OPTIONAL PURE / FURTHER CALCULUS

Reduction Formulae for Integrals

Worked solutions 14 questions · 152 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

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1. Let

$$I_n = \int_0^1 (1+2x)^n e^x \, \mathrm{d}x$$

where n is a non-negative integer.

(a) By integrating by parts, show that, for $n \geq 1$,

$$I_n = 3^n e - 1 - 2nI_{n-1}$$ **[3]**

(b) Hence find the exact value of I_3. **[4]**

Solution

(a) The integration by parts formula is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Take $u = (1+2x)^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2n(1+2x)^{n-1}$ and $v = e^x$. Then

$$\begin{aligned} I_n &= [(1+2x)^n e^x]_0^1 - \int_0^1 2n(1+2x)^{n-1} e^x \, \mathrm{d}x \\ &= \left(3^n e - 1^n e^0\right) - 2n \int_0^1 (1+2x)^{n-1} e^x \, \mathrm{d}x \\ &= 3^n e - 1 - 2nI_{n-1} \end{aligned}$$

as required.

(b) First find I_0:

$$I_0 = \int_0^1 e^x \, \mathrm{d}x = [e^x]_0^1 = e - 1$$

Using part (a) with $n = 1$, $n = 2$ and $n = 3$:

$$\begin{aligned} I_1 &= 3e - 1 - 2I_0 = 3e - 1 - 2(e-1) = e + 1 \\ I_2 &= 9e - 1 - 4I_1 = 9e - 1 - 4(e+1) = 5e - 5 \\ I_3 &= 27e - 1 - 6I_2 = 27e - 1 - 6(5e-5) = 29 - 3e \end{aligned}$$

2. For integers $n \geq 0$, let

$$I_n = \int_0^2 \left(8 - x^3\right)^n \, \mathrm{d}x$$

(a) Prove that, for $n \geq 1$,

$$I_n = \frac{24n}{3n+1} I_{n-1}$$ **[4]**

(b) Using the result from part (a) and showing all stages of your working, determine the value of n for which

$$I_n = \frac{1\,990\,656}{455}$$ **[4]**

Solution

(a) The integration by parts formula is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Take $u = \left(8 - x^3\right)^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = -3nx^2\left(8 - x^3\right)^{n-1}$ and $v = x$. Then

$$\begin{aligned} I_n &= \left[x\left(8 - x^3\right)^n\right]_0^2 - \int_0^2 x \times \left(-3nx^2\left(8 - x^3\right)^{n-1}\right) \mathrm{d}x \\ &= \left[x\left(8 - x^3\right)^n\right]_0^2 + 3n \int_0^2 x^3\left(8 - x^3\right)^{n-1} \, \mathrm{d}x \end{aligned}$$

For $n \geq 1$, the bracket is zero at $x = 2$, since $8 - 2^3 = 0$, and zero at $x = 0$, so

$$I_n = 3n \int_0^2 x^3\left(8 - x^3\right)^{n-1} \, \mathrm{d}x$$

Writing $x^3 = 8 - \left(8 - x^3\right)$:

$$\begin{aligned} I_n &= 3n \int_0^2 \left(8 - \left(8 - x^3\right)\right)\left(8 - x^3\right)^{n-1} \, \mathrm{d}x \\ &= 3n\left(8 \int_0^2 \left(8 - x^3\right)^{n-1} \, \mathrm{d}x - \int_0^2 \left(8 - x^3\right)^n \, \mathrm{d}x\right) \\ &= 24nI_{n-1} - 3nI_n \end{aligned}$$

Rearranging gives

$$(3n+1)I_n = 24nI_{n-1} \quad \Longrightarrow \quad I_n = \frac{24n}{3n+1} I_{n-1}$$

as required.

(b) First, find I_0:

$$I_0 = \int_0^2 1 \, \mathrm{d}x = 2$$

Applying the result from part (a) repeatedly:

$$I_1 = \frac{24}{4} \times 2 = 12$$

$$I_2 = \frac{48}{7} \times 12 = \frac{576}{7}$$

$$I_3 = \frac{72}{10} \times \frac{576}{7} = \frac{36 \times 576}{5 \times 7} = \frac{20\,736}{35}$$

$$I_4 = \frac{96}{13} \times \frac{20\,736}{35} = \frac{96 \times 20\,736}{13 \times 35} = \frac{1\,990\,656}{455}$$

Hence $I_n = \frac{1\,990\,656}{455}$ when $n = 4$.

3. The integral I_n is defined by

$$I_n = \int_0^{\frac{\pi}{6}} \sin^n x \, \mathrm{d}x \qquad (n \geq 0)$$

(a) Show that

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} - \frac{\sqrt{3}}{n\,(2^n)} \qquad (n \geq 2)$$ **[5]**

(b) Use the result from part **(a)** to show that

$$\int_0^{\frac{\pi}{6}} \sin^6 x \, \mathrm{d}x = \frac{a\pi + b\sqrt{3}}{96}$$

where a and b are integers to be found. **[4]**

Solution

(a) Write $\sin^n x = \sin^{n-1} x \sin x$ and integrate by parts. The formula is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Let $u = \sin^{n-1} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\sin^{n-2} x \cos x$ and $v = -\cos x$. Then

$$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{6}} + (n-1)\int_0^{\frac{\pi}{6}} \sin^{n-2} x \cos^2 x \, \mathrm{d}x$$

Since $n \geq 2$, $\sin^{n-1} 0 = 0$, so the bracket is

$$-\left(\frac{1}{2}\right)^{n-1} \times \frac{\sqrt{3}}{2} - 0 = -\frac{\sqrt{3}}{2^n}$$

Using $\cos^2 x = 1 - \sin^2 x$ in the remaining integral:

$$I_n = -\frac{\sqrt{3}}{2^n} + (n-1)\int_0^{\frac{\pi}{6}} \left(\sin^{n-2} x - \sin^n x\right)\mathrm{d}x$$

$$= -\frac{\sqrt{3}}{2^n} + (n-1)I_{n-2} - (n-1)I_n$$

Collecting the I_n terms gives

$$nI_n = (n-1)I_{n-2} - \frac{\sqrt{3}}{2^n}$$

and dividing by n:

$$I_n = \left(\frac{n-1}{n}\right)I_{n-2} - \frac{\sqrt{3}}{n(2^n)}$$

as required.

