OPTIONAL PURE / FURTHER CALCULUS

The Weierstrass Substitution

Questions by topic 10 questions · 82 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

Question 1	*6 marks*	2
Question 2	*6 marks*	3
Question 3	*7 marks*	4
Question 4	*6 marks*	5
Question 5	*6 marks*	6
Question 6	*9 marks*	7
Question 7	*9 marks*	8
Question 8	*11 marks*	9
Question 9	*11 marks*	10
Question 10	*11 marks*	11

1. Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$\int_0^{\frac{2}{3}\pi} \frac{1}{5 + 4\cos x}\,\mathrm{d}x = \frac{\pi}{9} \qquad \textbf{[6]}$$

2. The integral I is defined by

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x + 2\cos x} \, \mathrm{d}x$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of I, giving your answer in the form $p \ln q$, where p and q are rational numbers. **[6]**

3. **(a)** Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to find the integers a, b and c, with $b > 0$, such that

$$\int \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta = \int \frac{a}{(bt + c)^2 + 1}\,\mathrm{d}t \qquad \textbf{[3]}$$

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 - 2\sin\theta - \cos\theta}\,\mathrm{d}\theta \qquad \textbf{[4]}$$

4. By using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, find the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{1}{3 + 5\cos\theta}\,\mathrm{d}\theta \qquad \textbf{[6]}$$

5. Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find the exact value of

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5 + 4\cos x + 3\sin x}\,\mathrm{d}x \qquad \textbf{[6]}$$

6. Using the substitution $t = \tan\left(\frac{x}{2}\right)$, evaluate the integral

$$\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}x}{1 + 2\sin x}$$

giving your answer in the form $p\sqrt{3}\ln\left(q + \sqrt{3}\right)$, where p is a rational number and q is an integer, both to be determined. **[9]**

7. The integral I is defined by

$$I = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{2 - \cos x} \, \mathrm{d}x$$

(a) By first writing $\frac{\cos x}{2-\cos x}$ in the form $A + \frac{B}{2-\cos x}$, where A and B are constants, and then using the substitution $t = \tan\left(\frac{x}{2}\right)$, show that

$$I = -\frac{\pi}{2} + \int_0^1 \frac{4}{1+3t^2} \, \mathrm{d}t \qquad \textbf{[5]}$$

(b) Hence find the exact value of I, giving your answer in the form $p\pi + q\pi\sqrt{3}$, where p and q are rational numbers. **[4]**

8. The integral I is defined by

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x + \cos x} \, \mathrm{d}x$$

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \int_0^1 \frac{2t}{(1+t)\,(1+t^2)} \, \mathrm{d}t \qquad \textbf{[4]}$$

(b) Hence find the exact value of I, giving your answer in the form $a\pi + b\ln 2$, where a and b are rational. **[7]**

9. The integral I is defined by

$$I = \int_0^{\frac{1}{2}\pi} \frac{1-\cos\theta}{1+\sin\theta}\,\mathrm{d}\theta$$

(a) By using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, show that

$$I = \int_0^1 \frac{4t^2}{(1+t)^2(1+t^2)}\,\mathrm{d}t \qquad \textbf{[4]}$$

(b) Hence find the exact value of I, giving your answer in the form $a+b\ln 2$, where a and b are integers. **[7]**

10. The integral I is defined by

$$I = \int_0^{\frac{1}{3}\pi} \frac{\sin x}{1+\sin x}\,\mathrm{d}x$$

(a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{4t}{(1+t)^2(1+t^2)}\,\mathrm{d}t \qquad \textbf{[4]}$$

(b) Express $\frac{4t}{(1+t)^2(1+t^2)}$ in partial fractions. **[3]**

(c) Hence show that $I = \frac{1}{3}\pi + 1 - \sqrt{3}$. **[4]**