OPTIONAL PURE / FURTHER CALCULUS

Arc Length

Worked solutions 12 questions · 94 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

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1. The curve C has equation $y = \frac{1}{2}\ln(\sin 2x)$, where $0 < x < \frac{1}{2}\pi$.
Find the exact length of the arc of C from the point where $x = \frac{1}{6}\pi$ to the point where $x = \frac{1}{3}\pi$, giving your answer in the form $a\ln b$, where a is rational and b is an integer. **[6]**

Solution

The arc length of a curve from $x = x_1$ to $x = x_2$ is

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Differentiating $y = \frac{1}{2}\ln(\sin 2x)$ with the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \times \frac{2\cos 2x}{\sin 2x} = \cot 2x$$

Using the identity $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ gives

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \cot^2 2x = \operatorname{cosec}^2 2x$$

For $\frac{1}{6}\pi \le x \le \frac{1}{3}\pi$, we have $\frac{1}{3}\pi \le 2x \le \frac{2}{3}\pi$, so $\sin 2x > 0$ and $\sqrt{\operatorname{cosec}^2 2x} = \operatorname{cosec} 2x$. Hence

$$s = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2x\,\mathrm{d}x$$

Using the double angle formula $\sin 2x = 2\sin x\cos x$, and dividing the numerator and denominator by $\cos^2 x$:

$$\operatorname{cosec} 2x = \frac{1}{2\sin x\cos x} = \frac{\sec^2 x}{2\tan x}$$

The numerator $\sec^2 x$ is the derivative of $\tan x$, and $\tan x > 0$ on this interval, so

$$\begin{aligned} s &= \frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sec^2 x}{\tan x}\,\mathrm{d}x \\ &= \frac{1}{2}\left[\ln(\tan x)\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \\ &= \frac{1}{2}\left(\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{2}\ln\left(\sqrt{3}\times\sqrt{3}\right) \\ &= \frac{1}{2}\ln 3 \end{aligned}$$

This is in the form $a\ln b$ with $a = \frac{1}{2}$ and $b = 3$.

2. The curve C has equation

$$y = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\operatorname{arcosh} x, \qquad 2 \leq x \leq 5$$

Using calculus, find the exact length of the curve C. **[6]**

Solution

The arc length of a curve from $x = a$ to $x = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Differentiate, using the product rule on the first term and the standard result:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$

This gives

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\sqrt{x^2-1} + \frac{1}{2}x \times \frac{x}{\sqrt{x^2-1}} - \frac{1}{2\sqrt{x^2-1}} \\
&= \frac{1}{2}\sqrt{x^2-1} + \frac{x^2-1}{2\sqrt{x^2-1}} \\
&= \frac{1}{2}\sqrt{x^2-1} + \frac{1}{2}\sqrt{x^2-1} \\
&= \sqrt{x^2-1}
\end{aligned}$$

So

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + (x^2-1) = x^2$$

Since $x > 0$ on $2 \leq x \leq 5$, $\sqrt{x^2} = x$. The length of C is therefore

$$\begin{aligned}
s &= \int_2^5 x\,\mathrm{d}x \\
&= \left[\frac{x^2}{2}\right]_2^5 \\
&= \frac{25}{2} - 2 \\
&= \frac{21}{2}
\end{aligned}$$

3. The curve C has polar equation $r = \theta^2$ for $0 \leq \theta \leq \sqrt{5}$.
Find the length of C, giving your answer in an exact form. **[5]**

Solution

The arc length of a polar curve is

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Here $r = \theta^2$, so $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2\theta$ and

$$r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \theta^4 + 4\theta^2 = \theta^2\left(\theta^2 + 4\right)$$

Since $\theta \geq 0$ on the curve, $\sqrt{\theta^2(\theta^2+4)} = \theta\sqrt{\theta^2+4}$, so

$$s = \int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\,\mathrm{d}\theta$$

Let $u = \theta^2 + 4$, so $\mathrm{d}u = 2\theta\,\mathrm{d}\theta$. When $\theta = 0$, $u = 4$, and when $\theta = \sqrt{5}$, $u = 9$. Then

$$\begin{aligned} s &= \int_4^9 \frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u \\ &= \left[\frac{1}{3}u^{\frac{3}{2}}\right]_4^9 \\ &= \frac{1}{3}(27 - 8) \\ &= \frac{19}{3} \end{aligned}$$

Alternative Solution:

Writing the curve in parametric form, with parameter θ:

$$x = r\cos\theta = \theta^2\cos\theta, \qquad y = r\sin\theta = \theta^2\sin\theta$$

Differentiating with the product rule:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\theta} &= 2\theta\cos\theta - \theta^2\sin\theta \\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= 2\theta\sin\theta + \theta^2\cos\theta \end{aligned}$$

The arc length of a curve given parametrically is

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Squaring and adding, the terms in $\sin\theta\cos\theta$ cancel:

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \left(4\theta^2\cos^2\theta-4\theta^3\sin\theta\cos\theta+\theta^4\sin^2\theta\right)\\ &\quad+\left(4\theta^2\sin^2\theta+4\theta^3\sin\theta\cos\theta+\theta^4\cos^2\theta\right)\\ &= 4\theta^2\left(\cos^2\theta+\sin^2\theta\right)+\theta^4\left(\sin^2\theta+\cos^2\theta\right)\\ &= \theta^4+4\theta^2\end{aligned}$$

This is the expression $r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$ found above, so

$$s=\int_0^{\sqrt{5}}\theta\sqrt{\theta^2+4}\,\mathrm{d}\theta$$

and the integral is evaluated as above, giving $s=\frac{19}{3}$.

4. The curve C has parametric equations

$$x = \frac{4}{3}t^3 - \ln t, \qquad y = kt^{\frac{3}{2}}, \qquad 1 \leq t \leq 2$$

where k is a positive constant.
It is given that, for $1 \leq t \leq 2$,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(4t^2 + \frac{1}{t}\right)^2$$

Find the value of k, and find the exact length of C, giving your answer in the form $p + \ln q$, where p is rational and q is an integer. **[7]**

Solution

Differentiating with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4t^2 - \frac{1}{t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3k}{2}t^{\frac{1}{2}}$$

Squaring and adding:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(4t^2 - \frac{1}{t}\right)^2 + \frac{9k^2}{4}t$$

$$= 16t^4 - 8t + \frac{1}{t^2} + \frac{9k^2}{4}t$$

Expanding the given right-hand side:

$$\left(4t^2 + \frac{1}{t}\right)^2 = 16t^4 + 8t + \frac{1}{t^2}$$

For these to be equal for $1 \leq t \leq 2$, equating coefficients of t gives

$$-8 + \frac{9k^2}{4} = 8 \quad \Longrightarrow \quad k^2 = \frac{64}{9}$$

Since k is positive, $k = \frac{8}{3}$.

The arc length of a curve given parametrically is

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

For $1 \leq t \leq 2$, $4t^2 + \frac{1}{t} > 0$, so the square root of $\left(4t^2 + \frac{1}{t}\right)^2$ is $4t^2 + \frac{1}{t}$. Hence

$$s = \int_1^2 \left(4t^2 + \frac{1}{t}\right)\mathrm{d}t$$

$$= \left[\frac{4t^3}{3} + \ln t\right]_1^2$$

$$= \left(\frac{32}{3} + \ln 2\right) - \left(\frac{4}{3} + \ln 1\right)$$

$$= \frac{28}{3} + \ln 2$$

This is in the form $p + \ln q$ with $p = \frac{28}{3}$ and $q = 2$.

5. A curve has equation

$$y = \ln\left(\tanh\frac{x}{2}\right), \qquad x > 0$$

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{cosech} x$. **[2]**

(b) Hence show that the arc length of the curve from $x = a$ to $x = b$, where $0 < a < b$, is equal to

$$\ln\left(\frac{\sinh b}{\sinh a}\right)$$ **[4]**

Solution

(a) Using the chain rule, with $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh\frac{x}{2}\right) = \frac{1}{2}\operatorname{sech}^2\frac{x}{2}$:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{1}{2}\operatorname{sech}^2\frac{x}{2}}{\tanh\frac{x}{2}} \\ &= \frac{1}{2\cosh^2\frac{x}{2}} \times \frac{\cosh\frac{x}{2}}{\sinh\frac{x}{2}} \\ &= \frac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}\end{aligned}$$

The double angle formula $\sinh 2A = 2\sinh A\cosh A$, with $A = \frac{x}{2}$, gives $2\sinh\frac{x}{2}\cosh\frac{x}{2} = \sinh x$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sinh x} = \operatorname{cosech} x$$

as required.