(b) First, find I_0:

$$I_0 = \int_0^{\frac{\pi}{6}} 1 \, \mathrm{d}x = \frac{\pi}{6}$$

Using part **(a)** with $n = 2$, $n = 4$ and $n = 6$:

$$I_2 = \frac{1}{2}I_0 - \frac{\sqrt{3}}{2 \times 4} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

$$I_4 = \frac{3}{4}I_2 - \frac{\sqrt{3}}{4 \times 16} = \frac{\pi}{16} - \frac{3\sqrt{3}}{32} - \frac{\sqrt{3}}{64} = \frac{\pi}{16} - \frac{7\sqrt{3}}{64}$$

$$I_6 = \frac{5}{6}I_4 - \frac{\sqrt{3}}{6 \times 64} = \frac{5\pi}{96} - \frac{35\sqrt{3}}{384} - \frac{\sqrt{3}}{384} = \frac{5\pi}{96} - \frac{36\sqrt{3}}{384}$$

Since $\frac{36}{384} = \frac{9}{96}$, this gives

$$\int_0^{\frac{\pi}{6}} \sin^6 x \, \mathrm{d}x = I_6 = \frac{5\pi - 9\sqrt{3}}{96}$$

so $a = 5$ and $b = -9$.

4. The integral I_n is defined, for integers $n \geq 0$, by

$$I_n = \int_0^{\sqrt{2}} \left(4 - x^2\right)^n \, \mathrm{d}x$$

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4 - x^2\right)^n\right)$, or otherwise, show that, for $n \geq 1$,

$$(2n+1)I_n = 2^n\sqrt{2} + 8nI_{n-1}$$ **[5]**

(b) Hence find the exact value of I_3, giving your answer in the form $k\sqrt{2}$, where k is a rational number. **[4]**

Solution

(a) Using the product rule and the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4-x^2\right)^n\right) &= \left(4-x^2\right)^n + x \times n\left(4-x^2\right)^{n-1} \times (-2x) \\ &= \left(4-x^2\right)^n - 2nx^2\left(4-x^2\right)^{n-1}\end{aligned}$$

Writing $x^2 = 4 - \left(4 - x^2\right)$ in the second term gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4-x^2\right)^n\right) &= \left(4-x^2\right)^n - 2n\left(4-\left(4-x^2\right)\right)\left(4-x^2\right)^{n-1} \\ &= \left(4-x^2\right)^n - 8n\left(4-x^2\right)^{n-1} + 2n\left(4-x^2\right)^n \\ &= (2n+1)\left(4-x^2\right)^n - 8n\left(4-x^2\right)^{n-1}\end{aligned}$$

Integrating both sides from 0 to $\sqrt{2}$:

$$\left[x\left(4-x^2\right)^n\right]_0^{\sqrt{2}} = (2n+1)\int_0^{\sqrt{2}}\left(4-x^2\right)^n \mathrm{d}x - 8n\int_0^{\sqrt{2}}\left(4-x^2\right)^{n-1}\mathrm{d}x$$

The left-hand side is

$$\sqrt{2}(4-2)^n - 0 = 2^n\sqrt{2}$$

so

$$2^n\sqrt{2} = (2n+1)I_n - 8nI_{n-1}$$

and rearranging gives

$$(2n+1)I_n = 2^n\sqrt{2} + 8nI_{n-1}$$

as required.

(b) For $n = 0$:

$$I_0 = \int_0^{\sqrt{2}} 1 \, \mathrm{d}x = \sqrt{2}$$

Using the result of part (a) with $n = 1$, $n = 2$ and $n = 3$ in turn:

$$3I_1 = 2\sqrt{2} + 8I_0 = 2\sqrt{2} + 8\sqrt{2} = 10\sqrt{2} \quad \Longrightarrow \quad I_1 = \frac{10\sqrt{2}}{3}$$

$$5I_2 = 4\sqrt{2} + 16I_1 = 4\sqrt{2} + \frac{160\sqrt{2}}{3} = \frac{172\sqrt{2}}{3} \quad \Longrightarrow \quad I_2 = \frac{172\sqrt{2}}{15}$$

$$7I_3 = 8\sqrt{2} + 24I_2 = 8\sqrt{2} + \frac{4128\sqrt{2}}{15} = \frac{4248\sqrt{2}}{15} \quad \Longrightarrow \quad I_3 = \frac{4248\sqrt{2}}{105}$$

Dividing the numerator and denominator by 3 gives

$$I_3 = \frac{1416}{35}\sqrt{2}$$

so the answer is in the form $k\sqrt{2}$ with $k = \frac{1416}{35}$.

5. For integers $n \geq 0$, the integral I_n is defined by

$$I_n = \int_0^{\frac{1}{\sqrt{3}}} \frac{x^n}{1+x^2}\,\mathrm{d}x$$

(a) For integers $n \geq 2$, show that

$$I_n + I_{n-2} = \frac{1}{n-1}\left(\frac{1}{\sqrt{3}}\right)^{n-1} \quad \textbf{[3]}$$

(b) (i) Find the exact value of I_6, giving your answer in the form $a\sqrt{3} + b\pi$, where a and b are rational. **[4]**

(ii) Hence deduce that $\pi < \frac{82\sqrt{3}}{45}$. **[2]**

Solution

(a) Adding the two integrals, which have the same limits, gives

$$\begin{aligned} I_n + I_{n-2} &= \int_0^{\frac{1}{\sqrt{3}}} \frac{x^n + x^{n-2}}{1+x^2}\,\mathrm{d}x \\ &= \int_0^{\frac{1}{\sqrt{3}}} \frac{x^{n-2}\left(x^2+1\right)}{1+x^2}\,\mathrm{d}x \\ &= \int_0^{\frac{1}{\sqrt{3}}} x^{n-2}\,\mathrm{d}x \end{aligned}$$

Since $n \geq 2$, $n-1 \geq 1$, so

$$\begin{aligned} I_n + I_{n-2} &= \left[\frac{x^{n-1}}{n-1}\right]_0^{\frac{1}{\sqrt{3}}} \\ &= \frac{1}{n-1}\left(\frac{1}{\sqrt{3}}\right)^{n-1} - 0 \\ &= \frac{1}{n-1}\left(\frac{1}{\sqrt{3}}\right)^{n-1} \end{aligned}$$

as required.

