 3t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3\cos t - 3\cos 3t$$

Squaring and adding:

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 9\sin^2 t - 18\sin t\sin 3t + 9\sin^2 3t + 9\cos^2 t - 18\cos t\cos 3t + 9\cos^2 3t \\ &= 9\big(\sin^2 t + \cos^2 t\big) + 9\big(\sin^2 3t + \cos^2 3t\big) - 18(\cos 3t\cos t + \sin 3t\sin t) \\ &= 18 - 18\cos(3t - t) \\ &= 18(1 - \cos 2t) \\ &= 18 \times 2\sin^2 t \\ &= 36\sin^2 t\end{aligned}$$

using the addition formula for $\cos(A-B)$ and the double angle formula $\cos 2t = 1 - 2\sin^2 t$. For $0 \le t \le \pi$, $\sin t \ge 0$, so

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = 6\sin t$$

Next simplify y. Using the addition formula and the double angle formulae:

$$\begin{aligned}\sin 3t &= \sin 2t\cos t + \cos 2t \sin t \\ &= 2\sin t\cos^2 t + \left(1 - 2\sin^2 t\right)\sin t \\ &= 2\sin t\left(1 - \sin^2 t\right) + \sin t - 2\sin^3 t \\ &= 3\sin t - 4\sin^3 t\end{aligned}$$

so

$$y = 3\sin t - \left(3\sin t - 4\sin^3 t\right) = 4\sin^3 t$$

which is non-negative for $0 \le t \le \pi$, so this section of the curve lies on or above the x-axis. Substituting into the formula:

$$\begin{aligned}S &= 2\pi \int_0^{\pi} 4\sin^3 t \times 6\sin t \,\mathrm{d}t \\ &= 48\pi \int_0^{\pi} \sin^4 t \,\mathrm{d}t\end{aligned}$$

Now write $\sin^4 t$ in terms of multiple angles, using $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ and $\cos^2 2t = \frac{1}{2}(1 + \cos 4t)$:

$$\begin{aligned}\sin^4 t &= \frac{1}{4}(1 - \cos 2t)^2 \\ &= \frac{1}{4}\left(1 - 2\cos 2t + \cos^2 2t\right) \\ &= \frac{1}{4}\left(1 - 2\cos 2t + \frac{1}{2} + \frac{1}{2}\cos 4t\right) \\ &= \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t\end{aligned}$$

Integrating:

$$\begin{aligned}\int_0^{\pi} \sin^4 t \,\mathrm{d}t &= \left[\frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t\right]_0^{\pi} \\ &= \frac{3\pi}{8}\end{aligned}$$

since $\sin 2\pi = \sin 4\pi = \sin 0 = 0$. Therefore

$$S = 48\pi \times \frac{3\pi}{8} = 18\pi^2$$

so the area of the surface is $k\pi^2$ with $k = 18$.

10. A designer intends to manufacture a product using a 3-D printer. The product will take the form of a surface S which must meet a number of design specifications.
The designer chooses to model S with the equation $z = y^2 + \sinh^2 x$ for $-\ln 5 \leq x \leq \ln 5$, $-3 \leq y \leq 3$.

(a) Sketch, in the x-y plane, the curve C of the points (x, y) for which $z = 4$. **[1]**

(b) The part of C for which $y \geq 0$ is rotated through 2π radians about the x-axis to generate the closed surface T. The surface area of T is denoted by A.
Show that A can be written in the form

