OPTIONAL PURE / FURTHER CALCULUS

Leibnitz's Theorem

Worked solutions 10 questions · 78 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

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1. Given that k is a real non-zero constant and that

$$y = x^4 \cos kx$$

use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = \left(k^4 x^4 + Ak^2 x^2 + B\right) \cos kx + Ckx \left(k^2 x^2 + D\right) \sin kx$$

where A, B, C and D are integers to be determined. **[5]**

Solution

Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

With $n = 4$, the binomial coefficients are $1, 4, 6, 4, 1$, so

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = uv^{(4)} + 4u'v''' + 6u''v'' + 4u'''v' + u^{(4)}v$$

Take $u = x^4$ and $v = \cos kx$. Their derivatives are

$$u' = 4x^3, \qquad v' = -k \sin kx$$

$$u'' = 12x^2, \qquad v'' = -k^2 \cos kx$$

$$u''' = 24x, \qquad v''' = k^3 \sin kx$$

$$u^{(4)} = 24, \qquad v^{(4)} = k^4 \cos kx$$

Substituting into Leibnitz's theorem:

$$\begin{aligned}
\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} &= x^4\left(k^4 \cos kx\right) + 4\left(4x^3\right)\left(k^3 \sin kx\right) + 6\left(12x^2\right)\left(-k^2 \cos kx\right) + 4(24x)(-k \sin kx) + 24 \cos kx \\
&= k^4 x^4 \cos kx + 16k^3 x^3 \sin kx - 72k^2 x^2 \cos kx - 96kx \sin kx + 24 \cos kx \\
&= \left(k^4 x^4 - 72k^2 x^2 + 24\right) \cos kx + \left(16k^3 x^3 - 96kx\right) \sin kx \\
&= \left(k^4 x^4 - 72k^2 x^2 + 24\right) \cos kx + 16kx\left(k^2 x^2 - 6\right) \sin kx
\end{aligned}$$

This is in the required form with $A = -72$, $B = 24$, $C = 16$ and $D = -6$.

2. Given that

$$y = x^3 \sinh 2x$$

use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = Px\left(x^2 + Q\right)\cosh 2x + R\left(x^2 + 1\right)\sinh 2x$$

where P, Q and R are integers to be determined. **[5]**

Solution

Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

Take $u = x^3$ and $v = \sinh 2x$. The derivatives of u are

$$u' = 3x^2, \qquad u'' = 6x, \qquad u''' = 6, \qquad u^{(4)} = u^{(5)} = 0$$

and the derivatives of v are

$$v'' = 4\sinh 2x, \qquad v''' = 8\cosh 2x, \qquad v^{(4)} = 16\sinh 2x, \qquad v^{(5)} = 32\cosh 2x$$

With $n = 5$, the terms containing $u^{(4)}$ and $u^{(5)}$ are zero, so

$$\begin{aligned}
\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} &= uv^{(5)} + 5u'v^{(4)} + 10u''v''' + 10u'''v'' \\
&= x^3 \times 32\cosh 2x + 5 \times 3x^2 \times 16\sinh 2x + 10 \times 6x \times 8\cosh 2x + 10 \times 6 \times 4\sinh 2x \\
&= 32x^3 \cosh 2x + 240x^2 \sinh 2x + 480x \cosh 2x + 240\sinh 2x \\
&= 32x\left(x^2 + 15\right)\cosh 2x + 240\left(x^2 + 1\right)\sinh 2x
\end{aligned}$$

This is in the required form with $P = 32$, $Q = 15$ and $R = 240$.

