OPTIONAL PURE / FURTHER CALCULUS

Taylor Series

Questions by topic 15 questions · 132 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*7 marks*	2
Question 2	*6 marks*	3
Question 3	*6 marks*	4
Question 4	*8 marks*	5
Question 5	*7 marks*	6
Question 6	*6 marks*	7
Question 7	*6 marks*	8
Question 8	*10 marks*	9
Question 9	*11 marks*	10
Question 10	*10 marks*	11
Question 11	*10 marks*	12
Question 12	*9 marks*	13
Question 13	*8 marks*	14
Question 14	*14 marks*	15
Question 15	*14 marks*	16

1. The function f is defined by $f(x) = \sqrt{x}$, $x > 0$.

(a) Find $f'(x)$, $f''(x)$ and $f'''(x)$. **[2]**

(b) Hence find the Taylor series for $f(x)$ about $x = 4$, in ascending powers of $(x-4)$, up to and including the term in $(x-4)^3$. **[3]**

(c) Use your answer to part (b), with a suitable value of x, to find an estimate for $\sqrt{5}$, giving your answer as a fraction in its simplest form. **[2]**

2. The function f is defined by $f(x) = \cosh 2x$ for $x \in \mathbb{R}$.

(a) Show that $\cosh(2\ln 2) = \frac{17}{8}$, and find the exact value of $\sinh(2\ln 2)$. **[2]**

(b) Hence, by differentiation, find the Taylor series expansion of $f(x)$ about $x = \ln 2$, up to and including the term in $(x - \ln 2)^3$, giving each coefficient as an exact fraction. **[4]**

3. The function f is defined by $f(x) = \cos 2x$.

(a) By differentiating, find the Taylor series expansion of $f(x)$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$, giving each coefficient in exact form. **[4]**

(b) Use your answer to part (a) to obtain an estimate of $\cos 1$, giving your answer to 6 decimal places. **[2]**

4. Given that $y = \ln(\sin x)$, where $0 < x < \pi$,

(a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\,\mathrm{cosec}^2 x \cot x$. **[5]**

(b) Hence find the Taylor series expansion of $\ln(\sin x)$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in exact form. **[3]**

5. A function is defined by

$$y = \ln(2 + \ln x) \qquad x > \frac{1}{e}$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{3 + \ln x}{x^2(2 + \ln x)^2}$$ **[4]**

(b) Hence, or otherwise, find the Taylor series expansion of y about $x = e$, in ascending powers of $(x - e)$, up to and including the term in $(x - e)^2$, giving each coefficient in exact simplest form. **[3]**

6. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

(a) Use differentiation to determine the Taylor series expansion of $x \ln x$, in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$. **[4]**

(b) Hence find the exact value of

$$\lim_{x \to 1} \left(\frac{2x \ln x + 1 - x^2}{(x-1)^3} \right)$$ **[2]**

7. **(a)** By writing $\ln x = \ln 3 + \ln\left(1 + \frac{x-3}{3}\right)$ and using the series expansion of $\ln(1+u)$, find the Taylor series of $\ln x$ about $x = 3$ in ascending powers of $(x-3)$, up to and including the term in $(x-3)^3$. State the set of values of x for which the expansion is valid. **[4]**

(b) Use your series to find an approximation to $\ln 2.7$, giving your answer in the form $\ln 3 - p$, where p is a fraction in its lowest terms. **[2]**

8. The function f is defined by $f(x) = \arctan x$.

(a) Using repeated differentiation, find the Taylor series expansion of $f(x)$ about $x = 1$, up to and including the term in $(x-1)^3$. **[6]**

(b) Use your series to find an approximation to $\arctan 1.2$ in the form $\frac{\pi}{4} + \frac{p}{q}$, where p and q are positive integers with no common factor. **[2]**

(c) Given that $\arctan 1.2 = 0.876058$ to 6 decimal places, use your answer to part (b) to find an estimate for π, giving your answer to 4 decimal places. **[2]**

9. It is given that

$$y = \ln\left(3 + e^{2x}\right)$$

(a) Show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{12e^{2x}}{\left(3 + e^{2x}\right)^2} \quad \textbf{[4]}$$

(b) Hence determine the Maclaurin series expansion for y, in ascending powers of x, up to and including the term in x^2, giving each term in simplest form. **[3]**