(b) (i) The standard result is

$$\int \frac{1}{k^2+x^2}\,\mathrm{d}x = \frac{1}{k}\arctan\left(\frac{x}{k}\right) + c$$

Using it with $k = 1$:

$$I_0 = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+x^2}\,\mathrm{d}x = \left[\arctan x\right]_0^{\frac{1}{\sqrt{3}}} = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Using part (a) with $n = 2$:

$$I_2 + I_0 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \quad \Longrightarrow \quad I_2 = \frac{\sqrt{3}}{3} - \frac{\pi}{6}$$

With $n = 4$:

$$I_4 + I_2 = \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{1}{9\sqrt{3}} = \frac{\sqrt{3}}{27}$$

$$I_4 = \frac{\sqrt{3}}{27} - \frac{\sqrt{3}}{3} + \frac{\pi}{6} = -\frac{8\sqrt{3}}{27} + \frac{\pi}{6}$$

With $n = 6$:

$$I_6 + I_4 = \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 = \frac{1}{45\sqrt{3}} = \frac{\sqrt{3}}{135}$$

$$\begin{aligned} I_6 &= \frac{\sqrt{3}}{135} + \frac{8\sqrt{3}}{27} - \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{135} + \frac{40\sqrt{3}}{135} - \frac{\pi}{6} \\ &= \frac{41}{135}\sqrt{3} - \frac{1}{6}\pi \end{aligned}$$

This is in the form $a\sqrt{3} + b\pi$ with $a = \frac{41}{135}$, $b = -\frac{1}{6}$.

(ii) For $0 < x \leq \frac{1}{\sqrt{3}}$, both $x^6 > 0$ and $1 + x^2 > 0$, so the integrand $\frac{x^6}{1+x^2}$ is positive on the interval of integration, and hence $I_6 > 0$. Using part (b)(i):

$$\frac{41\sqrt{3}}{135} - \frac{\pi}{6} > 0$$

$$\frac{\pi}{6} < \frac{41\sqrt{3}}{135}$$

$$\pi < \frac{246\sqrt{3}}{135} = \frac{82\sqrt{3}}{45}$$

as required.

6. The integral I_n is defined, for $n \geq 0$, by

$$I_n = \int_0^{\frac{\pi}{6}} \cos^n 2x \, \mathrm{d}x$$

(a) Prove that, for $n \geq 2$,

$$nI_n = (n-1)I_{n-2} + \frac{\sqrt{3}}{2^{n+1}} \qquad \textbf{[5]}$$

(b) Hence determine the exact value of

$$\int_0^{\frac{\pi}{6}} 64\left(\cos^4 x - \sin^4 x\right)^5 \, \mathrm{d}x \qquad \textbf{[5]}$$

Solution

(a) Write $\cos^n 2x = \cos^{n-1} 2x \times \cos 2x$ and integrate by parts. The formula is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

with $u = \cos^{n-1} 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$, so

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2(n-1)\cos^{n-2} 2x \sin 2x, \qquad v = \frac{1}{2}\sin 2x$$

Then

$$I_n = \left[\frac{1}{2}\cos^{n-1} 2x \sin 2x\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{1}{2}\sin 2x \times \left(-2(n-1)\cos^{n-2} 2x \sin 2x\right)\mathrm{d}x$$

$$= \left[\frac{1}{2}\cos^{n-1} 2x \sin 2x\right]_0^{\frac{\pi}{6}} + (n-1)\int_0^{\frac{\pi}{6}} \cos^{n-2} 2x \sin^2 2x \, \mathrm{d}x$$

At $x = \frac{\pi}{6}$, $\cos 2x = \cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin 2x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, and at $x = 0$, $\sin 2x = 0$, so the bracket is

$$\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2^{n+1}}$$

Using $\sin^2 2x = 1 - \cos^2 2x$:

$$I_n = \frac{\sqrt{3}}{2^{n+1}} + (n-1)\int_0^{\frac{\pi}{6}} \left(\cos^{n-2} 2x - \cos^n 2x\right)\mathrm{d}x$$

$$= \frac{\sqrt{3}}{2^{n+1}} + (n-1)(I_{n-2} - I_n)$$

Collecting the I_n terms on the left:

$$I_n + (n-1)I_n = (n-1)I_{n-2} + \frac{\sqrt{3}}{2^{n+1}}$$

$$nI_n = (n-1)I_{n-2} + \frac{\sqrt{3}}{2^{n+1}}$$

as required.

(b) First simplify the integrand using the double angle formula $\cos 2x = \cos^2 x - \sin^2 x$:

$$\cos^4 x - \sin^4 x = \big(\cos^2 x - \sin^2 x\big)\big(\cos^2 x + \sin^2 x\big) = \cos 2x$$

so

$$\int_0^{\frac{\pi}{6}} 64\big(\cos^4 x - \sin^4 x\big)^5 \,\mathrm{d}x = 64\int_0^{\frac{\pi}{6}} \cos^5 2x \,\mathrm{d}x = 64I_5$$

Now

$$I_1 = \int_0^{\frac{\pi}{6}} \cos 2x \,\mathrm{d}x = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{4}$$

Using part (a) with $n = 3$:

$$3I_3 = 2I_1 + \frac{\sqrt{3}}{16} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{16} = \frac{9\sqrt{3}}{16}$$

$$I_3 = \frac{3\sqrt{3}}{16}$$

Using part (a) with $n = 5$:

$$5I_5 = 4I_3 + \frac{\sqrt{3}}{64} = \frac{48\sqrt{3}}{64} + \frac{\sqrt{3}}{64} = \frac{49\sqrt{3}}{64}$$

$$I_5 = \frac{49\sqrt{3}}{320}$$

Hence

$$\int_0^{\frac{\pi}{6}} 64\big(\cos^4 x - \sin^4 x\big)^5 \,\mathrm{d}x = 64 \times \frac{49\sqrt{3}}{320} = \frac{49\sqrt{3}}{5}$$

7. The integral I_n, where n is an integer, is defined by

$$I_n = \int_0^{2\sqrt{3}} \left(16 - x^2\right)^{-\frac{1}{2}n} \, \mathrm{d}x$$

(a) Find the exact value of I_1. **[2]**

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(16 - x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$16nI_{n+2} = 2^{1-n}\sqrt{3} + (n-1)I_n$$ **[5]**

(c) Find the exact value of I_7, giving your answer in the form $k\sqrt{3}$, where k is a rational number to be determined. **[4]**

Solution

(a) With $n = 1$, the integrand is $\left(16 - x^2\right)^{-\frac{1}{2}}$. The standard result is

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + c \qquad (|x| < a)$$

Using it with $a = 4$:

$$\begin{aligned} I_1 &= \int_0^{2\sqrt{3}} \frac{1}{\sqrt{16 - x^2}} \, \mathrm{d}x \\ &= \left[\arcsin\left(\frac{x}{4}\right)\right]_0^{2\sqrt{3}} \\ &= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin 0 \\ &= \frac{\pi}{3} \end{aligned}$$

