OPTIONAL PURE / FURTHER CALCULUS

Reduction Formulae for Integrals

Questions by topic 14 questions · 152 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

Question 1	*7 marks*	2
Question 2	*8 marks*	3
Question 3	*9 marks*	4
Question 4	*9 marks*	5
Question 5	*9 marks*	6
Question 6	*10 marks*	7
Question 7	*11 marks*	8
Question 8	*10 marks*	9
Question 9	*11 marks*	10
Question 10	*12 marks*	11
Question 11	*12 marks*	12
Question 12	*13 marks*	13
Question 13	*15 marks*	14
Question 14	*16 marks*	15

1. Let

$$I_n = \int_0^1 (1+2x)^n e^x \, \mathrm{d}x$$

where n is a non-negative integer.

(a) By integrating by parts, show that, for $n \geq 1$,

$$I_n = 3^n e - 1 - 2nI_{n-1}$$ **[3]**

(b) Hence find the exact value of I_3. **[4]**

2. For integers $n \geq 0$, let

$$I_n = \int_0^2 \left(8 - x^3\right)^n \, \mathrm{d}x$$

(a) Prove that, for $n \geq 1$,

$$I_n = \frac{24n}{3n+1} I_{n-1}$$

[4]

(b) Using the result from part (a) and showing all stages of your working, determine the value of n for which

$$I_n = \frac{1\,990\,656}{455}$$

[4]

3. The integral I_n is defined by

$$I_n = \int_0^{\frac{\pi}{6}} \sin^n x \, \mathrm{d}x \qquad (n \geq 0)$$

(a) Show that

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} - \frac{\sqrt{3}}{n\,(2^n)} \qquad (n \geq 2)$$ **[5]**

(b) Use the result from part **(a)** to show that

$$\int_0^{\frac{\pi}{6}} \sin^6 x \, \mathrm{d}x = \frac{a\pi + b\sqrt{3}}{96}$$

where a and b are integers to be found. **[4]**

4. The integral I_n is defined, for integers $n \geq 0$, by

$$I_n = \int_0^{\sqrt{2}} \left(4 - x^2\right)^n \, \mathrm{d}x$$

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4 - x^2\right)^n\right)$, or otherwise, show that, for $n \geq 1$,

$$(2n+1)I_n = 2^n\sqrt{2} + 8nI_{n-1}$$ **[5]**

(b) Hence find the exact value of I_3, giving your answer in the form $k\sqrt{2}$, where k is a rational number. **[4]**

5. For integers $n \geq 0$, the integral I_n is defined by

$$I_n = \int_0^{\frac{1}{\sqrt{3}}} \frac{x^n}{1+x^2}\,\mathrm{d}x$$

(a) For integers $n \geq 2$, show that

$$I_n + I_{n-2} = \frac{1}{n-1}\left(\frac{1}{\sqrt{3}}\right)^{n-1} \quad \textbf{[3]}$$

(b) (i) Find the exact value of I_6, giving your answer in the form $a\sqrt{3} + b\pi$, where a and b are rational. **[4]**

(ii) Hence deduce that $\pi < \frac{82\sqrt{3}}{45}$. **[2]**

6. The integral I_n is defined, for $n \geq 0$, by

$$I_n = \int_0^{\frac{\pi}{6}} \cos^n 2x \, \mathrm{d}x$$

(a) Prove that, for $n \geq 2$,

$$nI_n = (n-1)I_{n-2} + \frac{\sqrt{3}}{2^{n+1}} \qquad \textbf{[5]}$$

(b) Hence determine the exact value of

$$\int_0^{\frac{\pi}{6}} 64\left(\cos^4 x - \sin^4 x\right)^5 \, \mathrm{d}x \qquad \textbf{[5]}$$

7. The integral I_n, where n is an integer, is defined by

$$I_n = \int_0^{2\sqrt{3}} \left(16 - x^2\right)^{-\frac{1}{2}n} \, \mathrm{d}x$$

(a) Find the exact value of I_1. **[2]**

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(16 - x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$16nI_{n+2} = 2^{1-n}\sqrt{3} + (n-1)I_n$$ **[5]**

(c) Find the exact value of I_7, giving your answer in the form $k\sqrt{3}$, where k is a rational number to be determined. **[4]**

8. It is given that, for $n \geq 0$,

$$I_n = \int_0^{\ln\left(2+\sqrt{3}\right)} \operatorname{sech}^n x \, \mathrm{d}x$$

(a) Show that, for $n \geq 2$,

$$(n-1)I_n = \frac{\sqrt{3}}{2^{n-1}} + (n-2)I_{n-2}$$

You may use the result that $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$. **[5]**

(b) By first finding the exact value of I_1, find the exact value of I_5, giving your answer in the form $a\sqrt{3} + b\pi$, where a and b are rational. **[5]**

9. Let

$$I_n = \int_0^2 \frac{x^n}{\sqrt{9-x^3}}\,\mathrm{d}x$$

for integers $n \geq 0$.

