OPTIONAL PURE / FURTHER CALCULUS

L'Hôpital's Rule

Worked solutions 12 questions · 53 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · AQA 7367

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1. Which of the following limits is an indeterminate form to which L'Hôpital's rule can be applied directly?

$$\lim_{x\to 0}\left(\frac{e^{2x}}{\sin x}\right) \qquad \lim_{x\to \frac{\pi}{2}}\left(\frac{\cos x}{1+\sin x}\right) \qquad \lim_{x\to \infty}\left(\frac{\ln x}{x^2}\right) \qquad \lim_{x\to 0}\left(\frac{\sinh x}{e^x+1}\right) \qquad \textbf{[1]}$$

Solution

Substitute the limiting value into the numerator and the denominator of each limit.

For the first, $e^{2x}\to 1$ and $\sin x\to 0$ as $x\to 0$. This is the form $\frac{1}{0}$, which is not indeterminate.

For the second, $\cos x\to 0$ and $1+\sin x\to 2$ as $x\to \frac{\pi}{2}$. This is the form $\frac{0}{2}$, which is not indeterminate.

For the fourth, $\sinh x\to 0$ and $e^x+1\to 2$ as $x\to 0$. This is also the form $\frac{0}{2}$, which is not indeterminate.

For the third, $\ln x\to \infty$ and $x^2\to \infty$ as $x\to \infty$. This is the indeterminate form $\frac{\infty}{\infty}$, so L'Hôpital's rule can be applied directly.

The answer is

$$\lim_{x\to \infty}\left(\frac{\ln x}{x^2}\right)$$

2. Find

$$\lim_{x\to 0}\left(\frac{\tan 6x}{\ln(1+4x)}\right)$$

$\frac{2}{3}$ 1 $\frac{3}{2}$ 6 **[1]**

Solution

As $x \to 0$, $\tan 6x \to 0$ and $\ln(1+4x) \to 0$, so the limit has the form $\frac{0}{0}$. L'Hôpital's rule states that if $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$, then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Differentiating the numerator and the denominator separately:

$$\lim_{x\to 0}\left(\frac{\tan 6x}{\ln(1+4x)}\right) = \lim_{x\to 0}\left(\frac{6\sec^2 6x}{\frac{4}{1+4x}}\right)$$

$$= \frac{6\times 1}{4}$$

$$= \frac{3}{2}$$

3. Which one of the expressions below is equal to 1?

$$\lim_{x\to\infty}\left(\frac{\ln x}{x}\right) \qquad \lim_{x\to 0}\left(\frac{e^{2x}-1}{x}\right) \qquad \lim_{x\to 0^+}(x^x) \qquad \lim_{x\to 0}\left(\frac{\ln(1+3x)}{x}\right) \qquad \textbf{[1]}$$

Solution

Each limit has an indeterminate form. L'Hôpital's rule states that if $f(x)\to 0$ and $g(x)\to 0$, or $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$, then

$$\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

As $x\to\infty$, both $\ln x\to\infty$ and $x\to\infty$, so

$$\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)=\lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right)=\lim_{x\to\infty}\left(\frac{1}{x}\right)=0$$

As $x\to 0$, both $e^{2x}-1\to 0$ and $x\to 0$, so

$$\lim_{x\to 0}\left(\frac{e^{2x}-1}{x}\right)=\lim_{x\to 0}\left(\frac{2e^{2x}}{1}\right)=2$$

As $x\to 0$, both $\ln(1+3x)\to 0$ and $x\to 0$, so

$$\lim_{x\to 0}\left(\frac{\ln(1+3x)}{x}\right)=\lim_{x\to 0}\left(\frac{\frac{3}{1+3x}}{1}\right)=3$$

For the remaining limit, write $x^x=e^{x\ln x}$. As $x\to 0^+$, $x\ln x=\frac{\ln x}{1/x}$, where $\ln x\to-\infty$ and $\frac{1}{x}\to\infty$, so

$$\begin{aligned}\lim_{x\to 0^+}(x\ln x)&=\lim_{x\to 0^+}\left(\frac{\ln x}{\frac{1}{x}}\right)\\&=\lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)\\&=\lim_{x\to 0^+}(-x)\\&=0\end{aligned}$$

Hence

$$\lim_{x\to 0^+}(x^x)=e^0=1$$

The expression equal to 1 is $\lim_{x\to 0^+}(x^x)$.