(b) Differentiate using the product rule and the chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(16 - x^2\right)^{-\frac{1}{2}n}\right) &= \left(16 - x^2\right)^{-\frac{1}{2}n} + x \times \left(-\frac{n}{2}\right)\left(16 - x^2\right)^{-\frac{1}{2}n-1} \times (-2x) \\ &= \left(16 - x^2\right)^{-\frac{1}{2}n} + nx^2\left(16 - x^2\right)^{-\frac{1}{2}(n+2)} \end{aligned}$$

Writing $x^2 = 16 - \left(16 - x^2\right)$ in the second term gives

$$\begin{aligned} nx^2\left(16 - x^2\right)^{-\frac{1}{2}(n+2)} &= 16n\left(16 - x^2\right)^{-\frac{1}{2}(n+2)} - n\left(16 - x^2\right)\left(16 - x^2\right)^{-\frac{1}{2}(n+2)} \\ &= 16n\left(16 - x^2\right)^{-\frac{1}{2}(n+2)} - n\left(16 - x^2\right)^{-\frac{1}{2}n} \end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(16 - x^2\right)^{-\frac{1}{2}n}\right) = (1-n)\left(16 - x^2\right)^{-\frac{1}{2}n} + 16n\left(16 - x^2\right)^{-\frac{1}{2}(n+2)}$$

Integrating both sides from 0 to $2\sqrt{3}$:

$$\left[x\left(16 - x^2\right)^{-\frac{1}{2}n}\right]_0^{2\sqrt{3}} = (1-n)I_n + 16nI_{n+2}$$

When $x = 2\sqrt{3}$, $16 - x^2 = 4$, so the left-hand side is

$$\begin{aligned} 2\sqrt{3} \times 4^{-\frac{1}{2}n} - 0 &= 2\sqrt{3} \times 2^{-n} \\ &= 2^{1-n}\sqrt{3} \end{aligned}$$

Therefore

$$\begin{aligned} 2^{1-n}\sqrt{3} &= (1-n)I_n + 16nI_{n+2} \\ 16nI_{n+2} &= 2^{1-n}\sqrt{3} + (n-1)I_n \end{aligned}$$

as required.

(c) With $n = 1$, the coefficient of I_1 is zero:

$$\begin{aligned} 16I_3 &= 2^0\sqrt{3} + 0 \times I_1 \\ I_3 &= \frac{\sqrt{3}}{16} \end{aligned}$$

With $n = 3$:

$$\begin{aligned} 48I_5 &= 2^{-2}\sqrt{3} + 2I_3 \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} \\ &= \frac{3\sqrt{3}}{8} \\ I_5 &= \frac{\sqrt{3}}{128} \end{aligned}$$

With $n = 5$:

$$\begin{aligned} 80I_7 &= 2^{-4}\sqrt{3} + 4I_5 \\ &= \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{32} \\ &= \frac{3\sqrt{3}}{32} \\ I_7 &= \frac{3\sqrt{3}}{2560} \end{aligned}$$

This is in the form $k\sqrt{3}$ with $k = \frac{3}{2560}$.

8. It is given that, for $n \geq 0$,

$$I_n = \int_0^{\ln\left(2+\sqrt{3}\right)} \operatorname{sech}^n x \, \mathrm{d}x$$

(a) Show that, for $n \geq 2$,

$$(n-1)I_n = \frac{\sqrt{3}}{2^{n-1}} + (n-2)I_{n-2}$$

You may use the result that $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$. **[5]**

(b) By first finding the exact value of I_1, find the exact value of I_5, giving your answer in the form $a\sqrt{3} + b\pi$, where a and b are rational. **[5]**

Solution

(a) Write $\operatorname{sech}^n x = \operatorname{sech}^{n-2} x \operatorname{sech}^2 x$ and integrate by parts, using

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

with $u = \operatorname{sech}^{n-2} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \operatorname{sech}^2 x$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (n-2)\operatorname{sech}^{n-3} x(-\tanh x \operatorname{sech} x) = -(n-2)\operatorname{sech}^{n-2} x \tanh x, \qquad v = \tanh x$$

Let $L = \ln\left(2+\sqrt{3}\right)$. Integrating by parts gives

$$I_n = \left[\operatorname{sech}^{n-2} x \tanh x\right]_0^L + (n-2)\int_0^L \operatorname{sech}^{n-2} x \tanh^2 x \, \mathrm{d}x$$

Since $e^L = 2+\sqrt{3}$ and $e^{-L} = \frac{1}{2+\sqrt{3}} = 2-\sqrt{3}$, it follows that

$$\cosh L = \frac{\left(2+\sqrt{3}\right) + \left(2-\sqrt{3}\right)}{2} = 2$$

$$\sinh L = \frac{\left(2+\sqrt{3}\right) - \left(2-\sqrt{3}\right)}{2} = \sqrt{3}$$

so $\operatorname{sech} L = \frac{1}{2}$ and $\tanh L = \frac{\sqrt{3}}{2}$. As $\tanh 0 = 0$, the bracket is

$$\left(\frac{1}{2}\right)^{n-2} \times \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2^{n-1}}$$

Using $\tanh^2 x = 1 - \operatorname{sech}^2 x$ in the remaining integral:

$$I_n = \frac{\sqrt{3}}{2^{n-1}} + (n-2)\int_0^L \left(\operatorname{sech}^{n-2} x - \operatorname{sech}^n x\right)\mathrm{d}x$$

$$= \frac{\sqrt{3}}{2^{n-1}} + (n-2)I_{n-2} - (n-2)I_n$$

Collecting the I_n terms on the left, $I_n + (n-2)I_n = (n-1)I_n$, so

$$(n-1)I_n = \frac{\sqrt{3}}{2^{n-1}} + (n-2)I_{n-2}$$

as required.

(b) Write

$$\operatorname{sech} x = \frac{\cosh x}{\cosh^2 x} = \frac{\cosh x}{1+\sinh^2 x}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(\sinh x)) = \frac{\cosh x}{1+\sinh^2 x}$, it follows that

$$I_1 = \int_0^L \frac{\cosh x}{1+\sinh^2 x}\,\mathrm{d}x = [\arctan(\sinh x)]_0^L = \arctan\sqrt{3} - \arctan 0 = \frac{\pi}{3}$$

Using part (a) with $n = 3$:

$$2I_3 = \frac{\sqrt{3}}{4} + I_1 = \frac{\sqrt{3}}{4} + \frac{\pi}{3}$$

$$I_3 = \frac{\sqrt{3}}{8} + \frac{\pi}{6}$$

Using part (a) with $n = 5$:

$$4I_5 = \frac{\sqrt{3}}{16} + 3I_3 = \frac{\sqrt{3}}{16} + \frac{3\sqrt{3}}{8} + \frac{\pi}{2} = \frac{7\sqrt{3}}{16} + \frac{\pi}{2}$$

$$I_5 = \frac{7}{64}\sqrt{3} + \frac{1}{8}\pi$$

so the answer is in the form $a\sqrt{3} + b\pi$ with $a = \frac{7}{64}$, $b = \frac{1}{8}$.