(b) The arc length of a curve from $x = a$ to $x = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Dividing $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$ gives $\coth^2 x - 1 = \operatorname{cosech}^2 x$. Using part (a):

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \operatorname{cosech}^2 x = \coth^2 x$$

For $x > 0$, $\coth x > 0$, so $\sqrt{\coth^2 x} = \coth x$. Since $\coth x = \frac{\cosh x}{\sinh x}$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$, the integrand is the derivative of the denominator over the denominator. Also $\sinh x > 0$ for $x > 0$, so

$$\begin{aligned}s &= \int_a^b \frac{\cosh x}{\sinh x}\,\mathrm{d}x \\ &= \left[\ln(\sinh x)\right]_a^b \\ &= \ln(\sinh b) - \ln(\sinh a) \\ &= \ln\left(\frac{\sinh b}{\sinh a}\right)\end{aligned}$$

as required.

6. The curve C has parametric equations

$$x = 2\sinh t, \qquad y = \cosh 2t, \qquad 0 \leq t \leq \ln 2$$

The length of C is denoted by s.
By using the substitution $u = 2\sinh t$, show that

$$s = \int_0^{\frac{3}{2}} \sqrt{1+u^2}\,\mathrm{d}u \qquad \textbf{[5]}$$

Solution

The arc length of a curve given parametrically is

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Differentiating, and using the double angle formula $\sinh 2t = 2\sinh t\cosh t$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\cosh t$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sinh 2t = 4\sinh t\cosh t$$

So

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 4\cosh^2 t + 16\sinh^2 t\cosh^2 t$$

$$= 4\cosh^2 t\left(1 + 4\sinh^2 t\right)$$

Since $\cosh t > 0$, the square root is $2\cosh t\sqrt{1+4\sinh^2 t}$, and

$$s = \int_0^{\ln 2} 2\cosh t\sqrt{1+4\sinh^2 t}\,\mathrm{d}t$$

Now substitute $u = 2\sinh t$, so that $4\sinh^2 t = u^2$ and

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2\cosh t \quad \Longrightarrow \quad \mathrm{d}u = 2\cosh t\,\mathrm{d}t$$

For the limits, when $t = 0$, $u = 2\sinh 0 = 0$. When $t = \ln 2$:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4} \quad \Longrightarrow \quad u = 2 \times \frac{3}{4} = \frac{3}{2}$$

Hence

$$s = \int_0^{\frac{3}{2}} \sqrt{1+u^2}\,\mathrm{d}u$$

as required.

7. The curve C has polar equation $r = 4e^{-\frac{5}{12}\theta}$ for $0 \leq \theta \leq \beta$, where $\beta > 0$.
Given that the length of C is s, find β in terms of s, and state the range of possible values of s. **[6]**

Solution

The arc length of a polar curve is

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

Here $\alpha = 0$. Differentiating $r = 4e^{-\frac{5}{12}\theta}$ gives

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{5}{3}e^{-\frac{5}{12}\theta}$$

so

$$\begin{aligned} r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= 16e^{-\frac{5}{6}\theta} + \frac{25}{9}e^{-\frac{5}{6}\theta} \\ &= \frac{169}{9}e^{-\frac{5}{6}\theta} \end{aligned}$$

Taking the positive square root:

$$\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} = \frac{13}{3}e^{-\frac{5}{12}\theta}$$

Hence the length of C is

$$\begin{aligned} s &= \int_0^{\beta} \frac{13}{3}e^{-\frac{5}{12}\theta}\,\mathrm{d}\theta \\ &= \left[-\frac{13}{3} \times \frac{12}{5}e^{-\frac{5}{12}\theta}\right]_0^{\beta} \\ &= -\frac{52}{5}e^{-\frac{5}{12}\beta} + \frac{52}{5} \\ &= \frac{52}{5}\left(1 - e^{-\frac{5}{12}\beta}\right) \end{aligned}$$

Rearranging for β:

$$\begin{aligned} e^{-\frac{5}{12}\beta} &= 1 - \frac{5s}{52} = \frac{52 - 5s}{52} \\ -\frac{5}{12}\beta &= \ln\left(\frac{52 - 5s}{52}\right) \\ \beta &= \frac{12}{5}\ln\left(\frac{52}{52 - 5s}\right) \end{aligned}$$

Since $\beta > 0$, $0 < e^{-\frac{5}{12}\beta} < 1$, and $e^{-\frac{5}{12}\beta}$ takes every value in this interval as β runs over all positive values.
So $0 < 1 - e^{-\frac{5}{12}\beta} < 1$, and the possible values of s are

$$0 < s < \frac{52}{5}$$

Alternative Solution:

