OPTIONAL PURE / FURTHER CALCULUS

Surface Area of Revolution

Questions by topic 13 questions · 131 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

Question 1	*7 marks*	2
Question 2	*8 marks*	3
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Question 4	*7 marks*	5
Question 5	*13 marks*	6
Question 6	*14 marks*	7
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Question 10	*9 marks*	11
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1. A curve C has equation $y = 2\cosh\left(\frac{x}{2}\right)$, for $0 \leq x \leq 2\ln 2$.
The curve C is rotated through 2π radians about the x-axis.
Find the exact area of the surface generated, giving your answer in the form $\pi\left(a + b\ln 2\right)$, where a and b are rational numbers. **[7]**

2. A curve C has parametric equations $x = 2t$, $y = t^2$, for $0 \leq t \leq \sqrt{3}$, where both the x- and y-units are in cm. The area of the surface generated when C is rotated through 2π radians about the y-axis is $A\,\text{cm}^2$.

(a) Show that $A = \frac{56\pi}{3}$. **[6]**

The reflector of a car headlamp is made in the shape of this surface. The curved surface area of the reflector must be $58\,\text{cm}^2$, accurate to within two percent. The manufacturing process produces reflectors whose curved surface area is accurate to within one half of one percent of $A\,\text{cm}^2$.

(b) Determine whether all reflectors produced will be suitable for use in the headlamp. **[2]**

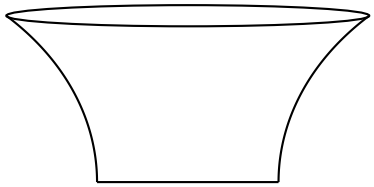
3. The curve C has parametric equations $x = 3t^2$, $y = 3t - t^3$, where $t \geq 0$.
The curve C meets the x-axis at the origin O and at one other point P.

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 9\left(1 + t^2\right)^2$$ **[2]**

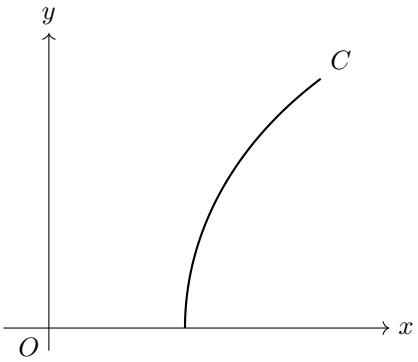
(b) The arc of C between O and P is rotated through 2π radians about the x-axis to form a surface of revolution.
Find the exact area of this surface. **[5]**

4. A curve has equation $y = \cosh x$, for $0 \leq x \leq \ln 2$.
The curve is rotated through 2π radians about the y-axis.
Find the area of the surface generated, giving your answer in the form $\frac{\pi}{2}(p \ln 2 + q)$, where p and q are integers. **[7]**



5. The diagram above shows a picture of a ceramic bowl.
The bowl has

- a flat circular base of radius 6 cm
- a height of 11 cm



The diagram above shows a sketch of the curve C with parametric equations

$$x = 6 + 6t^2 \qquad y = 12t - t^3 \qquad 0 \leq t \leq 1$$

The curved inner surface of the bowl is modelled by the surface of revolution formed by rotating C through 2π radians about the y-axis.

(a) Show that, according to the model, the area of the curved inner surface of the bowl is given by

$$36\pi \int_0^1 \left(4 + 5t^2 + t^4\right) \, \mathrm{d}t \qquad \textbf{[5]}$$

(b) Find, according to the model, the total area of the inner surface of the bowl, giving your answer as an exact multiple of π. **[4]**

The inner surface of each bowl is to be glazed. The curved inner surface needs two coats of glaze and the base needs one coat. One pot of glaze covers an area of $2\,\mathrm{m}^2$ with a single coat.

(c) Use your answer to part (b) to find, according to the model, the greatest number of bowls that can be glazed using one pot of glaze. **[3]**

(d) Give a reason why, in practice, one pot of glaze might cover fewer bowls than your answer to part (c). **[1]**

6. The points $P\left(1, \frac{3}{8}\right)$ and $Q\left(2, \frac{33}{16}\right)$ lie on the curve with equation $y = \frac{1}{8}x^4 + \frac{1}{4x^2}$.
The area of the surface generated when the arc PQ is rotated through 2π radians about the x-axis is denoted by A.