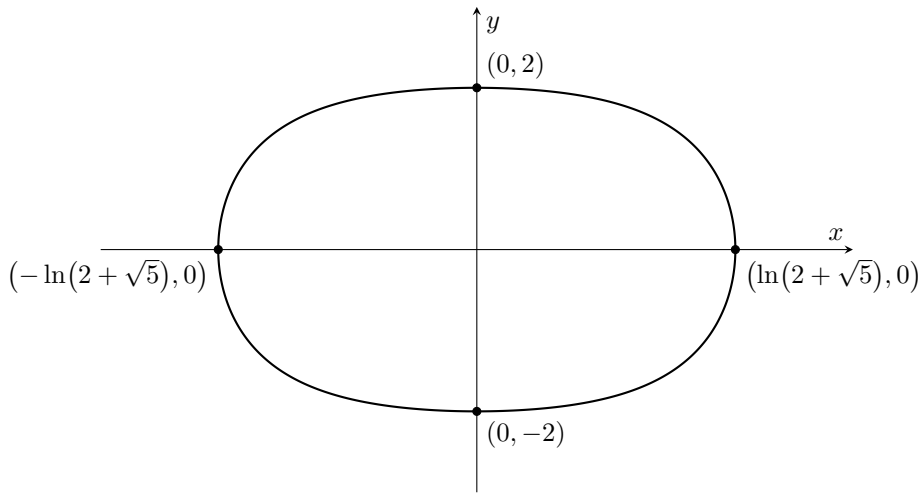
$$k\pi \int_0^{\ln\left(2+\sqrt{5}\right)} \sqrt{\sinh^4 x + 4}\,\mathrm{d}x$$

for some integer k to be determined. **[6]**

(c) A second design specification is that the surface area of T must not be greater than 45 square units.
Use your calculator to decide whether the product meets this requirement according to the model. **[2]**

Solution

(a) The sketch should look like:



The curve C is the closed curve $y^2 + \sinh^2 x = 4$, symmetric in both axes. When $x = 0$, $y = \pm 2$. When $y = 0$, $\sinh x = \pm 2$, so $x = \pm \operatorname{arsinh} 2$. The formula booklet gives

$$\operatorname{arsinh} u = \ln\left(u + \sqrt{u^2 + 1}\right)$$

so $x = \pm \ln\left(2 + \sqrt{5}\right) \approx \pm 1.44$, which lies inside $-\ln 5 \leq x \leq \ln 5$.

(b) The part of C with $y \geq 0$ is

$$y = \left(4 - \sinh^2 x\right)^{\frac{1}{2}}$$

and it meets the x-axis where $\sinh x = \pm 2$, so, by part (a), the curve runs from $x = -\ln\left(2 + \sqrt{5}\right)$ to $x = \ln\left(2 + \sqrt{5}\right)$.

Differentiating using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(4 - \sinh^2 x\right)^{-\frac{1}{2}} \times \left(-2 \sinh x \cosh x\right) = -\frac{\sinh x \cosh x}{\left(4 - \sinh^2 x\right)^{\frac{1}{2}}}$$

so

$$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+\frac{\sinh^2 x\cosh^2 x}{4-\sinh^2 x}$$
$$= \frac{4-\sinh^2 x+\sinh^2 x\cosh^2 x}{4-\sinh^2 x}$$

Hence

$$y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \left(4-\sinh^2 x\right)^{\frac{1}{2}} \times \frac{\sqrt{4-\sinh^2 x+\sinh^2 x\cosh^2 x}}{\left(4-\sinh^2 x\right)^{\frac{1}{2}}}$$
$$= \sqrt{4-\sinh^2 x+\sinh^2 x\left(1+\sinh^2 x\right)}$$
$$= \sqrt{\sinh^4 x+4}$$

using $\cosh^2 x = 1+\sinh^2 x$.

The surface area of revolution about the x-axis is

$$A = 2\pi\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

so

$$A = 2\pi\int_{-\ln\left(2+\sqrt{5}\right)}^{\ln\left(2+\sqrt{5}\right)} \sqrt{\sinh^4 x+4}\,\mathrm{d}x$$

Since $\sinh(-x) = -\sinh x$, $\sinh^4(-x) = \sinh^4 x$, so the integrand is even and the integral over the symmetric interval is twice the integral from 0:

$$A = 4\pi\int_0^{\ln\left(2+\sqrt{5}\right)} \sqrt{\sinh^4 x+4}\,\mathrm{d}x$$

This is in the required form with $k = 4$.