Writing the curve in parametric form, with parameter θ:

$$x = r\cos\theta = 4e^{-\frac{5}{12}\theta}\cos\theta, \qquad y = r\sin\theta = 4e^{-\frac{5}{12}\theta}\sin\theta$$

Differentiating with the product rule:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 4e^{-\frac{5}{12}\theta}\left(-\frac{5}{12}\cos\theta - \sin\theta\right) = -\frac{1}{3}e^{-\frac{5}{12}\theta}(5\cos\theta + 12\sin\theta)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 4e^{-\frac{5}{12}\theta}\left(-\frac{5}{12}\sin\theta + \cos\theta\right) = \frac{1}{3}e^{-\frac{5}{12}\theta}(12\cos\theta - 5\sin\theta)$$

The arc length of a curve given parametrically is

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

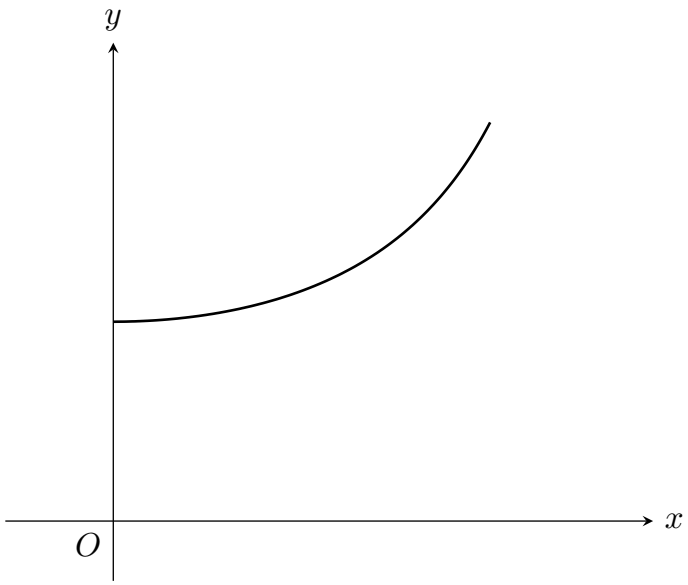
Squaring and adding, the terms in $\sin\theta\cos\theta$ cancel:

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= \frac{1}{9}e^{-\frac{5}{6}\theta}\left(25\cos^2\theta + 120\sin\theta\cos\theta + 144\sin^2\theta\right)\\ &\quad + \frac{1}{9}e^{-\frac{5}{6}\theta}\left(144\cos^2\theta - 120\sin\theta\cos\theta + 25\sin^2\theta\right)\\ &= \frac{1}{9}e^{-\frac{5}{6}\theta} \times 169\left(\cos^2\theta + \sin^2\theta\right)\\ &= \frac{169}{9}e^{-\frac{5}{6}\theta}\end{aligned}$$

This is the expression $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$ found above, so the length of C is

$$s = \int_0^{\beta} \frac{13}{3}e^{-\frac{5}{12}\theta}\,\mathrm{d}\theta$$

and the rest of the solution is as above.



8. The diagram above shows a sketch of the curve C with equation

$$y = \frac{1}{\cos^2 x - \sin^2 x}, \qquad 0 \leq x \leq \frac{\pi}{6}$$

(a) Show that the length of C is given by

$$\int_0^{\frac{\pi}{6}} \left(2\sec^2 2x - 1\right)\, \mathrm{d}x \qquad \textbf{[6]}$$

(b) Hence find the exact length of C. **[4]**

Solution

(a) Using the double angle formula $\cos 2x = \cos^2 x - \sin^2 x$, the equation of C becomes

$$y = \frac{1}{\cos 2x} = \sec 2x$$

Differentiating, using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sec 2x \tan 2x$$

The arc length of a curve from $x = a$ to $x = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$$

Using $\tan^2 2x = \sec^2 2x - 1$:

$$\begin{aligned} 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1 + 4\sec^2 2x \tan^2 2x \\ &= 1 + 4\sec^2 2x\left(\sec^2 2x - 1\right) \\ &= 4\sec^4 2x - 4\sec^2 2x + 1 \\ &= \left(2\sec^2 2x - 1\right)^2 \end{aligned}$$

For $0 \leq x \leq \frac{\pi}{6}$, $\sec^2 2x \geq 1$, so $2\sec^2 2x - 1 \geq 1 > 0$. Hence

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = 2\sec^2 2x - 1$$

and the length of C is

$$s = \int_0^{\frac{\pi}{6}} \left(2\sec^2 2x - 1\right) \mathrm{d}x$$

as required.