4. Use L'Hôpital's rule to determine

$$\lim_{x\to 0}\left(\frac{\sec(2x)-1}{x\sin(3x)}\right) \qquad \textbf{[5]}$$

Solution

As $x\to 0$, $\sec(2x)-1\to 1-1=0$ and $x\sin(3x)\to 0\times 0=0$, so the limit is of the form $\frac{0}{0}$. L'Hôpital's rule states that if $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$, then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Differentiate the numerator, and the denominator using the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\sec(2x)-1\big)=2\sec(2x)\tan(2x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x\sin(3x)\big)=\sin(3x)+3x\cos(3x)$$

so

$$\lim_{x\to 0}\left(\frac{\sec(2x)-1}{x\sin(3x)}\right)=\lim_{x\to 0}\left(\frac{2\sec(2x)\tan(2x)}{\sin(3x)+3x\cos(3x)}\right)$$

At $x=0$ the new numerator is $2\times 1\times 0=0$ and the new denominator is $0+0=0$, so this is still of the form $\frac{0}{0}$. Apply L'Hôpital's rule again, differentiating each part with the product rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big(2\sec(2x)\tan(2x)\big)&=2\big(2\sec(2x)\tan(2x)\big)\tan(2x)+2\sec(2x)\big(2\sec^2(2x)\big)\\&=4\sec(2x)\tan^2(2x)+4\sec^3(2x)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(3x)+3x\cos(3x)\big)&=3\cos(3x)+3\cos(3x)-9x\sin(3x)\\&=6\cos(3x)-9x\sin(3x)\end{aligned}$$

As $x\to 0$, $\sec(2x)\to 1$, $\tan(2x)\to 0$, $\cos(3x)\to 1$ and $x\sin(3x)\to 0$, so

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sec(2x)-1}{x\sin(3x)}\right)&=\lim_{x\to 0}\left(\frac{4\sec(2x)\tan^2(2x)+4\sec^3(2x)}{6\cos(3x)-9x\sin(3x)}\right)\\&=\frac{4\times 1\times 0+4\times 1}{6\times 1-0}\\&=\frac{2}{3}\end{aligned}$$

5. Use L'Hôpital's rule to prove that

$$\lim_{x\to 1}\left(\frac{x\ln x - x + 1}{\sin^2(\pi x)}\right) = \frac{1}{2\pi^2} \qquad \textbf{[5]}$$

Solution

Let $f(x) = x\ln x - x + 1$ and $g(x) = \sin^2(\pi x)$. At $x = 1$:

$$f(1) = 1 \times \ln 1 - 1 + 1 = 0, \qquad g(1) = \sin^2 \pi = 0$$

The limit therefore has the indeterminate form $\frac{0}{0}$. L'Hôpital's rule states that if $f(x) \to 0$ and $g(x) \to 0$ as $x \to 1$, then

$$\lim_{x\to 1}\frac{f(x)}{g(x)} = \lim_{x\to 1}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. Differentiating, using the product rule for $x\ln x$, the chain rule for $\sin^2(\pi x)$ and then the double angle formula:

$$f'(x) = \ln x + x \times \frac{1}{x} - 1 = \ln x$$

$$g'(x) = 2\sin(\pi x) \times \pi\cos(\pi x) = \pi\sin(2\pi x)$$

At $x = 1$, $f'(1) = \ln 1 = 0$ and $g'(1) = \pi \sin 2\pi = 0$. This is still of the form $\frac{0}{0}$, so L'Hôpital's rule is applied a second time. Differentiating again:

$$f''(x) = \frac{1}{x}$$

$$g''(x) = 2\pi^2\cos(2\pi x)$$

At $x = 1$, $f''(1) = 1$ and $g''(1) = 2\pi^2\cos 2\pi = 2\pi^2 \neq 0$, so the limit can now be evaluated by substitution. Hence

$$\begin{aligned}\lim_{x\to 1}\left(\frac{x\ln x - x + 1}{\sin^2(\pi x)}\right) &= \lim_{x\to 1}\left(\frac{\ln x}{\pi\sin(2\pi x)}\right)\\ &= \lim_{x\to 1}\left(\frac{\frac{1}{x}}{2\pi^2\cos(2\pi x)}\right)\\ &= \frac{1}{2\pi^2\cos 2\pi}\\ &= \frac{1}{2\pi^2}\end{aligned}$$

as required.

6. Use L'Hôpital's rule to show that

$$\lim_{x\to\frac{\pi}{6}} \frac{\ln(2\sin x)+e^{\cos(3x)}-1}{\sin(6x)} = \frac{3-\sqrt{3}}{6} \qquad \textbf{[5]}$$

Solution

Let $f(x)=\ln(2\sin x)+e^{\cos(3x)}-1$ and $g(x)=\sin(6x)$. Near $x=\frac{\pi}{6}$, $\sin x>0$, so $\ln(2\sin x)$ is defined.

First check the values at $x=\frac{\pi}{6}$:

$$f\left(\tfrac{\pi}{6}\right)=\ln\left(2\times\tfrac{1}{2}\right)+e^{\cos\left(\frac{\pi}{2}\right)}-1=\ln 1+e^{0}-1=0+1-1=0$$

$$g\left(\tfrac{\pi}{6}\right)=\sin\pi=0$$

So the limit is of the form $\frac{0}{0}$. L'Hôpital's rule states that if $f(x)\to 0$ and $g(x)\to 0$ as $x\to\frac{\pi}{6}$, then

$$\lim_{x\to\frac{\pi}{6}}\frac{f(x)}{g(x)}=\lim_{x\to\frac{\pi}{6}}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Differentiating with the chain rule:

$$f'(x)=\frac{2\cos x}{2\sin x}+(-3\sin(3x))e^{\cos(3x)}=\cot x-3\sin(3x)\,e^{\cos(3x)}$$

$$g'(x)=6\cos(6x)$$

At $x=\frac{\pi}{6}$, using $\cot\frac{\pi}{6}=\sqrt{3}$, $\sin\frac{\pi}{2}=1$ and $\cos\frac{\pi}{2}=0$:

$$f'\left(\tfrac{\pi}{6}\right)=\sqrt{3}-3\times 1\times e^{0}=\sqrt{3}-3$$

$$g'\left(\tfrac{\pi}{6}\right)=6\cos\pi=-6$$

Since $g'\left(\frac{\pi}{6}\right)\neq 0$, the limit is

$$\lim_{x\to\frac{\pi}{6}}\frac{\ln(2\sin x)+e^{\cos(3x)}-1}{\sin(6x)}=\frac{\sqrt{3}-3}{-6}=\frac{3-\sqrt{3}}{6}$$

as required.