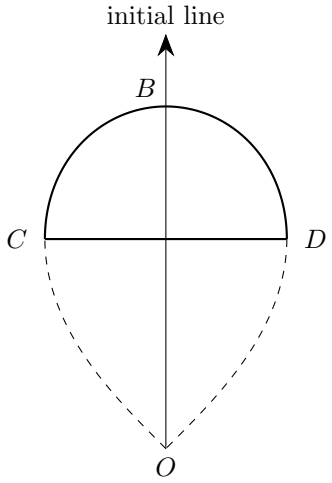
(c) Using the calculator to evaluate the integral:

$$\int_0^{\ln\left(2+\sqrt{5}\right)} \sqrt{\sinh^4 x+4}\,\mathrm{d}x \approx 3.5102$$

Using part (b):

$$A \approx 4\pi\times 3.5102 \approx 44.1$$

Since $44.1 < 45$, the surface area of T is not greater than 45 square units, so according to the model the product meets this requirement.



11. The diagram above shows the vertical cross-section of a glass paperweight with a flat circular base. The curve in the diagram, including its dashed sections, is the loop of the curve with polar equation

$$r^2 = 32\cos 2\theta, \qquad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

where r is measured in centimetres. The initial line points vertically upwards from the pole O, and B is the point where the loop meets the initial line.
The points C and D lie on the loop, the line CD is perpendicular to the initial line, and CD is the greatest width of the loop.
The paperweight is modelled as the solid formed when the region bounded by the arc CBD and the line segment CD is rotated through π radians about the initial line.

(a) Show that C and D are the points of the loop where $\theta = \pm\frac{\pi}{6}$, and find the length of CD. **[3]**

(b) Find, according to the model, the total surface area of the paperweight, including its base. Give your answer in the form $\pi\left(p - q\sqrt{3}\right)$ cm^2, where p and q are integers. **[7]**

Solution

(a) Take the initial line as the x-axis, with O as the origin, so that a point of the loop has

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Since $\cos 2\theta$ is unchanged when θ is replaced by $-\theta$, the points of the loop at θ and $-\theta$ have the same value of r and the same x-coordinate. So, for $0 < \theta \leq \frac{\pi}{4}$, the width of the loop perpendicular to the initial line is $W = 2r\sin\theta$. Using $r^2 = 32\cos 2\theta$ and the double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$:

$$\begin{aligned} W^2 &= 4r^2\sin^2\theta \\ &= 128\cos 2\theta\sin^2\theta \\ &= 128\left(1 - 2\sin^2\theta\right)\sin^2\theta \\ &= 128\sin^2\theta - 256\sin^4\theta \\ &= 16 - 256\left(\sin^2\theta - \tfrac{1}{4}\right)^2 \end{aligned}$$

This is greatest when $\sin^2\theta = \frac{1}{4}$, so $\sin\theta = \frac{1}{2}$ and, as $0 < \theta \leq \frac{\pi}{4}$, $\theta = \frac{\pi}{6}$. Hence C and D are the points of the loop where $\theta = \pm\frac{\pi}{6}$, and the greatest value of W^2 is 16, so

$$CD = 4\,\text{cm}$$

(b) The region is symmetric about the initial line, so rotating it through π radians gives the same solid as rotating its upper half through 2π radians. The curved surface is therefore formed by rotating the arc BC, for which $0 \leq \theta \leq \frac{\pi}{6}$, through 2π radians about the x-axis.