(b) Using part (a):

$$\begin{aligned} s &= \int_0^{\frac{\pi}{6}} \left(2\sec^2 2x - 1\right) \mathrm{d}x \\ &= \left[\tan 2x - x\right]_0^{\frac{\pi}{6}} \\ &= \left(\tan\frac{\pi}{3} - \frac{\pi}{6}\right) - (\tan 0 - 0) \\ &= \sqrt{3} - \frac{\pi}{6} \end{aligned}$$

9. The curve C has equation

$$y = 1 + 3\ln\left(x^2 - 9\right), \qquad 4 \leq x \leq 5$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence show that the length of C is given by

$$\int_4^5 \frac{x^2+9}{x^2-9}\,\mathrm{d}x \qquad \textbf{[4]}$$

(b) Express $\frac{x^2+9}{x^2-9}$ in partial fractions. **[4]**

(c) Hence, using algebraic integration, find the exact length of C, giving your answer in the form $p + q\ln r$, where p and q are integers and r is a rational number. **[4]**

Solution

(a) Differentiating using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \times \frac{2x}{x^2-9} = \frac{6x}{x^2-9}$$

Then

$$\begin{aligned}
1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1 + \frac{36x^2}{\left(x^2-9\right)^2} \\
&= \frac{\left(x^2-9\right)^2 + 36x^2}{\left(x^2-9\right)^2} \\
&= \frac{x^4 - 18x^2 + 81 + 36x^2}{\left(x^2-9\right)^2} \\
&= \frac{x^4 + 18x^2 + 81}{\left(x^2-9\right)^2} \\
&= \frac{\left(x^2+9\right)^2}{\left(x^2-9\right)^2}
\end{aligned}$$

For $4 \leq x \leq 5$, both $x^2 + 9$ and $x^2 - 9$ are positive, so

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{x^2+9}{x^2-9}$$

The arc length of a curve from $x = a$ to $x = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

so the length of C is

$$\int_4^5 \frac{x^2+9}{x^2-9}\,\mathrm{d}x$$

as required.

(b) The numerator and denominator have the same degree, so first divide:

$$\frac{x^2+9}{x^2-9} = \frac{(x^2-9)+18}{x^2-9} = 1 + \frac{18}{(x-3)(x+3)}$$

Now set up the partial fractions:

$$\frac{18}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$

Multiplying through by $(x-3)(x+3)$ gives

$$18 = A(x+3) + B(x-3)$$

Putting $x = 3$ gives $18 = 6A$, so $A = 3$. Putting $x = -3$ gives $18 = -6B$, so $B = -3$. Hence

$$\frac{x^2+9}{x^2-9} = 1 + \frac{3}{x-3} - \frac{3}{x+3}$$

(c) Using part (a) and part (b), the length of C is

$$\int_4^5 \left(1 + \frac{3}{x-3} - \frac{3}{x+3}\right) \mathrm{d}x$$

For $4 \leq x \leq 5$, both $x-3$ and $x+3$ are positive, so the logarithms can be written with round brackets:

$$\begin{aligned}
\int_4^5 \left(1 + \frac{3}{x-3} - \frac{3}{x+3}\right) \mathrm{d}x &= [x + 3\ln(x-3) - 3\ln(x+3)]_4^5 \\
&= (5 + 3\ln 2 - 3\ln 8) - (4 + 3\ln 1 - 3\ln 7) \\
&= 1 + 3\ln 2 - 3\ln 8 + 3\ln 7 \\
&= 1 + 3\ln\left(\frac{2\times 7}{8}\right) \\
&= 1 + 3\ln\frac{7}{4}
\end{aligned}$$

This is in the form $p + q\ln r$ with $p = 1$, $q = 3$, $r = \frac{7}{4}$.

