OPTIONAL PURE / FURTHER CALCULUS

The t-Formulae

Questions by topic 10 questions · 93 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*10 marks*	2
Question 2	*8 marks*	3
Question 3	*9 marks*	4
Question 4	*7 marks*	5
Question 5	*9 marks*	6
Question 6	*10 marks*	7
Question 7	*7 marks*	8
Question 8	*13 marks*	9
Question 9	*9 marks*	10
Question 10	*11 marks*	11

1. **(a)** Write down the t-formula for $\tan x$, where $t = \tan\left(\frac{x}{2}\right)$. **[1]**

(b) (i) Given that $\cos x = \frac{1-t^2}{1+t^2}$, use your answer to part (a) to show that

$$\sin x = \frac{2t}{1+t^2}$$ **[2]**

(ii) Given that $\tan\left(\frac{x}{2}\right) = \frac{1}{\sqrt{5}}$, find the exact values of $\tan x$ and $\sin x$. **[2]**

(c) Use the t-formulae to solve the equation

$$4\sin\theta + 3\cos\theta + 3 = 0$$

for $-180° < \theta \leq 180°$, giving your answers to 1 decimal place where appropriate. Explain why the substitution $t = \tan\left(\frac{\theta}{2}\right)$ does not give every solution of this equation. **[5]**

2. Consider the equation

$$7\sin x + \cos x = 5$$

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that the equation can be written as

$$3t^2 - 7t + 2 = 0$$ **[3]**

(b) Hence solve the equation for $0° \leq x \leq 720°$, giving your answers to 1 decimal place. Explain why no solutions of the equation in this interval are lost by using the substitution $t = \tan\left(\frac{x}{2}\right)$. **[5]**

3. **(a)** Show that $\sin 2\theta + 3\cos 2\theta + 2$ can be written as

$$\frac{5 + 2t - t^2}{1 + t^2}$$

where $t = \tan\theta$. **[3]**

(b) Hence find the general solution of the equation

$$\sin 2\theta + 3\cos 2\theta + 2 = 3$$

giving your answers in exact form. **[6]**

4. Throughout this question, $t = \tan\left(\frac{x}{2}\right)$, where $x \neq n\pi$, $n \in \mathbb{Z}$.

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\operatorname{cosec} x + \cot x \equiv \frac{1}{t} \quad \textbf{[2]}$$

(b) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ and your answer to part (a) to prove that

$$\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x - \cot x} \equiv \frac{1 + \cos x}{1 - \cos x} \quad \textbf{[3]}$$

(c) Hence solve, for $0 < x < 2\pi$, the equation

$$\operatorname{cosec} x + \cot x = 3(\operatorname{cosec} x - \cot x) \quad \textbf{[2]}$$

5. During 2031, the mean daily temperature in a city, $T\,^\circ\text{C}$, is modelled by the equation

$$T = 1.2\sin\left(\frac{x}{60}\right) - 6\cos\left(\frac{x}{60}\right) + 11 \qquad 0 \leq x < 365$$

where x is the number of days after 1st January 2031 and the angle is in radians.

(a) Find, according to the model, the mean temperature on 1st March 2031, giving your answer to 3 significant figures. **[1]**

(b) Use the substitution $t = \tan\left(\frac{x}{120}\right)$ to show that, for $x \neq 60\pi$, T can be written as

$$T = \frac{at^2 + bt + c}{1 + t^2}$$

where a, b and c are constants to be determined. **[3]**

(c) Hence find, according to the model, the date of the first day and the date of the last day in 2031 on which the mean temperature is at least $15\,^\circ\text{C}$. **[5]**

6. The depth of water, D metres, at the entrance to a harbour on a particular day is modelled by the equation

$$D = 6 + \frac{12}{5}\sin(30x)^\circ + \frac{16}{5}\cos(30x)^\circ$$

where x is the number of hours after midnight and $0 \leq x < 12$.

(a) Use the model to find the depth of water at the harbour entrance at 06:00. **[1]**

(b) Using the substitution $t = \tan(15x)^\circ$, show that the model can be written as

$$D = \frac{At^2 + Bt + C}{5\left(1 + t^2\right)}$$

where A, B and C are integers to be determined. **[3]**

(c) Hence find, according to the model, the two times between midnight and noon at which the depth of water at the harbour entrance is 4 m. Give your answers to the nearest minute. **[5]**

(d) Explain why the expression for D found in part (b) cannot be used to find the depth of water at 06:00. **[1]**

7. **(a)** Use $t = \tan\left(\frac{\theta}{2}\right)$ to show that, where both sides are defined,

$$\frac{13 + 12\cos\theta + 5\sin\theta}{13 - 12\cos\theta - 5\sin\theta} \equiv \left(\frac{t+5}{5t-1}\right)^2 \quad \textbf{[4]}$$

(b) The acute angle α is such that $\tan\alpha = \frac{1}{5}$.
Hence show that, where both sides are defined,

$$\frac{13 + 12\cos\theta + 5\sin\theta}{13 - 12\cos\theta - 5\sin\theta} \equiv \cot^2\left(\frac{\theta}{2} - \alpha\right) \quad \textbf{[3]}$$

8. Throughout this question, $t = \tan\left(\frac{x}{2}\right)$.

(a) Prove that

$$\sin x \equiv \frac{2t}{1+t^2}$$ **[3]**

(b) You may use without proof the corresponding results for $\cos x$ and $\tan x$ in terms of t.
Show that the equation

$$4\tan x + 3\sin x + 6\cos x + 6 = 0$$

can be written as

$$t^3 - 6t^2 + 7t + 6 = 0$$ **[4]**

(c) Hence solve, for $-180° < x < 180°$, the equation

$$4\tan x + 3\sin x + 6\cos x + 6 = 0$$

giving each answer to 1 decimal place. **[6]**

9.

$$f(x) = \frac{1 + \cos x}{3 + 2\sin x + \cos x}, \qquad 0 < x < 2\pi$$

Using the substitution $t = \tan\left(\frac{x}{2}\right)$,

(a) show that, for $x \neq \pi$, $f(x)$ can be written in the form

$$\frac{1}{(t+1)^2 + 1}$$ **[3]**

(b) Hence solve the equation $f(x) = \frac{1}{5}$, giving each answer exactly where possible and otherwise to 2 decimal places. **[4]**

(c) Find the greatest value of $f(x)$ and state the exact value of x at which it occurs. **[2]**

10. Using the substitution $t = \tan\left(\frac{x}{2}\right)$, find the general solution, in radians, to the equation

$$4\sin x + \tan x = 3\cot\left(\frac{x}{2}\right)$$

You should determine whether any solutions of the equation are lost by using this substitution, and give your final answer in as simple a form as possible. **[11]**