9. Let

$$I_n = \int_0^2 \frac{x^n}{\sqrt{9-x^3}}\,\mathrm{d}x$$

for integers $n \geq 0$.

(a) By considering the derivative of $\sqrt{9-x^3}$ with respect to x, find the exact value of I_2. **[2]**

(b) Given that $n \geq 3$, show that

$$(2n-1)I_n = 18(n-2)I_{n-3} - 2^{n-1}$$ **[5]**

(c) Hence find the exact value of

$$\int_0^2 x^5\sqrt{9-x^3}\,\mathrm{d}x$$ **[4]**

Solution

(a) Using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{9-x^3}\right) = \frac{1}{2}(9-x^3)^{-\frac{1}{2}} \times (-3x^2) = -\frac{3x^2}{2\sqrt{9-x^3}}$$

so $\frac{x^2}{\sqrt{9-x^3}}$ is the derivative of $-\frac{2}{3}\sqrt{9-x^3}$. Hence

$$\begin{aligned} I_2 &= \int_0^2 \frac{x^2}{\sqrt{9-x^3}}\,\mathrm{d}x \\ &= \left[-\frac{2}{3}\sqrt{9-x^3}\right]_0^2 \\ &= -\frac{2}{3}\sqrt{1} + \frac{2}{3}\sqrt{9} \\ &= \frac{4}{3} \end{aligned}$$

(b) Write

$$I_n = \int_0^2 x^{n-2} \times \frac{x^2}{\sqrt{9-x^3}}\,\mathrm{d}x$$

and integrate by parts. The formula is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Take $u = x^{n-2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x^2}{\sqrt{9-x^3}}$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = (n-2)x^{n-3}$ and, by part (a), $v = -\frac{2}{3}\sqrt{9-x^3}$. Then

$$I_n = \left[-\frac{2}{3}x^{n-2}\sqrt{9-x^3}\right]_0^2 + \frac{2}{3}(n-2)\int_0^2 x^{n-3}\sqrt{9-x^3}\,\mathrm{d}x$$

Since $n \geq 3$, the power x^{n-2} is zero at $x = 0$, so the bracket is

$$-\frac{2}{3} \times 2^{n-2} \times \sqrt{1} - 0 = -\frac{2^{n-1}}{3}$$

For the remaining integral, write $\sqrt{9-x^3} = \frac{9-x^3}{\sqrt{9-x^3}}$:

$$\begin{aligned}\int_0^2 x^{n-3}\sqrt{9-x^3}\,\mathrm{d}x &= \int_0^2 \frac{x^{n-3}\left(9-x^3\right)}{\sqrt{9-x^3}}\,\mathrm{d}x \\ &= 9\int_0^2 \frac{x^{n-3}}{\sqrt{9-x^3}}\,\mathrm{d}x - \int_0^2 \frac{x^n}{\sqrt{9-x^3}}\,\mathrm{d}x \\ &= 9I_{n-3} - I_n\end{aligned}$$

Substituting both results:

$$\begin{aligned}I_n &= -\frac{2^{n-1}}{3} + \frac{2}{3}(n-2)(9I_{n-3} - I_n) \\ 3I_n &= -2^{n-1} + 18(n-2)I_{n-3} - 2(n-2)I_n \\ (3+2n-4)I_n &= 18(n-2)I_{n-3} - 2^{n-1} \\ (2n-1)I_n &= 18(n-2)I_{n-3} - 2^{n-1}\end{aligned}$$

as required.

(c) As in part (b), write $\sqrt{9-x^3} = \frac{9-x^3}{\sqrt{9-x^3}}$:

$$\begin{aligned}\int_0^2 x^5\sqrt{9-x^3}\,\mathrm{d}x &= \int_0^2 \frac{x^5\left(9-x^3\right)}{\sqrt{9-x^3}}\,\mathrm{d}x \\ &= 9I_5 - I_8\end{aligned}$$

Using part (b) with $n = 5$, and $I_2 = \frac{4}{3}$ from part (a):

$$9I_5 = 54I_2 - 2^4 = 54 \times \frac{4}{3} - 16 = 56$$

$$I_5 = \frac{56}{9}$$

Using part (b) with $n = 8$:

$$15I_8 = 108I_5 - 2^7 = 108 \times \frac{56}{9} - 128 = 544$$

$$I_8 = \frac{544}{15}$$

Therefore

$$\int_0^2 x^5\sqrt{9-x^3}\,\mathrm{d}x = 56 - \frac{544}{15} = \frac{296}{15}$$

10. For integers $n \geq 1$, let

$$I_n = \int \frac{\sin(nx)}{\cos x}\,\mathrm{d}x$$

(a) Show that, for $n \geq 1$,

$$I_{n+2} = -\frac{2\cos\big((n+1)x\big)}{n+1} - I_n$$ **[6]**

(b) Hence find the exact value of the integral below, giving your answer in the form $a + b\ln c$, where a, b and c are rational numbers to be found.

$$\int_0^{\frac{\pi}{4}} \frac{\sin(7x)}{\cos x}\,\mathrm{d}x$$ **[6]**

Solution

(a) Write $(n+2)x = (n+1)x + x$ and $nx = (n+1)x - x$. By the addition formulae:

$$\sin\big((n+2)x\big) = \sin\big((n+1)x\big)\cos x + \cos\big((n+1)x\big)\sin x$$
$$\sin(nx) = \sin\big((n+1)x\big)\cos x - \cos\big((n+1)x\big)\sin x$$

Adding these gives

$$\sin\big((n+2)x\big) + \sin(nx) = 2\sin\big((n+1)x\big)\cos x$$

Therefore

$$\begin{aligned} I_{n+2} + I_n &= \int \frac{\sin\big((n+2)x\big) + \sin(nx)}{\cos x}\,\mathrm{d}x \\ &= \int \frac{2\sin\big((n+1)x\big)\cos x}{\cos x}\,\mathrm{d}x \\ &= \int 2\sin\big((n+1)x\big)\,\mathrm{d}x \\ &= -\frac{2\cos\big((n+1)x\big)}{n+1} + c \end{aligned}$$

Since I_{n+2} and I_n are indefinite integrals, the constant c can be absorbed into I_{n+2}, so

$$I_{n+2} = -\frac{2\cos\big((n+1)x\big)}{n+1} - I_n$$

as required.