10. The curve C has parametric equations

$$x = \frac{1}{3}t^3 - t(\ln t)^2 + 2t\ln t - 2t + \frac{5}{3}, \qquad y = \frac{1}{2}t^2(2\ln t - 1) + \frac{1}{2}, \qquad 1 \le t \le e$$

Find the exact length of C, giving your answer in terms of e. **[8]**

Solution

The arc length of a curve given parametrically is

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Differentiating x with respect to t, using the product rule on $t(\ln t)^2$ and $2t\ln t$:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= t^2 - \left((\ln t)^2 + t \times \frac{2\ln t}{t}\right) + \left(2\ln t + 2t \times \frac{1}{t}\right) - 2\\ &= t^2 - (\ln t)^2 - 2\ln t + 2\ln t + 2 - 2\\ &= t^2 - (\ln t)^2\end{aligned}$$

Writing $y = t^2\ln t - \frac{1}{2}t^2 + \frac{1}{2}$ and differentiating:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= 2t\ln t + t^2 \times \frac{1}{t} - t\\ &= 2t\ln t\end{aligned}$$

So

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \left(t^2 - (\ln t)^2\right)^2 + 4t^2(\ln t)^2\\ &= t^4 - 2t^2(\ln t)^2 + (\ln t)^4 + 4t^2(\ln t)^2\\ &= t^4 + 2t^2(\ln t)^2 + (\ln t)^4\\ &= \left(t^2 + (\ln t)^2\right)^2\end{aligned}$$

Since $t^2 + (\ln t)^2 > 0$, the square root is $t^2 + (\ln t)^2$, and

$$s = \int_1^e \left(t^2 + (\ln t)^2\right)\mathrm{d}t$$

The integral of $(\ln t)^2$ is found by integration by parts, which is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

With $u = (\ln t)^2$, $\frac{\mathrm{d}v}{\mathrm{d}t} = 1$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{2\ln t}{t}$, $v = t$:

$$\begin{aligned}\int (\ln t)^2\,\mathrm{d}t &= t(\ln t)^2 - \int t \times \frac{2\ln t}{t}\,\mathrm{d}t\\ &= t(\ln t)^2 - 2\int \ln t\,\mathrm{d}t\end{aligned}$$

Integrating by parts again, with $u = \ln t$, $\frac{\mathrm{d}v}{\mathrm{d}t} = 1$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{t}$, $v = t$:

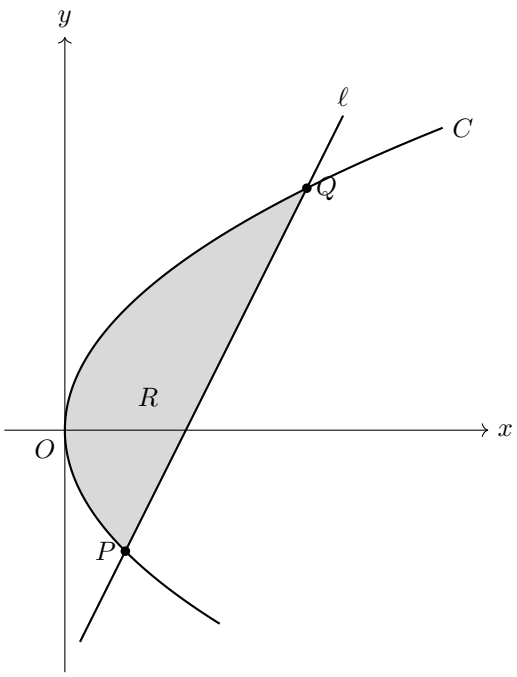
$$\begin{aligned}\int \ln t \,\mathrm{d}t &= t\ln t - \int t \times \frac{1}{t}\,\mathrm{d}t \\ &= t\ln t - t + c\end{aligned}$$

Hence

$$\int (\ln t)^2 \,\mathrm{d}t = t(\ln t)^2 - 2t\ln t + 2t + c$$

Therefore, using $\ln e = 1$ and $\ln 1 = 0$:

$$\begin{aligned}s &= \left[\frac{1}{3}t^3 + t(\ln t)^2 - 2t\ln t + 2t\right]_1^e \\ &= \left(\frac{1}{3}e^3 + e - 2e + 2e\right) - \left(\frac{1}{3} + 0 - 0 + 2\right) \\ &= \frac{1}{3}e^3 + e - \frac{7}{3} \\ &= \frac{1}{3}\left(e^3 + 3e - 7\right)\end{aligned}$$



11. The diagram above shows a sketch of part of the curve C with equation $y^2 = 4x$ and part of the line ℓ with equation $y = 2x - 4$.
The line ℓ meets C at the points P and Q. The region R, shown shaded in the diagram above, is bounded by C and ℓ.