For a curve given in terms of a parameter θ, the area of the surface formed by rotating it through 2π radians about the x-axis is

$$S = 2\pi \int_{\theta_0}^{\theta_1} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta$$

Differentiating $x = r\cos\theta$ and $y = r\sin\theta$ with respect to θ, where r depends on θ:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$$

When these are squared and added, the terms in $r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta$ cancel, so

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\sin^2\theta + \cos^2\theta\right)$$

$$= r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$$

Differentiating $r^2 = 32\cos 2\theta$ implicitly with respect to θ:

$$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -64\sin 2\theta \quad \Longrightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{32\sin 2\theta}{r}$$

So

$$r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = r^2 + \frac{1024\sin^2 2\theta}{r^2}$$

$$= \frac{r^4 + 1024\sin^2 2\theta}{r^2}$$

$$= \frac{1024\cos^2 2\theta + 1024\sin^2 2\theta}{r^2}$$

$$= \frac{1024}{r^2}$$

and, since $r > 0$, the square root is $\frac{32}{r}$. The curved surface area is then

$$S = 2\pi \int_0^{\frac{\pi}{6}} r\sin\theta \times \frac{32}{r}\, \mathrm{d}\theta$$

$$= 64\pi \int_0^{\frac{\pi}{6}} \sin\theta\, \mathrm{d}\theta$$

$$= 64\pi\left[-\cos\theta\right]_0^{\frac{\pi}{6}}$$

$$= 64\pi\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= \left(64\pi - 32\sqrt{3}\,\pi\right) \mathrm{cm}^2$$

The base is a circle whose diameter is $CD = 4\,\text{cm}$, from part (a), so its radius is $2\,\text{cm}$ and its area is $\pi \times 2^2 = 4\pi\,\text{cm}^2$. The total surface area is

$$\begin{aligned}\text{Total} &= 64\pi - 32\sqrt{3}\,\pi + 4\pi \\ &= \pi\left(68 - 32\sqrt{3}\right)\,\text{cm}^2\end{aligned}$$

so the answer is in the form $\pi\left(p - q\sqrt{3}\right)$ with $p = 68$, $q = 32$.

12. The curve C has parametric equations

$$x = 4\cosh t, \qquad y = 3t - \sinh 2t, \qquad 0 \leq t \leq \ln 2$$

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (2\cosh 2t - 1)^2 \qquad \textbf{[4]}$$

The curve C is rotated through 2π radians about the x-axis. The area of the surface generated is denoted by A.

(b) Find A, giving your answer in the form $\pi\left(p + q\ln 2 + r(\ln 2)^2\right)$, where p, q and r are rational numbers. **[8]**

Solution

(a) Differentiating with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4\sinh t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3 - 2\cosh 2t$$

The double angle formula $\cosh 2t = 1 + 2\sinh^2 t$ gives $\sinh^2 t = \frac{1}{2}(\cosh 2t - 1)$, so

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 16\sinh^2 t = 8\cosh 2t - 8$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 9 - 12\cosh 2t + 4\cosh^2 2t$$

Adding:

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 8\cosh 2t - 8 + 9 - 12\cosh 2t + 4\cosh^2 2t \\ &= 4\cosh^2 2t - 4\cosh 2t + 1 \\ &= (2\cosh 2t - 1)^2\end{aligned}$$

as required.

(b) The area of the surface generated when a curve with $y \geq 0$ is rotated through 2π radians about the x-axis is

$$A = 2\pi\int_{t_0}^{t_1} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

At $t = \ln 2$, $e^{2t} = 4$, so

$$\sinh(2\ln 2) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}, \qquad \cosh(2\ln 2) = \frac{1}{2}\left(4 + \frac{1}{4}\right) = \frac{17}{8}$$

The curve starts at $y = 0$ when $t = 0$, and y increases while $2\cosh 2t < 3$ and then decreases to

$y = 3\ln 2 - \frac{15}{8} > 0$ at $t = \ln 2$, so $y \geq 0$ throughout. Since $\cosh 2t \geq 1$, $2\cosh 2t - 1 > 0$, so by part (a) the square root is $2\cosh 2t - 1$. Hence

$$\begin{aligned} A &= 2\pi \int_0^{\ln 2} (3t - \sinh 2t)(2\cosh 2t - 1)\mathrm{d}t \\ &= 2\pi \int_0^{\ln 2} (6t\cosh 2t - 3t - 2\sinh 2t\cosh 2t + \sinh 2t)\mathrm{d}t \end{aligned}$$