7. Prove that

$$\lim_{x\to 0}\left(\frac{\ln(\cosh x)-\frac{1}{2}\tan^2 x}{x^4}\right)=-\frac{5}{12} \qquad \textbf{[5]}$$

Solution

Let $f(x)=\ln(\cosh x)-\frac{1}{2}\tan^2 x$ and $g(x)=x^4$. Since $\cosh 0=1$ and $\tan 0=0$, we have $f(0)=\ln 1-0=0$ and $g(0)=0$, so the limit has the form $\frac{0}{0}$. L'Hôpital's rule states that if $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$, then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Differentiating once:

$$f'(x)=\frac{\sinh x}{\cosh x}-\tan x\sec^2 x=\tanh x-\tan x\sec^2 x$$
$$g'(x)=4x^3$$

At $x=0$, $f'(0)=0-0=0$ and $g'(0)=0$, so this is again of the form $\frac{0}{0}$. Differentiate again, using the product rule on $\tan x\sec^2 x$:

$$\begin{aligned} f''(x)&=\operatorname{sech}^2 x-\left(\sec^2 x\times\sec^2 x+\tan x\times 2\sec^2 x\tan x\right)\\ &=\operatorname{sech}^2 x-\sec^4 x-2\tan^2 x\sec^2 x\end{aligned}$$
$$g''(x)=12x^2$$

Now write everything in terms of $\tan x$ and $\tanh x$, using $\sec^2 x=1+\tan^2 x$ and $\operatorname{sech}^2 x=1-\tanh^2 x$:

$$\begin{aligned}\sec^4 x+2\tan^2 x\sec^2 x&=\left(1+\tan^2 x\right)^2+2\tan^2 x\left(1+\tan^2 x\right)\\ &=1+4\tan^2 x+3\tan^4 x\end{aligned}$$

so

$$f''(x)=1-\tanh^2 x-1-4\tan^2 x-3\tan^4 x=-\tanh^2 x-4\tan^2 x-3\tan^4 x$$

Dividing numerator and denominator by x^2:

$$\frac{f''(x)}{g''(x)}=-\frac{1}{12}\left[\left(\frac{\tanh x}{x}\right)^2+4\left(\frac{\tan x}{x}\right)^2+3\left(\frac{\tan x}{x}\right)^2\tan^2 x\right]$$

Both $\frac{\tanh x}{x}$ and $\frac{\tan x}{x}$ are of the form $\frac{0}{0}$ at $x=0$, so by L'Hôpital's rule:

$$\lim_{x\to 0}\frac{\tanh x}{x}=\lim_{x\to 0}\frac{\operatorname{sech}^2 x}{1}=1,\qquad \lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sec^2 x}{1}=1$$

Also $\tan^2 x\to 0$ as $x\to 0$. Therefore

$$\lim_{x\to 0}\frac{f''(x)}{g''(x)}=-\frac{1}{12}(1+4\times 1+3\times 1\times 0)=-\frac{5}{12}$$

This limit exists, so applying L'Hôpital's rule twice gives

$$\lim_{x\to 0}\left(\frac{\ln(\cosh x)-\frac{1}{2}\tan^2 x}{x^4}\right)=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=-\frac{5}{12}$$

as required.

8. Use L'Hôpital's rule to show that

$$\lim_{x\to 0^+}\left(x(\ln x)^2\right) = 0$$

Fully justify your answer. **[6]**

Solution

L'Hôpital's rule states that if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to 0^+$, with f and g differentiable and $g'(x) \neq 0$ for small $x > 0$, then

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

The expression is a product, so first write it as a quotient:

$$x(\ln x)^2 = \frac{(\ln x)^2}{\frac{1}{x}}$$

As $x \to 0^+$, $\ln x \to -\infty$, so $(\ln x)^2 \to +\infty$. Also $\frac{1}{x} \to +\infty$. The limit is therefore of the form $\frac{\infty}{\infty}$, and the derivative of the denominator, $-\frac{1}{x^2}$, is non-zero for $x > 0$, so L'Hôpital's rule applies. Differentiating the numerator and the denominator:

$$\begin{aligned}\lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}} &= \lim_{x\to 0^+}\frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}}\\ &= \lim_{x\to 0^+}\left(\frac{2\ln x}{x}\times\left(-x^2\right)\right)\\ &= \lim_{x\to 0^+}\left(-2x\ln x\right)\\ &= \lim_{x\to 0^+}\frac{-2\ln x}{\frac{1}{x}}\end{aligned}$$

provided this last limit exists.