(a) Show that the perimeter of R is given by

$$\alpha + \int_{\gamma}^{\beta} \sqrt{1 + \frac{y^2}{4}}\,\mathrm{d}y$$

where α, β and γ are constants to be determined. **[4]**

(b) Use the substitution $y = 2\sinh u$ and algebraic integration to determine the exact perimeter of R, giving your answer in simplest form. **[7]**

Solution

(a) From ℓ, $x = \frac{y+4}{2}$. Substituting into $y^2 = 4x$:

$$y^2 = 2(y+4)$$
$$y^2 - 2y - 8 = 0$$
$$(y-4)(y+2) = 0$$

so $y = -2$ or $y = 4$, with $x = 1$ or $x = 4$ respectively. Hence $P(1, -2)$ and $Q(4, 4)$.

The perimeter of R is the chord PQ together with the arc of C from P to Q. The length of the chord is

$$PQ = \sqrt{(4-1)^2 + (4-(-2))^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

On C, $x = \frac{y^2}{4}$, so

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2} \quad \Longrightarrow \quad \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = \frac{y^2}{4}$$

The arc length of a curve from $y = y_1$ to $y = y_2$ is

$$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Along the arc, y runs from -2 at P to 4 at Q, so the perimeter of R is

$$3\sqrt{5} + \int_{-2}^{4} \sqrt{1 + \frac{y^2}{4}}\,\mathrm{d}y$$

This is the given form with $\alpha = 3\sqrt{5}$, $\beta = 4$ and $\gamma = -2$.

(b) With $y = 2\sinh u$, $\mathrm{d}y = 2\cosh u\,\mathrm{d}u$. Since $\cosh^2 u - \sinh^2 u = 1$ and $\cosh u > 0$, the integrand becomes

$$\sqrt{1 + \frac{y^2}{4}} = \sqrt{1 + \sinh^2 u} = \sqrt{\cosh^2 u} = \cosh u$$

For the limits, $\sinh u = \frac{y}{2}$, and

$$\operatorname{arsinh} t = \ln\left(t + \sqrt{t^2 + 1}\right)$$

When $y = 4$, $\sinh u = 2$, so $u = \ln\left(2 + \sqrt{5}\right)$ and $\cosh u = \sqrt{1 + 2^2} = \sqrt{5}$.

When $y = -2$, $\sinh u = -1$, so $u = \ln\left(\sqrt{2} - 1\right)$ and $\cosh u = \sqrt{1 + (-1)^2} = \sqrt{2}$.

Using $\cosh 2u = 2\cosh^2 u - 1$ and $\sinh 2u = 2\sinh u \cosh u$:

$$\begin{aligned}
\int_{-2}^{4} \sqrt{1 + \frac{y^2}{4}}\,\mathrm{d}y &= \int_{\ln(\sqrt{2}-1)}^{\ln(2+\sqrt{5})} \cosh u \times 2\cosh u\,\mathrm{d}u \\
&= \int_{\ln(\sqrt{2}-1)}^{\ln(2+\sqrt{5})} (1 + \cosh 2u)\,\mathrm{d}u \\
&= \left[u + \tfrac{1}{2}\sinh 2u\right]_{\ln(\sqrt{2}-1)}^{\ln(2+\sqrt{5})} \\
&= \left[u + \sinh u \cosh u\right]_{\ln(\sqrt{2}-1)}^{\ln(2+\sqrt{5})} \\
&= \left(\ln\left(2 + \sqrt{5}\right) + 2\sqrt{5}\right) - \left(\ln\left(\sqrt{2} - 1\right) - \sqrt{2}\right) \\
&= 2\sqrt{5} + \sqrt{2} + \ln\left(2 + \sqrt{5}\right) - \ln\left(\sqrt{2} - 1\right)
\end{aligned}$$

Since $\left(\sqrt{2} - 1\right)\left(\sqrt{2} + 1\right) = 1$, $-\ln\left(\sqrt{2} - 1\right) = \ln\left(\sqrt{2} + 1\right)$, so the arc length is

$$2\sqrt{5} + \sqrt{2} + \ln\left(\left(2 + \sqrt{5}\right)\left(1 + \sqrt{2}\right)\right)$$

Adding the chord $PQ = 3\sqrt{5}$ from part (a), the perimeter of R is

$$5\sqrt{5} + \sqrt{2} + \ln\left(\left(2 + \sqrt{5}\right)\left(1 + \sqrt{2}\right)\right)$$

12. A curve has equation $y = 16\sqrt{x}$, where $x > 0$. The length of the arc of the curve between the points on the curve where $x = 36$ and $x = 225$ is denoted by s.