Integration by parts is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

With $u = 6t$, $\frac{\mathrm{d}v}{\mathrm{d}t} = \cosh 2t$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = 6$, $v = \frac{1}{2}\sinh 2t$:

$$\begin{aligned} \int_0^{\ln 2} 6t\cosh 2t\,\mathrm{d}t &= [3t\sinh 2t]_0^{\ln 2} - \int_0^{\ln 2} 3\sinh 2t\,\mathrm{d}t \\ &= \left[3t\sinh 2t - \frac{3}{2}\cosh 2t\right]_0^{\ln 2} \end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\left(\sinh^2 2t\right) = 4\sinh 2t\cosh 2t$, the term $2\sinh 2t\cosh 2t$ integrates to $\frac{1}{2}\sinh^2 2t$. Therefore

$$\begin{aligned} A &= 2\pi\left[3t\sinh 2t - \frac{3}{2}\cosh 2t - \frac{3}{2}t^2 - \frac{1}{2}\sinh^2 2t + \frac{1}{2}\cosh 2t\right]_0^{\ln 2} \\ &= 2\pi\left[3t\sinh 2t - \cosh 2t - \frac{3}{2}t^2 - \frac{1}{2}\sinh^2 2t\right]_0^{\ln 2} \\ &= 2\pi\left(\left(\frac{45}{8}\ln 2 - \frac{17}{8} - \frac{3}{2}(\ln 2)^2 - \frac{225}{128}\right) - (-1)\right) \\ &= 2\pi\left(-\frac{369}{128} + \frac{45}{8}\ln 2 - \frac{3}{2}(\ln 2)^2\right) \\ &= \pi\left(-\frac{369}{64} + \frac{45}{4}\ln 2 - 3(\ln 2)^2\right) \end{aligned}$$

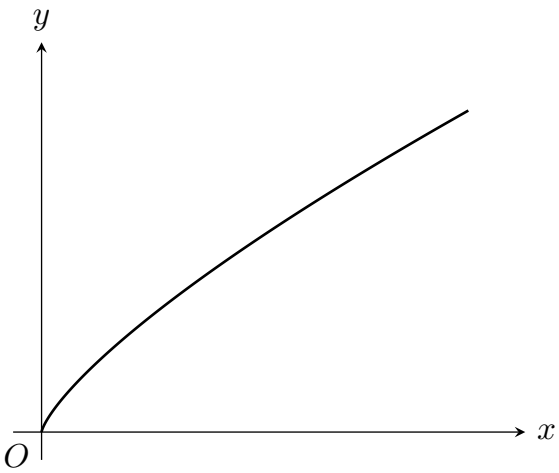
so the answer is in the form $\pi\left(p + q\ln 2 + r(\ln 2)^2\right)$ with $p = -\frac{369}{64}$, $q = \frac{45}{4}$, $r = -3$.

13. For integers $n \geq 0$, let

$$I_n = \int_0^1 t^n \sqrt{1+3t^2}\,\mathrm{d}t$$

(a) Show that, for $n \geq 2$,

$$3(n+2)I_n = 8 - (n-1)I_{n-2}$$ **[5]**



The curve shown in the diagram above has parametric equations

$$x = \frac{3}{4}t^4 \qquad y = \frac{1}{\sqrt{3}}t^3 \qquad 0 \leq t \leq 1$$

The curve is rotated through 2π radians about the x-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by

$$2\pi \int_0^1 t^5 \sqrt{1+3t^2}\,\mathrm{d}t$$ **[5]**

(c) Using the results of parts (a) and (b), find the exact value of the external surface area of the shell, giving your answer in the form $k\pi$, where k is a rational number. Make each step of your working clear. **[5]**