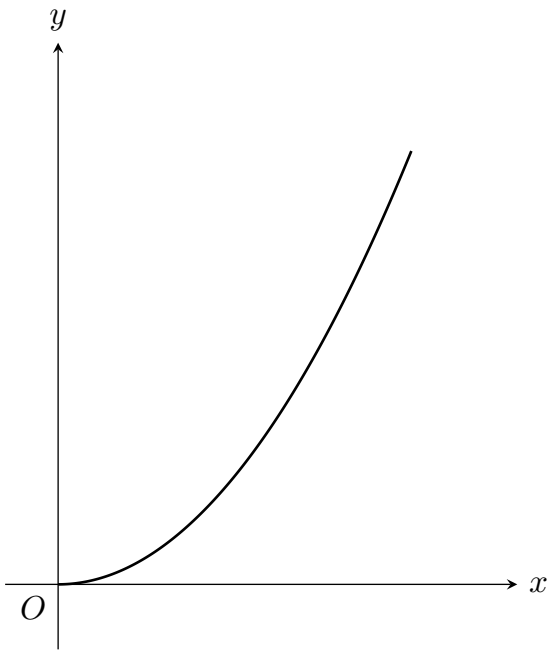
(a) Find the exact value of A, giving your answer as a rational multiple of π. **[6]**

Student X finds an approximation to A by modelling the arc PQ as the straight line segment PQ, then rotating this segment through 2π radians about the x-axis to form the curved surface of a frustum of a cone.

(b) Find the approximation to A obtained by student X, giving your answer in the form $k\pi\sqrt{n}$, where k is rational and n is a positive integer. **[4]**

Student Y finds a second approximation to A by modelling the arc PQ as the line $y = M$, where M is the mean value of $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2}$ over the interval $1 \leq x \leq 2$, then rotating the part of this line between $x = 1$ and $x = 2$ through 2π radians about the x-axis.

(c) Find the percentage error in the approximation to A obtained by student Y, giving your answer to 3 significant figures. **[4]**



7. The diagram above shows a sketch of the curve with equation

$$y = x^2, \qquad 0 \leq x \leq 1$$

The curve is rotated through 2π radians about the x-axis.

(a) Show that the area of the curved surface generated is given by

$$S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2}\, \mathrm{d}x$$ **[2]**

(b) Hence, using the substitution $2x = \sinh\theta$, show that

$$S = \frac{\pi}{32}\left(a\sqrt{5} - \ln\left(b + \sqrt{5}\right)\right)$$

where a and b are integers to be determined. **[8]**

8. **(a)** Show that

$$\int_1^2 \left(t + \frac{1}{t}\right) \ln t \, \mathrm{d}t = a + b \ln 2 + c(\ln 2)^2$$

where a, b and c are rational numbers to be found. **[5]**

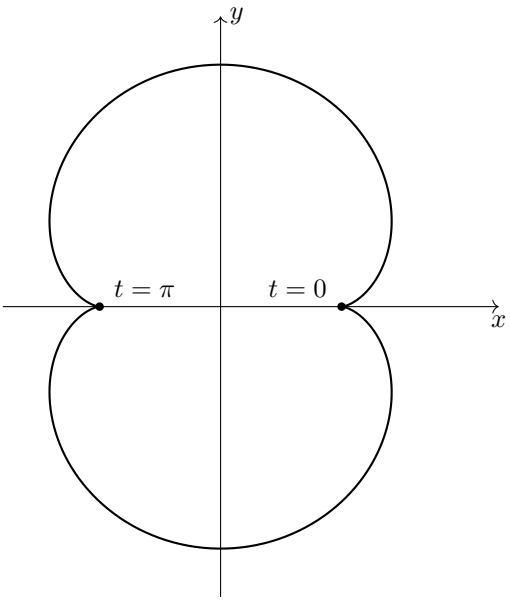
(b) A curve C is defined parametrically for $t > 0$ by

$$x = t^2 - 2 \ln t \qquad y = 4t$$

The arc of C from $t = 1$ to $t = 2$ is rotated through 2π radians about the y-axis to generate a surface with area S.
Using your answer to part (a), find the exact value of S, giving your answer in the form

$$S = \pi \left(p + q \ln 2 + r(\ln 2)^2\right)$$

where p, q and r are rational numbers to be found. **[6]**



9. The diagram above shows a curve known as a nephroid.
The curve has parametric equations

$$x = 3\cos t - \cos 3t$$
$$y = 3\sin t - \sin 3t$$
$$(0 \leq t < 2\pi)$$

The section of the curve from $t = 0$ to $t = \pi$ is rotated through 2π radians about the x-axis.
Show that the area of the surface generated is $k\pi^2$, where k is an integer to be found. **[8]**

10. A designer intends to manufacture a product using a 3-D printer. The product will take the form of a surface S which must meet a number of design specifications.
The designer chooses to model S with the equation $z = y^2 + \sinh^2 x$ for $-\ln 5 \leq x \leq \ln 5$, $-3 \leq y \leq 3$.