3. It is given that $y = (3x - 1)e^{2x}$.

(a) Use Leibnitz's theorem to show that, for all positive integers n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 2^{n-1}\,(6x + 3n - 2)\,e^{2x} \qquad \textbf{[5]}$$

(b) Find the value of $\frac{\mathrm{d}^{10} y}{\mathrm{d}x^{10}}$ when $x = 0$. **[2]**

Solution

(a) Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

Let $u = 3x - 1$ and $v = e^{2x}$. Then

$$u' = 3, \qquad u'' = 0$$

so every derivative of u of order 2 or more is zero. Each differentiation of v multiplies it by 2, so for each positive integer k

$$v^{(k)} = 2^k e^{2x}$$

Only the terms with $r = 0$ and $r = 1$ are non-zero, and $\binom{n}{1} = n$, so

$$\begin{aligned}\frac{\mathrm{d}^n y}{\mathrm{d}x^n} &= (3x - 1) \times 2^n e^{2x} + n \times 3 \times 2^{n-1} e^{2x} \\ &= 2^{n-1} e^{2x}(2(3x - 1) + 3n) \\ &= 2^{n-1}(6x + 3n - 2)e^{2x}\end{aligned}$$

as required.

(b) Using part (a) with $n = 10$ and $x = 0$:

$$\begin{aligned}\frac{\mathrm{d}^{10} y}{\mathrm{d}x^{10}} &= 2^9(0 + 30 - 2)e^0 \\ &= 512 \times 28 \\ &= 14\,336\end{aligned}$$

4.

$$f(x) = x^3 \cos(2x)$$

The Taylor series expansion of $f(x)$ about $x = k$ is given by

$$f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \cdots + \frac{(x-k)^r}{r!}f^{(r)}(k) + \cdots$$

Use Leibnitz's theorem to show that the coefficient of $\left(x - \frac{\pi}{2}\right)^7$ in the Taylor series expansion of $f(x)$ about $x = \frac{\pi}{2}$ can be written in the form $p\pi^2 + q$, where p and q are rational numbers to be determined. **[8]**

Solution

Comparing with the given Taylor series, the coefficient of $\left(x - \frac{\pi}{2}\right)^7$ is

$$\frac{f^{(7)}\left(\frac{\pi}{2}\right)}{7!}$$

Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

Take $u = x^3$ and $v = \cos 2x$. The derivatives of u are

$$u' = 3x^2, \qquad u'' = 6x, \qquad u''' = 6, \qquad u^{(4)} = 0$$

so every derivative of u of order 4 or more is zero, and only the terms with $r = 0, 1, 2, 3$ survive. The derivatives of v needed are

$$v^{(4)} = 16\cos 2x$$

$$v^{(5)} = -32\sin 2x$$

$$v^{(6)} = -64\cos 2x$$

$$v^{(7)} = 128\sin 2x$$

Applying Leibnitz's theorem with $n = 7$:

$$\begin{aligned} f^{(7)}(x) &= \binom{7}{0}uv^{(7)} + \binom{7}{1}u'v^{(6)} + \binom{7}{2}u''v^{(5)} + \binom{7}{3}u'''v^{(4)} \\ &= x^3(128\sin 2x) + 7(3x^2)(-64\cos 2x) + 21(6x)(-32\sin 2x) + 35(6)(16\cos 2x) \\ &= 128x^3\sin 2x - 1344x^2\cos 2x - 4032x\sin 2x + 3360\cos 2x \end{aligned}$$

At $x = \frac{\pi}{2}$, $\sin 2x = \sin\pi = 0$ and $\cos 2x = \cos\pi = -1$, so the first and third terms vanish:

$$\begin{aligned} f^{(7)}\left(\tfrac{\pi}{2}\right) &= -1344\left(\frac{\pi^2}{4}\right)(-1) + 3360(-1) \\ &= 336\pi^2 - 3360 \end{aligned}$$

Since $7! = 5040$, the coefficient is

$$\begin{aligned} \frac{336\pi^2 - 3360}{5040} &= \frac{336}{5040}\pi^2 - \frac{3360}{5040} \\ &= \frac{1}{15}\pi^2 - \frac{2}{3} \end{aligned}$$

This is in the form $p\pi^2 + q$ with $p = \frac{1}{15}$ and $q = -\frac{2}{3}$.