As $x \to 0^+$, $-2\ln x \to +\infty$ and $\frac{1}{x} \to +\infty$, so this is again of the form $\frac{\infty}{\infty}$, and the denominator's derivative is again $-\frac{1}{x^2} \neq 0$. Applying L'Hôpital's rule a second time:

$$\begin{aligned}\lim_{x\to 0^+}\frac{-2\ln x}{\frac{1}{x}} &= \lim_{x\to 0^+}\frac{-\frac{2}{x}}{-\frac{1}{x^2}}\\ &= \lim_{x\to 0^+}\left(\frac{2}{x}\times x^2\right)\\ &= \lim_{x\to 0^+} 2x\\ &= 0\end{aligned}$$

This limit exists, so both applications of L'Hôpital's rule are valid, and therefore

$$\lim_{x\to 0^+}\left(x(\ln x)^2\right) = 0$$

as required.

9. Use L'Hôpital's rule to find the exact value of

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right) \qquad \textbf{[6]}$$

Solution

Write the expression as a single fraction:

$$\frac{1}{\ln(1+x)}-\frac{1}{x}=\frac{x-\ln(1+x)}{x\ln(1+x)}$$

As $x\to 0$, $\ln(1+x)\to 0$, so the numerator and the denominator both tend to 0. This is an indeterminate form of type $\frac{0}{0}$. L'Hôpital's rule states that if $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$, then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Differentiate the numerator and the denominator, using the product rule for the denominator:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x-\ln(1+x)\big)=1-\frac{1}{1+x}=\frac{x}{1+x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x\ln(1+x)\big)=\ln(1+x)+\frac{x}{1+x}$$

So

$$\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)}=\lim_{x\to 0}\frac{\dfrac{x}{1+x}}{\ln(1+x)+\dfrac{x}{1+x}}$$

Multiplying the numerator and the denominator by $1+x$ gives

$$\lim_{x\to 0}\frac{x}{(1+x)\ln(1+x)+x}$$

As $x\to 0$, the numerator and the denominator both still tend to 0, so apply L'Hôpital's rule again. The derivative of the numerator is 1, and by the product rule the derivative of the denominator is

$$\ln(1+x)+(1+x)\times\frac{1}{1+x}+1=\ln(1+x)+2$$

Therefore

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)&=\lim_{x\to 0}\frac{1}{\ln(1+x)+2}\\&=\frac{1}{0+2}\\&=\frac{1}{2}\end{aligned}$$

10. Given that k is a positive constant, use the substitution $a = \frac{1}{p}$ to find

$$\lim_{p\to\infty}\left[p\ln\left(\frac{p+k}{p-k}\right)\right] \qquad \textbf{[3]}$$

Solution

With $a = \frac{1}{p}$, we have $p = \frac{1}{a}$, and $a \to 0^+$ as $p \to \infty$. Multiplying the numerator and denominator of the fraction by a:

$$\frac{p+k}{p-k} = \frac{\frac{1}{a}+k}{\frac{1}{a}-k} = \frac{1+ka}{1-ka}$$

so the limit becomes

$$\lim_{p\to\infty}\left[p\ln\left(\frac{p+k}{p-k}\right)\right] = \lim_{a\to 0^+}\frac{\ln(1+ka)-\ln(1-ka)}{a}$$

For small positive a, both $1+ka$ and $1-ka$ are positive, so the logarithms are defined. As $a \to 0^+$, the numerator tends to $\ln 1 - \ln 1 = 0$ and the denominator tends to 0. This is of the form $\frac{0}{0}$. L'Hôpital's rule states that if $f(a) \to 0$ and $g(a) \to 0$ as $a \to 0^+$, then

$$\lim_{a\to 0^+}\frac{f(a)}{g(a)} = \lim_{a\to 0^+}\frac{f'(a)}{g'(a)}$$

provided the limit on the right exists. Differentiating the numerator and denominator with respect to a:

$$\begin{aligned}\lim_{a\to 0^+}\frac{\ln(1+ka)-\ln(1-ka)}{a} &= \lim_{a\to 0^+}\frac{\frac{k}{1+ka}+\frac{k}{1-ka}}{1}\\ &= \frac{k}{1}+\frac{k}{1}\\ &= 2k\end{aligned}$$