(b) Let

$$J_n = \int_0^{\frac{\pi}{4}} \frac{\sin(nx)}{\cos x}\,\mathrm{d}x$$

Applying the result of part (a) between the limits 0 and $\frac{\pi}{4}$:

$$\begin{aligned} J_{n+2} &= \left[-\frac{2\cos\big((n+1)x\big)}{n+1}\right]_0^{\frac{\pi}{4}} - J_n \\ &= \frac{2}{n+1}\left(1-\cos\frac{(n+1)\pi}{4}\right) - J_n \end{aligned}$$

For the starting value, $\cos x > 0$ for $0 \le x \le \frac{\pi}{4}$, so

$$\begin{aligned} J_1 &= \int_0^{\frac{\pi}{4}} \tan x \,\mathrm{d}x \\ &= \Big[-\ln(\cos x)\Big]_0^{\frac{\pi}{4}} \\ &= -\ln\left(\frac{1}{\sqrt{2}}\right) + \ln 1 \\ &= \frac{1}{2}\ln 2 \end{aligned}$$

Using the reduction formula with $n = 1$, $n = 3$ and $n = 5$:

$$J_3 = \frac{2}{2}\left(1-\cos\frac{\pi}{2}\right) - J_1 = 1 - \frac{1}{2}\ln 2$$

$$J_5 = \frac{2}{4}(1-\cos\pi) - J_3 = 1 - \left(1 - \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2$$

$$J_7 = \frac{2}{6}\left(1-\cos\frac{3\pi}{2}\right) - J_5 = \frac{1}{3} - \frac{1}{2}\ln 2$$

So

$$\int_0^{\frac{\pi}{4}} \frac{\sin(7x)}{\cos x}\,\mathrm{d}x = \frac{1}{3} - \frac{1}{2}\ln 2$$

which is in the form $a + b\ln c$ with $a = \frac{1}{3}$, $b = -\frac{1}{2}$, $c = 2$.

11. For non-negative integers n, let

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x \, \mathrm{d}x$$

(a) Show that $I_n + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$. **[3]**

(b) Explain why $I_n < I_{n-2}$ for $n \geq 2$. **[2]**

(c) Using parts (a) and (b), show that, for $n \geq 2$,

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

By taking $n = 8$, find an interval within which the value of π lies, giving the end-points of your interval as fractions. **[7]**

Solution

(a) For $n \geq 2$, write $\tan^n x = \tan^{n-2} x \tan^2 x$ and use $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned} I_n &= \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \big(\sec^2 x - 1\big) \mathrm{d}x \\ &= \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \sec^2 x \, \mathrm{d}x - \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \, \mathrm{d}x \\ &= \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \sec^2 x \, \mathrm{d}x - I_{n-2} \end{aligned}$$

Using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\tan^{n-1} x\big) = (n-1)\tan^{n-2} x \sec^2 x$$

As $n - 1 \geq 1$, the remaining integral is

$$\begin{aligned} \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \sec^2 x \, \mathrm{d}x &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{1}{4}\pi} \\ &= \frac{1-0}{n-1} \\ &= \frac{1}{n-1} \end{aligned}$$

Substituting this back gives

$$I_n = \frac{1}{n-1} - I_{n-2} \quad \Longrightarrow \quad I_n + I_{n-2} = \frac{1}{n-1}$$

as required.

(b) For $0 < x < \frac{1}{4}\pi$, $0 < \tan x < 1$, so $0 < \tan^2 x < 1$. Multiplying by $\tan^{n-2} x > 0$ gives

$$\tan^n x < \tan^{n-2} x \quad \text{for } 0 < x < \tfrac{1}{4}\pi$$

So the integrand of I_n is less than the integrand of I_{n-2} throughout the interval of integration, and therefore $I_n < I_{n-2}$.

(c) From part (b), $I_n < I_{n-2}$, so using part (a):

$$2I_n < I_n + I_{n-2} = \frac{1}{n-1} \quad \Longrightarrow \quad I_n < \frac{1}{2(n-1)}$$

Replacing n by $n+2$ in part (a), which is valid since $n+2 \geq 2$, gives

$$I_{n+2} + I_n = \frac{1}{n+1}$$

By part (b) with $n+2$ in place of n, $I_{n+2} < I_n$. Therefore

$$\frac{1}{n+1} = I_{n+2} + I_n < 2I_n \quad \Longrightarrow \quad I_n > \frac{1}{2(n+1)}$$

Combining the two inequalities:

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

as required.

To use $n = 8$, first find I_8. Starting from

$$I_0 = \int_0^{\frac{1}{4}\pi} 1 \, \mathrm{d}x = \frac{\pi}{4}$$

apply part (a) with $n = 2$, 4, 6 and 8:

$$I_2 = 1 - I_0 = 1 - \frac{\pi}{4}$$

$$I_4 = \frac{1}{3} - I_2 = \frac{\pi}{4} - \frac{2}{3}$$

$$I_6 = \frac{1}{5} - I_4 = \frac{13}{15} - \frac{\pi}{4}$$

$$I_8 = \frac{1}{7} - I_6 = \frac{\pi}{4} - \frac{76}{105}$$

Taking $n = 8$ in the inequality gives

$$\frac{1}{18} < \frac{\pi}{4} - \frac{76}{105} < \frac{1}{14}$$

Adding $\frac{76}{105}$ throughout:

$$\frac{1}{18} + \frac{76}{105} < \frac{\pi}{4} < \frac{1}{14} + \frac{76}{105}$$

$$\frac{491}{630} < \frac{\pi}{4} < \frac{167}{210}$$

Multiplying by 4 gives

$$\frac{982}{315} < \pi < \frac{334}{105}$$

12. For a positive constant a, the integral C_n is defined by

$$C_n = \int_0^a x^n \cosh 2x \, \mathrm{d}x \qquad (n \geq 0)$$

(a) Show that, for $n \geq 2$,

$$4C_n = n(n-1)C_{n-2} + 2a^n \sinh 2a - na^{n-1} \cosh 2a \qquad \textbf{[7]}$$

(b) Hence show that

$$\int_0^1 x^3 \cosh 2x \, \mathrm{d}x = \frac{1}{16}\left(e^2 + 6 - 19e^{-2}\right) \qquad \textbf{[6]}$$

Solution

(a) The integration by parts formula is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Take $u = x^n$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cosh 2x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$, $v = \frac{1}{2}\sinh 2x$. Since $\sinh 0 = 0$, the lower limit contributes nothing, so

$$\begin{aligned} C_n &= \left[\frac{1}{2}x^n \sinh 2x\right]_0^a - \int_0^a \frac{1}{2} n x^{n-1} \sinh 2x \, \mathrm{d}x \\ &= \frac{1}{2}a^n \sinh 2a - \frac{n}{2}\int_0^a x^{n-1} \sinh 2x \, \mathrm{d}x \end{aligned}$$