(c) Use the series expansion for $\cos x$ in ascending powers of x to determine, in simplest form, the first 2 non-zero terms in ascending powers of x of the series for $\cos 2x$. **[1]**

(d) Use your answers to parts (b) and (c) to find the exact value of

$$\lim_{x \to 0} \frac{\ln\left(3 + e^{2x}\right) - 2\ln 2 - \frac{1}{2}x}{1 - \cos 2x} \quad \textbf{[3]}$$

10. It is given that $y = \tan x$.

(a) Show that

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = a\tan^4 x + b\tan^2 x + c$$

where a, b and c are integers to be determined. **[4]**

(b) Hence find the Taylor series expansion of $\tan x$ about $\frac{\pi}{6}$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$, up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$, giving each coefficient in exact simplest form. **[3]**

(c) Use your answer to part (b) to find an approximate value of $\tan 0.5$, giving your answer to four significant figures. Explain why this approximation would be expected to be accurate. **[3]**

11. Given that $y = \sec^3 x$,

(a) show that

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 3\sec^3 x \tan x \left(p\sec^2 x + q\right)$$

where p and q are integers to be determined. **[5]**

(b) Hence find the Taylor series expansion of $\sec^3 x$ about $x = \frac{\pi}{4}$, in ascending powers of $\left(x - \frac{\pi}{4}\right)$, up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in exact simplest form. **[3]**

(c) Use your expansion to estimate the value of $\sec^3\left(\frac{\pi}{4} + 0.1\right)$, giving your answer to 3 decimal places. **[2]**

12. The function f is defined by $f(x) = \ln(1 + \tan x)$ for $0 \leq x < \frac{\pi}{2}$.

(a) Find the Taylor series expansion of $f(x)$ about $x = \frac{\pi}{4}$, in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, giving each coefficient in simplest form. **[7]**

(b) Hence show that

$$\ln\left(1 + \sqrt{3}\right) \approx \ln 2 + \frac{\pi}{12} + \frac{\pi^2}{A} + \frac{\pi^3}{B}$$

where A and B are integers to be determined. **[2]**

13. **(a)** Using the substitution $t = x - \frac{\pi}{4}$ and the standard series for $\sin t$ and $\cos t$, show that the Taylor series of $\sin x$ about $x = \frac{\pi}{4}$, up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$, is

$$\frac{\sqrt{2}}{2}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right] \quad \textbf{[3]}$$

(b) Hence find the value of the constant k for which the limit below exists, and find the value of this limit.

$$\lim_{x \to \frac{\pi}{4}} \frac{2\sqrt{2}\sin x + \cos 2x - 2 + k\left(x - \frac{\pi}{4}\right)^2}{\left(x - \frac{\pi}{4}\right)^3} \quad \textbf{[5]}$$

14. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

It is given that

$$y = x^x, \qquad x > 0$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^x\left[(1+\ln x)^2 + \frac{1}{x}\right] \quad \textbf{[4]}$$

(b) Hence find the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ when $x = 1$. **[3]**

(c) Find the Taylor series expansion of x^x about $x = 1$ in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$, giving each coefficient in simplest form. **[3]**

(d) Use your answer to part (c), explaining your reasoning clearly, to evaluate

$$\lim_{x\to 1}\frac{x^x - x^2 + x - 1}{(x-1)^2\ln x} \quad \textbf{[4]}$$

15.

$$y = \ln(\cos x - \sin x) \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sec 2x - \tan 2x$$

[3]

(b) Find an expression for $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}$ in terms of $\sec 2x$ and $\tan 2x$. **[3]**

(c) Hence determine the first four non-zero terms, in ascending powers of x, of the Maclaurin series expansion of $\ln(\cos x - \sin x)$, giving each coefficient in simplest form. **[3]**

(d) Use the Maclaurin series expansion for $\ln(1+x)$ to write down the first three non-zero terms, in ascending powers of x, of the Maclaurin series expansion of $\ln(1+kx)$, where k is a non-zero constant. **[1]**

(e) Hence determine the value of k for which

$$\lim_{x\to 0}\left(\frac{1}{x^2}\ln\left((1+kx)(\cos x - \sin x)\right)\right)$$

exists, and find the value of this limit. **[4]**