11. **(a)** Write down the expansion of $\cos 3x$ in ascending powers of x up to and including the term in x^4. **[1]**

(b) The constant a is such that the following limit exists and is finite.

$$\lim_{x\to 0}\left[\frac{ax^2+2\cos 3x-2}{x^3\ln(1+x)}\right]$$

Find the value of a and the value of the limit. **[5]**

Solution

(a) The formula booklet gives

$$\cos u = 1-\frac{u^2}{2!}+\frac{u^4}{4!}-\cdots \qquad \text{for all } u$$

Replacing u by $3x$:

$$\begin{aligned}\cos 3x &= 1-\frac{(3x)^2}{2}+\frac{(3x)^4}{24}-\cdots\\ &= 1-\frac{9}{2}x^2+\frac{27}{8}x^4-\cdots\end{aligned}$$

(b) Split the expression into two factors:

$$\frac{ax^2+2\cos 3x-2}{x^3\ln(1+x)} = \frac{ax^2+2\cos 3x-2}{x^4}\times\frac{x}{\ln(1+x)}$$

L'Hôpital's rule states that if $f(x)\to 0$ and $g(x)\to 0$ as $x\to 0$, then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

For the second factor, $x\to 0$ and $\ln(1+x)\to 0$, so L'Hôpital's rule gives

$$\lim_{x\to 0}\frac{x}{\ln(1+x)} = \lim_{x\to 0}\frac{1}{\frac{1}{1+x}} = \lim_{x\to 0}(1+x) = 1$$

So the required limit is finite exactly when

$$L = \lim_{x\to 0}\frac{ax^2+2\cos 3x-2}{x^4}$$

is finite, and the required limit then equals L.

As $x\to 0$, the numerator tends to $2\cos 0-2=0$ and the denominator tends to 0. Applying L'Hôpital's rule twice, each time to a limit of the form $\frac{0}{0}$:

$$\begin{aligned}L &= \lim_{x\to 0}\frac{2ax-6\sin 3x}{4x^3}\\ &= \lim_{x\to 0}\frac{2a-18\cos 3x}{12x^2}\end{aligned}$$

As $x\to 0$, the numerator tends to $2a-18$ while the denominator tends to 0. If $2a-18\neq 0$ the expression grows without bound, so for a finite limit

$$2a-18=0 \quad\Longrightarrow\quad a=9$$

With $a = 9$, the limit is again of the form $\frac{0}{0}$, so L'Hôpital's rule can be applied twice more:

$$\begin{aligned} L &= \lim_{x \to 0} \frac{18 - 18\cos 3x}{12x^2} \\ &= \lim_{x \to 0} \frac{54 \sin 3x}{24x} \\ &= \lim_{x \to 0} \frac{162 \cos 3x}{24} \\ &= \frac{162}{24} = \frac{27}{4} \end{aligned}$$

The required limit is therefore

$$\frac{27}{4} \times 1 = \frac{27}{4}$$

So $a = 9$ and the value of the limit is $\frac{27}{4}$.