5.

$$f(x) = x^3 \ln x, \qquad x > 0$$

The Taylor series expansion of $f(x)$ about $x = k$ is given by

$$f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \cdots + \frac{(x-k)^r}{r!}f^{(r)}(k) + \cdots$$

(a) Use Leibnitz's theorem to show that

$$f^{(6)}(x) = \frac{12}{x^3}$$ **[5]**

(b) Hence find the coefficients of $(x-1)^6$ and $(x-1)^7$ in the Taylor series expansion of $f(x)$ about $x = 1$, giving each as a fraction in its simplest form. **[3]**

Solution

(a) Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

Let $u = x^3$ and $v = \ln x$. The derivatives of u are

$$u' = 3x^2, \qquad u'' = 6x, \qquad u''' = 6, \qquad u^{(r)} = 0 \text{ for } r \geq 4$$

so only the terms with $r = 0, 1, 2, 3$ are non-zero. The derivatives of v needed are

$$v' = \frac{1}{x}, \qquad v'' = -\frac{1}{x^2}, \qquad v''' = \frac{2}{x^3}$$

$$v^{(4)} = -\frac{6}{x^4}, \qquad v^{(5)} = \frac{24}{x^5}, \qquad v^{(6)} = -\frac{120}{x^6}$$

Applying Leibnitz's theorem with $n = 6$:

$$\begin{aligned} f^{(6)}(x) &= \binom{6}{0} uv^{(6)} + \binom{6}{1} u'v^{(5)} + \binom{6}{2} u''v^{(4)} + \binom{6}{3} u'''v''' \\ &= 1 \times x^3 \times \left(-\frac{120}{x^6}\right) + 6 \times 3x^2 \times \frac{24}{x^5} + 15 \times 6x \times \left(-\frac{6}{x^4}\right) + 20 \times 6 \times \frac{2}{x^3} \\ &= -\frac{120}{x^3} + \frac{432}{x^3} - \frac{540}{x^3} + \frac{240}{x^3} \\ &= \frac{12}{x^3} \end{aligned}$$

as required.

(b) The coefficient of $(x-1)^r$ in the Taylor series about $x = 1$ is $\frac{f^{(r)}(1)}{r!}$. Using part (a):

$$f^{(6)}(1) = \frac{12}{1^3} = 12 \quad \Longrightarrow \quad \text{coefficient of } (x-1)^6 = \frac{12}{6!} = \frac{12}{720} = \frac{1}{60}$$

Differentiating the result of part (a):

$$f^{(7)}(x) = -\frac{36}{x^4} \quad \Longrightarrow \quad f^{(7)}(1) = -36$$

so

$$\text{coefficient of } (x-1)^7 = -\frac{36}{7!} = -\frac{36}{5040} = -\frac{1}{140}$$

The coefficients of $(x-1)^6$ and $(x-1)^7$ are $\frac{1}{60}$ and $-\frac{1}{140}$ respectively.

6. A curve has equation

$$y = e^{-x}\cos x$$

(a) Use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -4y \qquad \textbf{[5]}$$

(b) Hence find $\frac{\mathrm{d}^{20}y}{\mathrm{d}x^{20}}$ in terms of y, and find the exact value of $\frac{\mathrm{d}^{22}y}{\mathrm{d}x^{22}}$ when $x = \frac{\pi}{2}$. **[4]**

Solution

(a) Write $y = uv$ with $u = e^{-x}$ and $v = \cos x$. Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n}\binom{n}{r}u^{(r)}v^{(n-r)}$$

The derivatives needed are

$$u' = -e^{-x}, \quad u'' = e^{-x}, \quad u''' = -e^{-x}, \quad u^{(4)} = e^{-x}$$

$$v' = -\sin x, \quad v'' = -\cos x, \quad v''' = \sin x, \quad v^{(4)} = \cos x$$

With $n = 4$, the binomial coefficients are $1, 4, 6, 4, 1$, so

$$\begin{aligned}\frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= uv^{(4)} + 4u'v''' + 6u''v'' + 4u'''v' + u^{(4)}v \\ &= e^{-x}\cos x + 4\big(-e^{-x}\big)\sin x + 6e^{-x}(-\cos x) + 4\big(-e^{-x}\big)(-\sin x) + e^{-x}\cos x \\ &= e^{-x}\cos x - 4e^{-x}\sin x - 6e^{-x}\cos x + 4e^{-x}\sin x + e^{-x}\cos x \\ &= -4e^{-x}\cos x \\ &= -4y\end{aligned}$$

as required.