(a) Show that

$$s = \int_{36}^{225} \sqrt{\frac{x+64}{x}}\,\mathrm{d}x \qquad \textbf{[3]}$$

(b) (i) Show that $\operatorname{arsinh}\frac{15}{8} = 2\operatorname{arsinh}\frac{3}{4}$. **[3]**

(ii) Use the substitution $x = 64\sinh^2\theta$ to show that $s = p + q\ln 2$, where p and q are positive integers to be determined. **[6]**

Solution

(a) Differentiating $y = 16x^{\frac{1}{2}}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^{-\frac{1}{2}} = \frac{8}{\sqrt{x}}$$

so

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \frac{64}{x} = \frac{x+64}{x}$$

The arc length of a curve from $x = a$ to $x = b$ is

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

With limits $x = 36$ and $x = 225$, this gives

$$s = \int_{36}^{225} \sqrt{\frac{x+64}{x}}\,\mathrm{d}x$$

as required.

(b) (i) The standard result is

$$\operatorname{arsinh} t = \ln\left(t + \sqrt{t^2+1}\right)$$

so

$$\operatorname{arsinh}\frac{15}{8} = \ln\left(\frac{15}{8} + \sqrt{\frac{225}{64}+1}\right) = \ln\left(\frac{15}{8} + \frac{17}{8}\right) = \ln 4$$

$$\operatorname{arsinh}\frac{3}{4} = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$$

Since $\ln 4 = 2\ln 2$, it follows that

$$\operatorname{arsinh}\frac{15}{8} = 2\ln 2 = 2\operatorname{arsinh}\frac{3}{4}$$

as required.

(ii) With $x = 64\sinh^2\theta$:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 128\sinh\theta\cosh\theta$$

When $x = 36$, $\sinh^2\theta = \frac{36}{64}$, so taking $\theta > 0$, $\sinh\theta = \frac{3}{4}$ and $\theta = \operatorname{arsinh}\frac{3}{4} = \ln 2$ by part (b)(i).
When $x = 225$, $\sinh^2\theta = \frac{225}{64}$, so $\sinh\theta = \frac{15}{8}$ and $\theta = \operatorname{arsinh}\frac{15}{8} = \ln 4$.

Using $\cosh^2\theta - \sinh^2\theta = 1$:

$$x + 64 = 64\sinh^2\theta + 64 = 64\cosh^2\theta$$

For $\ln 2 \leq \theta \leq \ln 4$, both $\sinh\theta$ and $\cosh\theta$ are positive, so

$$\sqrt{\frac{x+64}{x}} = \sqrt{\frac{64\cosh^2\theta}{64\sinh^2\theta}} = \frac{\cosh\theta}{\sinh\theta}$$

The double angle formula $\cosh 2\theta = 2\cosh^2\theta - 1$ gives $\cosh^2\theta = \frac{1}{2}(1 + \cosh 2\theta)$. Substituting:

$$\begin{aligned} s &= \int_{\ln 2}^{\ln 4} \frac{\cosh\theta}{\sinh\theta} \times 128\sinh\theta\cosh\theta\,\mathrm{d}\theta \\ &= \int_{\ln 2}^{\ln 4} 128\cosh^2\theta\,\mathrm{d}\theta \\ &= \int_{\ln 2}^{\ln 4} 64(1 + \cosh 2\theta)\,\mathrm{d}\theta \\ &= [64\theta + 32\sinh 2\theta]_{\ln 2}^{\ln 4} \\ &= [64\theta + 64\sinh\theta\cosh\theta]_{\ln 2}^{\ln 4} \end{aligned}$$

At $\theta = \ln 4$, $\sinh\theta = \frac{15}{8}$ and $\cosh\theta = \sqrt{1 + \frac{225}{64}} = \frac{17}{8}$. At $\theta = \ln 2$, $\sinh\theta = \frac{3}{4}$ and $\cosh\theta = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$. So

$$\begin{aligned} s &= 64(\ln 4 - \ln 2) + 64\left(\frac{15}{8} \times \frac{17}{8} - \frac{3}{4} \times \frac{5}{4}\right) \\ &= 64\ln 2 + 255 - 60 \\ &= 195 + 64\ln 2 \end{aligned}$$

so s is in the form $p + q\ln 2$ with $p = 195$, $q = 64$.