Solution

(a) Write the integrand as $t^{n-1} \times t\sqrt{1+3t^2}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1+3t^2\right)^{\frac{3}{2}} = \frac{3}{2} \times 6t\left(1+3t^2\right)^{\frac{1}{2}} = 9t\sqrt{1+3t^2}$$

it follows that

$$\int t\sqrt{1+3t^2}\,\mathrm{d}t = \frac{1}{9}\left(1+3t^2\right)^{\frac{3}{2}} + c$$

Integration by parts is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

Take $u = t^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = t\sqrt{1+3t^2}$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = (n-1)t^{n-2}$ and $v = \frac{1}{9}\left(1+3t^2\right)^{\frac{3}{2}}$. Then

$$\begin{aligned} I_n &= \left[\frac{1}{9}t^{n-1}\left(1+3t^2\right)^{\frac{3}{2}}\right]_0^1 - \frac{n-1}{9}\int_0^1 t^{n-2}\left(1+3t^2\right)^{\frac{3}{2}}\,\mathrm{d}t \\ &= \frac{1}{9}\times 4^{\frac{3}{2}} - 0 - \frac{n-1}{9}\int_0^1 t^{n-2}\left(1+3t^2\right)\sqrt{1+3t^2}\,\mathrm{d}t \\ &= \frac{8}{9} - \frac{n-1}{9}\left(\int_0^1 t^{n-2}\sqrt{1+3t^2}\,\mathrm{d}t + 3\int_0^1 t^{n}\sqrt{1+3t^2}\,\mathrm{d}t\right) \\ &= \frac{8}{9} - \frac{n-1}{9}(I_{n-2}+3I_n) \end{aligned}$$

where the lower limit gives 0 because $n-1 \geq 1$. Multiplying by 9:

$$\begin{aligned} 9I_n &= 8 - (n-1)I_{n-2} - 3(n-1)I_n \\ (9+3n-3)I_n &= 8-(n-1)I_{n-2} \\ 3(n+2)I_n &= 8-(n-1)I_{n-2} \end{aligned}$$

as required.

(b) The surface area formed by rotating a parametric curve about the x-axis is

$$S = 2\pi\int_{t_0}^{t_1} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Differentiating the parametric equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^3, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3t^2}{\sqrt{3}} = \sqrt{3}\,t^2$$

so

$$\begin{aligned} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 9t^6 + 3t^4 \\ &= 3t^4\left(1+3t^2\right) \end{aligned}$$

Since $t \geq 0$, $\sqrt{3t^4} = \sqrt{3}\,t^2$, so

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \sqrt{3}\,t^2\sqrt{1+3t^2}$$

Hence

$$\begin{aligned} S &= 2\pi\int_0^1 \frac{1}{\sqrt{3}}t^3 \times \sqrt{3}\,t^2\sqrt{1+3t^2}\,\mathrm{d}t \\ &= 2\pi\int_0^1 t^5\sqrt{1+3t^2}\,\mathrm{d}t \end{aligned}$$

as required.

(c) By part (b), the surface area is $2\pi I_5$. Using the result from part (a):

$$I_1 = \int_0^1 t\sqrt{1+3t^2}\,\mathrm{d}t = \left[\frac{1}{9}\left(1+3t^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$$

Putting $n = 3$ into the reduction formula from part (a):

$$15I_3 = 8 - 2I_1 = 8 - \frac{14}{9} = \frac{58}{9} \quad \Longrightarrow \quad I_3 = \frac{58}{135}$$

Putting $n = 5$:

$$21I_5 = 8 - 4I_3 = 8 - \frac{232}{135} = \frac{848}{135} \quad \Longrightarrow \quad I_5 = \frac{848}{2835}$$

Therefore the external surface area is

$$2\pi I_5 = \frac{1696}{2835}\pi$$

so it is in the form $k\pi$ with $k = \frac{1696}{2835}$.