(b) Using part (a), differentiate both sides of $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -4y$ four times:

$$\frac{\mathrm{d}^8y}{\mathrm{d}x^8} = -4\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = (-4)^2y$$

Repeating this, each further four differentiations multiply by -4, so $\frac{\mathrm{d}^{4k}y}{\mathrm{d}x^{4k}} = (-4)^k y$. With $k = 5$:

$$\frac{\mathrm{d}^{20}y}{\mathrm{d}x^{20}} = (-4)^5 y = -1024y$$

Differentiating twice more gives

$$\frac{\mathrm{d}^{22}y}{\mathrm{d}x^{22}} = -1024\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

By Leibnitz's theorem with $n = 2$, using the derivatives from part (a):

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= uv'' + 2u'v' + u''v \\ &= e^{-x}(-\cos x) + 2\big(-e^{-x}\big)(-\sin x) + e^{-x}\cos x \\ &= 2e^{-x}\sin x\end{aligned}$$

When $x = \frac{\pi}{2}$, $\sin x = 1$, so $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2e^{-\pi/2}$. Therefore

$$\frac{\mathrm{d}^{22}y}{\mathrm{d}x^{22}} = -1024 \times 2e^{-\pi/2} = -2048e^{-\pi/2}$$

7. A curve has equation

$$y = e^{2x}\sin x$$

(a) Use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = e^{2x}\left(41\cos x - 38\sin x\right) \qquad \textbf{[5]}$$

(b) Hence express $\frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$ in the form

$$Re^{2x}\cos(x+\alpha)$$

where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give R as an exact surd in its simplest form and α to 3 significant figures. **[3]**

Solution

(a) Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n}\binom{n}{r}u^{(r)}v^{(n-r)}$$

Let $u = e^{2x}$ and $v = \sin x$. Each differentiation of u multiplies it by 2, so

$$u^{(k)} = 2^k e^{2x}$$

The derivatives of v are

$$v' = \cos x, \qquad v'' = -\sin x, \qquad v''' = -\cos x, \qquad v^{(4)} = \sin x, \qquad v^{(5)} = \cos x$$

With $n = 5$, the binomial coefficients are $1, 5, 10, 10, 5, 1$, so

$$\begin{aligned}\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} &= e^{2x}\cos x + 5\left(2e^{2x}\right)\sin x + 10\left(4e^{2x}\right)(-\cos x)\\ &\quad + 10\left(8e^{2x}\right)(-\sin x) + 5\left(16e^{2x}\right)\cos x + \left(32e^{2x}\right)\sin x\\ &= e^{2x}(\cos x + 10\sin x - 40\cos x - 80\sin x + 80\cos x + 32\sin x)\\ &= e^{2x}((1 - 40 + 80)\cos x + (10 - 80 + 32)\sin x)\\ &= e^{2x}(41\cos x - 38\sin x)\end{aligned}$$

as required.

(b) Using the addition formulae:

$$R\cos(x+\alpha) = R\cos\alpha\cos x - R\sin\alpha\sin x$$

Comparing coefficients with $41\cos x - 38\sin x$ from part (a):

$$R\cos\alpha = 41, \qquad R\sin\alpha = 38$$

Squaring and adding gives

$$R = \sqrt{41^2 + 38^2} = \sqrt{3125} = 25\sqrt{5}$$

Dividing gives

$$\tan\alpha = \frac{38}{41} \quad \Longrightarrow \quad \alpha = \arctan\frac{38}{41} = 0.747 \qquad (3 \text{ s.f.})$$

This lies in $0 < \alpha < \frac{\pi}{2}$, so

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = 25\sqrt{5}\, e^{2x}\cos(x + 0.747)$$

which is in the form $Re^{2x}\cos(x+\alpha)$ with $R = 25\sqrt{5}$ and $\alpha = 0.747$.