Integrate by parts again, with $u = x^{n-1}$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \sinh 2x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2}$, $v = \frac{1}{2}\cosh 2x$. For $n \geq 2$, $x^{n-1} = 0$ at $x = 0$, so again the lower limit contributes nothing:

$$\begin{aligned} \int_0^a x^{n-1} \sinh 2x \, \mathrm{d}x &= \left[\frac{1}{2}x^{n-1} \cosh 2x\right]_0^a - \int_0^a \frac{1}{2}(n-1)x^{n-2} \cosh 2x \, \mathrm{d}x \\ &= \frac{1}{2}a^{n-1} \cosh 2a - \frac{n-1}{2} C_{n-2} \end{aligned}$$

Substituting this into the expression for C_n:

$$\begin{aligned} C_n &= \frac{1}{2}a^n \sinh 2a - \frac{n}{2}\left(\frac{1}{2}a^{n-1} \cosh 2a - \frac{n-1}{2} C_{n-2}\right) \\ &= \frac{1}{2}a^n \sinh 2a - \frac{1}{4}na^{n-1} \cosh 2a + \frac{1}{4}n(n-1)C_{n-2} \end{aligned}$$

Multiplying by 4 gives

$$4C_n = n(n-1)C_{n-2} + 2a^n \sinh 2a - na^{n-1} \cosh 2a$$

as required.

(b) With $a = 1$, the integral is C_3. Using part (a) with $n = 3$ and $a = 1$:

$$4C_3 = 6C_1 + 2\sinh 2 - 3\cosh 2$$

Part (a) holds only for $n \geq 2$, so C_1 is found directly by parts, with $u = x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cosh 2x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, $v = \frac{1}{2}\sinh 2x$:

$$\begin{aligned} C_1 &= \left[\frac{1}{2}x\sinh 2x\right]_0^1 - \int_0^1 \frac{1}{2}\sinh 2x\,\mathrm{d}x \\ &= \frac{1}{2}\sinh 2 - \left[\frac{1}{4}\cosh 2x\right]_0^1 \\ &= \frac{1}{2}\sinh 2 - \frac{1}{4}\cosh 2 + \frac{1}{4} \end{aligned}$$

Substituting:

$$\begin{aligned} 4C_3 &= 6\left(\frac{1}{2}\sinh 2 - \frac{1}{4}\cosh 2 + \frac{1}{4}\right) + 2\sinh 2 - 3\cosh 2 \\ &= 5\sinh 2 - \frac{9}{2}\cosh 2 + \frac{3}{2} \end{aligned}$$

Now write $\sinh 2 = \frac{1}{2}\left(e^2 - e^{-2}\right)$ and $\cosh 2 = \frac{1}{2}\left(e^2 + e^{-2}\right)$:

$$\begin{aligned} 4C_3 &= \frac{5}{2}\left(e^2 - e^{-2}\right) - \frac{9}{4}\left(e^2 + e^{-2}\right) + \frac{3}{2} \\ &= \frac{1}{4}e^2 + \frac{6}{4} - \frac{19}{4}e^{-2} \end{aligned}$$

Dividing by 4 gives

$$\int_0^1 x^3 \cosh 2x\,\mathrm{d}x = C_3 = \frac{1}{16}\left(e^2 + 6 - 19e^{-2}\right)$$

as required.

13. It is given that

$$I_n = \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^n x \, \mathrm{d}x$$

where n is a positive integer.

(a) By writing $\operatorname{cosec}^n x = \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$, or otherwise, show that

$$(n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2} \qquad (n \geq 2)$$ **[5]**

(b) Show that $I_8 = \frac{464}{35}\sqrt{3}$. **[4]**

(c) Prove by induction that, for every positive integer n, I_{2n} is a rational multiple of $\sqrt{3}$. **[6]**

Solution

(a) Write the integrand as $\operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$ and integrate by parts. The integration by parts formula is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Take $u = \operatorname{cosec}^{n-2} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \operatorname{cosec}^2 x$, so

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (n-2)\operatorname{cosec}^{n-3} x(-\operatorname{cosec} x \cot x) = -(n-2)\operatorname{cosec}^{n-2} x \cot x$$

$$v = -\cot x$$

Then

$$I_n = \left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} - \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} (-\cot x)\big(-(n-2)\operatorname{cosec}^{n-2} x \cot x\big)\mathrm{d}x$$

$$= \left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} - (n-2)\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^{n-2} x \cot^2 x \, \mathrm{d}x$$

At $x = \frac{1}{2}\pi$, $\cot x = 0$ and $\operatorname{cosec} x = 1$; at $x = \frac{1}{6}\pi$, $\cot x = \sqrt{3}$ and $\operatorname{cosec} x = 2$. So

$$\left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} = 0 - \left(-\sqrt{3} \times 2^{n-2}\right) = 2^{n-2}\sqrt{3}$$

Using $\cot^2 x = \operatorname{cosec}^2 x - 1$:

$$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^{n-2} x \cot^2 x \, \mathrm{d}x = \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \big(\operatorname{cosec}^n x - \operatorname{cosec}^{n-2} x\big)\mathrm{d}x$$

$$= I_n - I_{n-2}$$

Therefore

$$I_n = 2^{n-2}\sqrt{3} - (n-2)(I_n - I_{n-2})$$

$$I_n + (n-2)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}$$

$$(n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}$$

as required.

(b) First, find I_2:

$$I_2 = \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^2 x \, \mathrm{d}x = \left[-\cot x\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} = 0 - \left(-\sqrt{3}\right) = \sqrt{3}$$

Using part (a) with $n = 4$, $n = 6$ and $n = 8$ in turn:

$$3I_4 = 2^2\sqrt{3} + 2I_2 = 4\sqrt{3} + 2\sqrt{3} = 6\sqrt{3} \quad \Longrightarrow \quad I_4 = 2\sqrt{3}$$

$$5I_6 = 2^4\sqrt{3} + 4I_4 = 16\sqrt{3} + 8\sqrt{3} = 24\sqrt{3} \quad \Longrightarrow \quad I_6 = \frac{24}{5}\sqrt{3}$$

$$7I_8 = 2^6\sqrt{3} + 6I_6 = 64\sqrt{3} + \frac{144}{5}\sqrt{3} = \frac{464}{5}\sqrt{3} \quad \Longrightarrow \quad I_8 = \frac{464}{35}\sqrt{3}$$

as required.