(a) By considering the derivative of $\sqrt{9-x^3}$ with respect to x, find the exact value of I_2. **[2]**

(b) Given that $n \geq 3$, show that

$$(2n-1)I_n = 18(n-2)I_{n-3} - 2^{n-1}$$ **[5]**

(c) Hence find the exact value of

$$\int_0^2 x^5\sqrt{9-x^3}\,\mathrm{d}x$$ **[4]**

10. For integers $n \geq 1$, let

$$I_n = \int \frac{\sin(nx)}{\cos x}\,\mathrm{d}x$$

(a) Show that, for $n \geq 1$,

$$I_{n+2} = -\frac{2\cos\big((n+1)x\big)}{n+1} - I_n$$ **[6]**

(b) Hence find the exact value of the integral below, giving your answer in the form $a + b\ln c$, where a, b and c are rational numbers to be found.

$$\int_0^{\frac{\pi}{4}} \frac{\sin(7x)}{\cos x}\,\mathrm{d}x$$ **[6]**

11. For non-negative integers n, let

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x \, \mathrm{d}x$$

(a) Show that $I_n + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$. **[3]**

(b) Explain why $I_n < I_{n-2}$ for $n \geq 2$. **[2]**

(c) Using parts (a) and (b), show that, for $n \geq 2$,

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

By taking $n = 8$, find an interval within which the value of π lies, giving the end-points of your interval as fractions. **[7]**

12. For a positive constant a, the integral C_n is defined by

$$C_n = \int_0^a x^n \cosh 2x \, \mathrm{d}x \qquad (n \geq 0)$$

(a) Show that, for $n \geq 2$,

$$4C_n = n(n-1)C_{n-2} + 2a^n \sinh 2a - na^{n-1} \cosh 2a \qquad \textbf{[7]}$$

(b) Hence show that

$$\int_0^1 x^3 \cosh 2x \, \mathrm{d}x = \frac{1}{16}\left(e^2 + 6 - 19e^{-2}\right) \qquad \textbf{[6]}$$

13. It is given that

$$I_n = \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^n x \, \mathrm{d}x$$

where n is a positive integer.

(a) By writing $\operatorname{cosec}^n x = \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x$, or otherwise, show that

$$(n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2} \qquad (n \geq 2)$$ **[5]**

(b) Show that $I_8 = \frac{464}{35}\sqrt{3}$. **[4]**

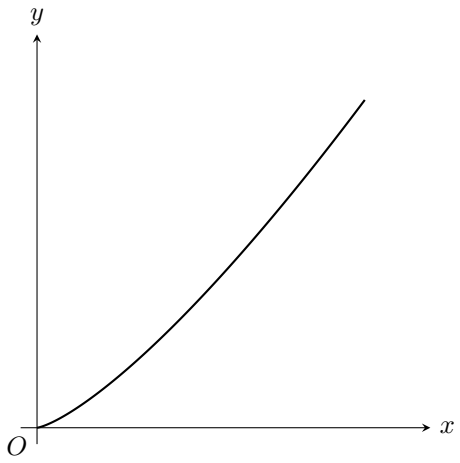
(c) Prove by induction that, for every positive integer n, I_{2n} is a rational multiple of $\sqrt{3}$. **[6]**

14.

$$I_n = \int_0^1 t^n \sqrt{9 + 16t^2}\, \mathrm{d}t \qquad (n \geq 0)$$

(a) Show that, for $n \geq 2$,

$$16(n+2)I_n = 125 - 9(n-1)I_{n-2}$$ **[6]**



The curve shown in the diagram above is defined by the parametric equations

$$x = t^3 \qquad y = t^4 \qquad 0 \leq t \leq 1$$

This curve is rotated through 2π radians about the y-axis to form a hollow open shell.

(b) Show that the external surface area of the shell is given by

$$2\pi \int_0^1 t^5 \sqrt{9 + 16t^2}\, \mathrm{d}t$$ **[5]**

(c) Using the results of parts (a) and (b), and making each step of your working clear, find the exact value of the external surface area of the shell, giving your answer in the form $\frac{p}{q}\pi$, where p and q are positive integers. **[5]**