12. **(a)** (i) Write down the expansion of $\ln(1+3x)$ in ascending powers of x up to and including the term in x^4. **[1]**

(ii) Hence, or otherwise, find the first two non-zero terms in the expansion of

$$\ln\left(\frac{1+3x}{1-3x}\right)$$

in ascending powers of x, and state the range of values of x for which the expansion is valid. **[3]**

(b) Find

$$\lim_{x\to 0}\left[\frac{\ln\left(\frac{1+3x}{1-3x}\right)-6x}{x\left(2-\sqrt{4-x^2}\right)}\right] \quad \textbf{[5]}$$

Solution

(a) (i) The formula booklet gives

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots \qquad (-1 < u \leq 1)$$

Replacing u by $3x$:

$$\begin{aligned}\ln(1+3x) &= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \cdots \\ &= 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \cdots\end{aligned}$$

(ii) The laws of logarithms give

$$\ln\left(\frac{1+3x}{1-3x}\right) = \ln(1+3x) - \ln(1-3x)$$

Replacing x by $-x$ in the result of part (a)(i):

$$\ln(1-3x) = -3x - \frac{9}{2}x^2 - 9x^3 - \frac{81}{4}x^4 - \cdots$$

Subtracting, the even powers cancel and the odd powers double:

$$\ln\left(\frac{1+3x}{1-3x}\right) = 6x + 18x^3 + \cdots$$

So the first two non-zero terms are $6x + 18x^3$.

The expansion of $\ln(1+3x)$ is valid for $-1 < 3x \leq 1$, that is $-\frac{1}{3} < x \leq \frac{1}{3}$, and the expansion of $\ln(1-3x)$ is valid for $-1 < -3x \leq 1$, that is $-\frac{1}{3} \leq x < \frac{1}{3}$. Both are needed, so the expansion is valid for

$$-\frac{1}{3} < x < \frac{1}{3}$$

(b) First rationalise the bracket in the denominator:

$$\begin{aligned} 2-\sqrt{4-x^2} &= \frac{\left(2-\sqrt{4-x^2}\right)\left(2+\sqrt{4-x^2}\right)}{2+\sqrt{4-x^2}} \\ &= \frac{4-\left(4-x^2\right)}{2+\sqrt{4-x^2}} \\ &= \frac{x^2}{2+\sqrt{4-x^2}} \end{aligned}$$

Hence, for $x \neq 0$, the expression becomes

$$\frac{\ln\left(\dfrac{1+3x}{1-3x}\right)-6x}{x\left(2-\sqrt{4-x^2}\right)} = \left(2+\sqrt{4-x^2}\right) \times \frac{\ln\left(\dfrac{1+3x}{1-3x}\right)-6x}{x^3}$$

As $x \to 0$, $2+\sqrt{4-x^2} \to 4$, so the required limit is

$$4 \lim_{x\to 0} \frac{f(x)}{g(x)}, \quad \text{where } f(x) = \ln\left(\frac{1+3x}{1-3x}\right) - 6x, \quad g(x) = x^3$$

Since $f(0) = \ln 1 - 0 = 0$ and $g(0) = 0$, the limit is of the form $\frac{0}{0}$. L'Hôpital's rule states that if $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

For x near 0, $1+3x > 0$ and $1-3x > 0$, so $f(x) = \ln(1+3x) - \ln(1-3x) - 6x$. Differentiating:

$$\begin{aligned} f'(x) &= \frac{3}{1+3x} + \frac{3}{1-3x} - 6 \\ &= \frac{3(1-3x)+3(1+3x)}{1-9x^2} - 6 \\ &= \frac{6}{1-9x^2} - 6 \\ &= \frac{6-6\left(1-9x^2\right)}{1-9x^2} \\ &= \frac{54x^2}{1-9x^2} \end{aligned}$$

Also $g'(x) = 3x^2$, so for $x \neq 0$

$$\frac{f'(x)}{g'(x)} = \frac{54x^2}{3x^2(1-9x^2)} = \frac{18}{1-9x^2}$$

This tends to 18 as $x \to 0$. Therefore

$$\lim_{x\to 0}\left[\frac{\ln\left(\dfrac{1+3x}{1-3x}\right)-6x}{x\left(2-\sqrt{4-x^2}\right)}\right] = 4 \times 18 = 72$$