8. It is given that

$$y = \left(x^2 + 3\right) e^{-2x}$$

(a) Use Leibnitz's theorem to show that, for all positive integers k,

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-2)^{k-2} e^{-2x} \left(Ax^2 + Bkx + Ck(k-1) + D\right)$$

where A, B, C and D are integers to be determined. **[5]**

(b) Hence find the least value of k for which the equation

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = 0$$

has a real solution, and find this solution. **[3]**

Solution

(a) Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(uv) = \sum_{r=0}^{k} \binom{k}{r} u^{(r)} v^{(k-r)}$$

Take $u = x^2 + 3$ and $v = e^{-2x}$. Each differentiation of v multiplies it by -2, so for any $m \geq 0$

$$v^{(m)} = (-2)^m e^{-2x}$$

The derivatives of u are

$$u' = 2x, \qquad u'' = 2, \qquad u^{(r)} = 0 \text{ for } r \geq 3$$

so only the terms with $r = 0, 1, 2$ are non-zero:

$$\begin{aligned}\frac{\mathrm{d}^k y}{\mathrm{d}x^k} &= \binom{k}{0}(-2)^k e^{-2x}\left(x^2+3\right) + \binom{k}{1}(-2)^{k-1} e^{-2x}(2x) + \binom{k}{2}(-2)^{k-2} e^{-2x}(2) \\ &= (-2)^k e^{-2x}\left(x^2+3\right) + 2kx(-2)^{k-1} e^{-2x} + k(k-1)(-2)^{k-2} e^{-2x}\end{aligned}$$

Taking out the factor $(-2)^{k-2} e^{-2x}$, and using $(-2)^k = 4(-2)^{k-2}$ and $(-2)^{k-1} = -2(-2)^{k-2}$:

$$\begin{aligned}\frac{\mathrm{d}^k y}{\mathrm{d}x^k} &= (-2)^{k-2} e^{-2x}\left(4\left(x^2+3\right) - 4kx + k(k-1)\right) \\ &= (-2)^{k-2} e^{-2x}\left(4x^2 - 4kx + k(k-1) + 12\right)\end{aligned}$$

This is in the required form with $A = 4$, $B = -4$, $C = 1$ and $D = 12$.

(b) Using part (a), since $e^{-2x} > 0$ for all x and $(-2)^{k-2} \neq 0$, the equation $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = 0$ holds exactly when

$$4x^2 - 4kx + k^2 - k + 12 = 0$$

This quadratic in x has a real solution when its discriminant is non-negative:

$$\begin{aligned} b^2 - 4ac &= (-4k)^2 - 4 \times 4 \times \left(k^2 - k + 12\right) \\ &= 16k^2 - 16k^2 + 16k - 192 \\ &= 16(k - 12) \end{aligned}$$

So a real solution exists if and only if $k \geq 12$, and the least value is $k = 12$.

When $k = 12$:

$$4x^2 - 48x + 144 = 0 \quad \Longrightarrow \quad 4(x-6)^2 = 0 \quad \Longrightarrow \quad x = 6$$

The least value is $k = 12$, and the solution is $x = 6$.