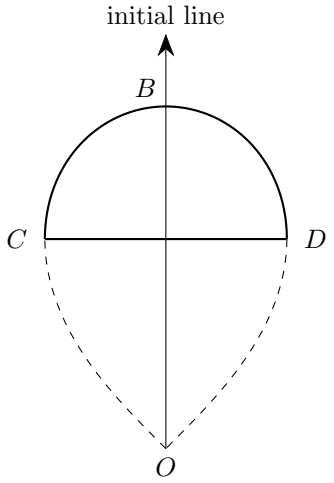
(a) Sketch, in the x-y plane, the curve C of the points (x, y) for which $z = 4$. **[1]**

(b) The part of C for which $y \geq 0$ is rotated through 2π radians about the x-axis to generate the closed surface T. The surface area of T is denoted by A.
Show that A can be written in the form

$$k\pi \int_0^{\ln\left(2+\sqrt{5}\right)} \sqrt{\sinh^4 x + 4}\,\mathrm{d}x$$

for some integer k to be determined. **[6]**

(c) A second design specification is that the surface area of T must not be greater than 45 square units.
Use your calculator to decide whether the product meets this requirement according to the model. **[2]**



11. The diagram above shows the vertical cross-section of a glass paperweight with a flat circular base. The curve in the diagram, including its dashed sections, is the loop of the curve with polar equation

$$r^2 = 32\cos 2\theta, \qquad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

where r is measured in centimetres. The initial line points vertically upwards from the pole O, and B is the point where the loop meets the initial line.
The points C and D lie on the loop, the line CD is perpendicular to the initial line, and CD is the greatest width of the loop.
The paperweight is modelled as the solid formed when the region bounded by the arc CBD and the line segment CD is rotated through π radians about the initial line.

(a) Show that C and D are the points of the loop where $\theta = \pm\frac{\pi}{6}$, and find the length of CD. **[3]**

(b) Find, according to the model, the total surface area of the paperweight, including its base. Give your answer in the form $\pi\left(p - q\sqrt{3}\right)$ cm^2, where p and q are integers. **[7]**

12. The curve C has parametric equations

$$x = 4\cosh t, \qquad y = 3t - \sinh 2t, \qquad 0 \leq t \leq \ln 2$$

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (2\cosh 2t - 1)^2 \qquad \textbf{[4]}$$

The curve C is rotated through 2π radians about the x-axis. The area of the surface generated is denoted by A.

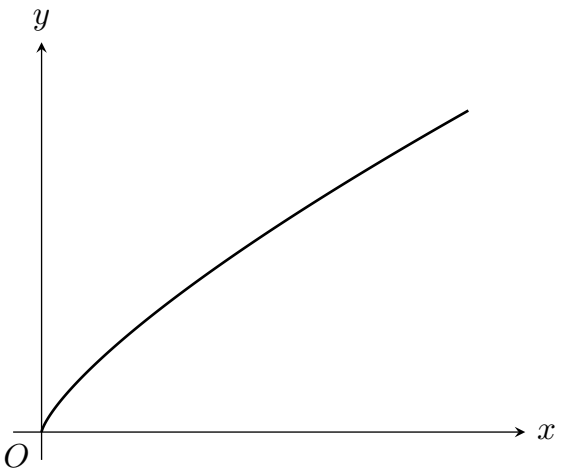
(b) Find A, giving your answer in the form $\pi\left(p + q\ln 2 + r(\ln 2)^2\right)$, where p, q and r are rational numbers. **[8]**

13. For integers $n \geq 0$, let

$$I_n = \int_0^1 t^n \sqrt{1+3t^2}\,\mathrm{d}t$$

(a) Show that, for $n \geq 2$,

$$3(n+2)I_n = 8 - (n-1)I_{n-2}$$ **[5]**



The curve shown in the diagram above has parametric equations

$$x = \frac{3}{4}t^4 \qquad y = \frac{1}{\sqrt{3}}t^3 \qquad 0 \leq t \leq 1$$

The curve is rotated through 2π radians about the x-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by

$$2\pi \int_0^1 t^5 \sqrt{1+3t^2}\,\mathrm{d}t$$ **[5]**

(c) Using the results of parts (a) and (b), find the exact value of the external surface area of the shell, giving your answer in the form $k\pi$, where k is a rational number. Make each step of your working clear. **[5]**