(c) **Base case:** When $n = 1$, part (b) gives $I_2 = \sqrt{3} = 1 \times \sqrt{3}$, and 1 is rational, so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, where k is a positive integer, so that

$$I_{2k} = q\sqrt{3} \quad \text{for some rational number } q$$

Inductive step: Using part (a) with $2k + 2$ in place of n, which is valid since $2k + 2 \geq 2$, and then the inductive hypothesis:

$$\begin{aligned}(2k+1)I_{2k+2} &= 2^{2k}\sqrt{3} + 2kI_{2k} \\ &= 4^k\sqrt{3} + 2kq\sqrt{3} \\ I_{2k+2} &= \frac{4^k + 2kq}{2k+1}\sqrt{3}\end{aligned}$$

Since 4^k, $2k$ and $2k + 1$ are integers with $2k + 1 \neq 0$, and q is rational, $\frac{4^k+2kq}{2k+1}$ is rational. So $I_{2(k+1)}$ is a rational multiple of $\sqrt{3}$, and the statement is true for $n = k + 1$.

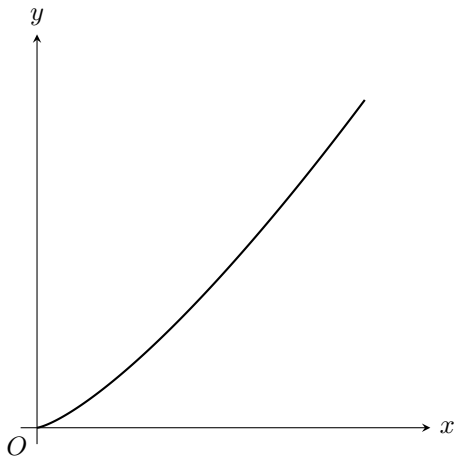
Conclusion: If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for every positive integer n.

14.

$$I_n = \int_0^1 t^n \sqrt{9+16t^2}\,\mathrm{d}t \qquad (n \geq 0)$$

(a) Show that, for $n \geq 2$,

$$16(n+2)I_n = 125 - 9(n-1)I_{n-2}$$ **[6]**



The curve shown in the diagram above is defined by the parametric equations

$$x = t^3 \qquad y = t^4 \qquad 0 \leq t \leq 1$$

This curve is rotated through 2π radians about the y-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by

$$2\pi \int_0^1 t^5 \sqrt{9+16t^2}\,\mathrm{d}t$$ **[5]**

(c) Using the results of parts (a) and (b), and making each step of your working clear, find the exact value of the external surface area of the shell, giving your answer in the form $\frac{p}{q}\pi$, where p and q are positive integers. **[5]**

Solution

(a) Write the integrand as $t^{n-1} \times t\sqrt{9+16t^2}$ and integrate by parts. The integration by parts formula is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

Take $u = t^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = t\left(9+16t^2\right)^{\frac{1}{2}}$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\left(9+16t^2\right)^{\frac{3}{2}} = \frac{3}{2} \times 32t\left(9+16t^2\right)^{\frac{1}{2}} = 48t\left(9+16t^2\right)^{\frac{1}{2}}$, this gives

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (n-1)t^{n-2}, \qquad v = \frac{1}{48}\left(9+16t^2\right)^{\frac{3}{2}}$$

Then

$$I_n = \left[\frac{1}{48}t^{n-1}\left(9+16t^2\right)^{\frac{3}{2}}\right]_0^1 - \frac{n-1}{48}\int_0^1 t^{n-2}\left(9+16t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$$

$$= \frac{125}{48} - \frac{n-1}{48}\int_0^1 t^{n-2}\left(9+16t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$$

since $25^{\frac{3}{2}} = 125$ at the upper limit, and $t^{n-1} = 0$ at $t = 0$ because $n \geq 2$.

Writing $\left(9+16t^2\right)^{\frac{3}{2}} = \left(9+16t^2\right)\sqrt{9+16t^2}$, the remaining integral becomes

$$\int_0^1 t^{n-2}\left(9+16t^2\right)^{\frac{3}{2}}\,\mathrm{d}t = \int_0^1 \left(9t^{n-2}+16t^n\right)\sqrt{9+16t^2}\,\mathrm{d}t$$

$$= 9I_{n-2} + 16I_n$$

Substituting this and multiplying through by 48:

$$48I_n = 125 - (n-1)(9I_{n-2} + 16I_n)$$

$$48I_n + 16(n-1)I_n = 125 - 9(n-1)I_{n-2}$$

$$16(n+2)I_n = 125 - 9(n-1)I_{n-2}$$

as required.

(b) The surface area generated when a parametric curve is rotated through 2π radians about the y-axis is

$$S = 2\pi\int x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Differentiating $x = t^3$ and $y = t^4$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4t^3$$

so

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 9t^4 + 16t^6$$

$$= t^4\left(9+16t^2\right)$$

Since $t^2 \geq 0$, the square root is $t^2\sqrt{9+16t^2}$. The curve runs from $t = 0$ to $t = 1$, with $x = t^3 \geq 0$ throughout, so

$$S = 2\pi\int_0^1 t^3 \times t^2\sqrt{9+16t^2}\,\mathrm{d}t$$

$$= 2\pi\int_0^1 t^5\sqrt{9+16t^2}\,\mathrm{d}t$$

as required.

(c) By part (b), the surface area is $S = 2\pi I_5$. Using the recurrence from part (a) with $n = 5$ and then $n = 3$ needs I_1, which is found directly. From the derivative found in part (a):

$$\begin{aligned} I_1 &= \int_0^1 t\sqrt{9+16t^2}\,\mathrm{d}t \\ &= \left[\frac{1}{48}\left(9+16t^2\right)^{\frac{3}{2}}\right]_0^1 \\ &= \frac{125-27}{48} \\ &= \frac{49}{24} \end{aligned}$$

Putting $n = 3$ into the result of part (a):

$$\begin{aligned} 80I_3 &= 125 - 18I_1 \\ &= 125 - 18 \times \frac{49}{24} \\ &= 125 - \frac{147}{4} \\ &= \frac{353}{4} \end{aligned}$$

so

$$I_3 = \frac{353}{320}$$

Putting $n = 5$ into the result of part (a):

$$\begin{aligned} 112I_5 &= 125 - 36I_3 \\ &= 125 - 36 \times \frac{353}{320} \\ &= 125 - \frac{3177}{80} \\ &= \frac{6823}{80} \end{aligned}$$

so

$$I_5 = \frac{6823}{8960}$$

Hence the external surface area is

$$S = 2\pi \times \frac{6823}{8960} = \frac{6823}{4480}\pi$$

This is in the form $\frac{p}{q}\pi$ with $p = 6823$ and $q = 4480$.