9. It is given that $y = e^{2x}u$, where u is a function of x. The rth derivatives $\frac{\mathrm{d}^r y}{\mathrm{d}x^r}$ and $\frac{\mathrm{d}^r u}{\mathrm{d}x^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively, with $u^{(0)} = u$.
You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^{2x}\sum_{r=0}^{n}\binom{n}{r}2^{n-r}u^{(r)} \qquad \textbf{[7]}$$

(b) Hence find $\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(x^3e^{2x}\right)$, giving your answer in the form $e^{2x}p(x)$, where $p(x)$ is a polynomial. **[2]**

Solution

(a) **Base case:** When $n = 1$, the product rule gives

$$y^{(1)} = 2e^{2x}u + e^{2x}u^{(1)} = e^{2x}\left(2u + u^{(1)}\right)$$

The right-hand side of the statement with $n = 1$ is

$$e^{2x}\left(\binom{1}{0}2^1u^{(0)} + \binom{1}{1}2^0u^{(1)}\right) = e^{2x}\left(2u + u^{(1)}\right)$$

so the statement is true for $n = 1$.

Inductive hypothesis: Assume the statement is true for $n = k$, where k is a positive integer, so that

$$y^{(k)} = e^{2x}\sum_{r=0}^{k}\binom{k}{r}2^{k-r}u^{(r)}$$

Inductive step: Differentiate the hypothesis using the product rule, with $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}\right) = 2e^{2x}$ and the derivative of $u^{(r)}$ being $u^{(r+1)}$:

$$\begin{aligned} y^{(k+1)} &= 2e^{2x}\sum_{r=0}^{k}\binom{k}{r}2^{k-r}u^{(r)} + e^{2x}\sum_{r=0}^{k}\binom{k}{r}2^{k-r}u^{(r+1)} \\ &= e^{2x}\left[\sum_{r=0}^{k}\binom{k}{r}2^{k+1-r}u^{(r)} + \sum_{r=0}^{k}\binom{k}{r}2^{k-r}u^{(r+1)}\right] \end{aligned}$$

In the second sum, replace r by $r-1$, so that it runs from $r = 1$ to $r = k+1$:

$$\sum_{r=0}^{k}\binom{k}{r}2^{k-r}u^{(r+1)} = \sum_{r=1}^{k+1}\binom{k}{r-1}2^{k+1-r}u^{(r)}$$

Taking the $r = 0$ term out of the first sum and the $r = k+1$ term out of the second, with $\binom{k}{0} = \binom{k}{k} = 1$, and then using the given result:

$$\begin{aligned} y^{(k+1)} &= e^{2x}\left[2^{k+1}u + \sum_{r=1}^{k}\left(\binom{k}{r} + \binom{k}{r-1}\right)2^{k+1-r}u^{(r)} + u^{(k+1)}\right] \\ &= e^{2x}\left[2^{k+1}u + \sum_{r=1}^{k}\binom{k+1}{r}2^{k+1-r}u^{(r)} + u^{(k+1)}\right] \end{aligned}$$

Since $\binom{k+1}{0} = \binom{k+1}{k+1} = 1$, the term $2^{k+1}u$ is the $r = 0$ term $\binom{k+1}{0}2^{k+1}u^{(0)}$, and $u^{(k+1)}$ is the $r = k+1$ term $\binom{k+1}{k+1}2^0 u^{(k+1)}$. Hence

$$y^{(k+1)} = e^{2x}\sum_{r=0}^{k+1}\binom{k+1}{r}2^{(k+1)-r}u^{(r)}$$

which is the statement for $n = k+1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k+1$. Since it is true for $n = 1$, by the principle of mathematical induction the statement is true for all positive integers n.

(b) Take $u = x^3$, so that

$$u^{(1)} = 3x^2, \qquad u^{(2)} = 6x, \qquad u^{(3)} = 6, \qquad u^{(4)} = 0$$

Using part (a) with $n = 4$:

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(x^3e^{2x}\right) &= e^{2x}\left[\binom{4}{0}2^4x^3 + \binom{4}{1}2^3(3x^2) + \binom{4}{2}2^2(6x) + \binom{4}{3}2^1(6) + \binom{4}{4}2^0(0)\right]\\ &= e^{2x}\left(16x^3 + 4\times 8\times 3x^2 + 6\times 4\times 6x + 4\times 2\times 6\right)\\ &= e^{2x}\left(16x^3 + 96x^2 + 144x + 48\right)\end{aligned}$$

10. It is given that $y = x^2 \ln(1+x)$, where $x > -1$.

(a) Show that

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \frac{2\left(x^2+3x+3\right)}{(1+x)^3} \qquad \textbf{[5]}$$

(b) Prove by mathematical induction that, for every integer $n \geq 3$,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1}\frac{(n-3)!\left(2x^2+2nx+n(n-1)\right)}{(1+x)^n} \qquad \textbf{[6]}$$

Solution

(a) Leibnitz's theorem, with $u^{(r)}$ denoting the rth derivative of u, is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}$$

Take $u = x^2$ and $v = \ln(1+x)$. The derivatives are

$$u' = 2x, \qquad u'' = 2, \qquad u''' = 0$$

$$v' = \frac{1}{1+x}, \qquad v'' = -\frac{1}{(1+x)^2}, \qquad v''' = \frac{2}{(1+x)^3}$$

With $n = 3$, the term containing $u''' = 0$ vanishes, so

$$\begin{aligned}
\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} &= uv''' + 3u'v'' + 3u''v' \\
&= \frac{2x^2}{(1+x)^3} - \frac{6x}{(1+x)^2} + \frac{6}{1+x} \\
&= \frac{2x^2 - 6x(1+x) + 6(1+x)^2}{(1+x)^3} \\
&= \frac{2x^2 - 6x - 6x^2 + 6 + 12x + 6x^2}{(1+x)^3} \\
&= \frac{2x^2 + 6x + 6}{(1+x)^3} \\
&= \frac{2\left(x^2+3x+3\right)}{(1+x)^3}
\end{aligned}$$

as required.

(b) **Base case:** When $n = 3$, the right-hand side is

$$(-1)^2 \frac{0!\left(2x^2+6x+6\right)}{(1+x)^3} = \frac{2\left(x^2+3x+3\right)}{(1+x)^3}$$

which equals $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ by part (a). So the statement is true for $n = 3$.

Inductive hypothesis: Assume the statement is true for $n = k$, where $k \geq 3$, so that

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^{k-1}(k-3)!\big(2x^2 + 2kx + k(k-1)\big)(1+x)^{-k}$$

Inductive step: Differentiating the assumed result using the product rule:

$$\begin{aligned}\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} &= (-1)^{k-1}(k-3)!\big[(4x+2k)(1+x)^{-k} - k\big(2x^2+2kx+k(k-1)\big)(1+x)^{-k-1}\big] \\ &= (-1)^{k-1}(k-3)!\,\frac{(4x+2k)(1+x) - k\big(2x^2+2kx+k^2-k\big)}{(1+x)^{k+1}}\end{aligned}$$

The numerator is

$$\begin{aligned}&4x^2 + (4+2k)x + 2k - \big(2kx^2 + 2k^2x + k^3 - k^2\big) \\ &= (4-2k)x^2 + \big(4+2k-2k^2\big)x + \big(2k + k^2 - k^3\big) \\ &= -(k-2)\big[2x^2 + 2(k+1)x + (k+1)k\big]\end{aligned}$$

since $(k-2)(k+1) = k^2 - k - 2$. Since $k \geq 3$, $(k-2)(k-3)! = (k-2)!$, so

$$\begin{aligned}\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} &= (-1)^{k-1} \times (-1)(k-2)(k-3)!\,\frac{2x^2 + 2(k+1)x + (k+1)k}{(1+x)^{k+1}} \\ &= (-1)^{(k+1)-1}\frac{((k+1)-3)!\big(2x^2 + 2(k+1)x + (k+1)((k+1)-1)\big)}{(1+x)^{k+1}}\end{aligned}$$

which is the statement for $n = k+1$.

Conclusion: If the statement is true for $n = k$, then it is true for $n = k+1$. Since it is true for $n = 3$, by the principle of mathematical induction the statement is true for every integer $n \geq 3$.