OPTIONAL PURE / FURTHER CALCULUS

Leibnitz's Theorem

Questions by topic 10 questions · 78 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*5 marks*	2
Question 2	*5 marks*	3
Question 3	*7 marks*	4
Question 4	*8 marks*	5
Question 5	*8 marks*	6
Question 6	*9 marks*	7
Question 7	*8 marks*	8
Question 8	*8 marks*	9
Question 9	*9 marks*	10
Question 10	*11 marks*	11

1. Given that k is a real non-zero constant and that

$$y = x^4 \cos kx$$

use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = \left(k^4 x^4 + Ak^2 x^2 + B\right) \cos kx + Ckx \left(k^2 x^2 + D\right) \sin kx$$

where A, B, C and D are integers to be determined. **[5]**

2. Given that

$$y = x^3 \sinh 2x$$

use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = Px\left(x^2 + Q\right)\cosh 2x + R\left(x^2 + 1\right)\sinh 2x$$

where P, Q and R are integers to be determined. **[5]**

3. It is given that $y = (3x - 1)e^{2x}$.

(a) Use Leibnitz's theorem to show that, for all positive integers n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 2^{n-1}\left(6x + 3n - 2\right)e^{2x} \qquad \textbf{[5]}$$

(b) Find the value of $\frac{\mathrm{d}^{10} y}{\mathrm{d}x^{10}}$ when $x = 0$. **[2]**

4.

$$f(x) = x^3 \cos(2x)$$

The Taylor series expansion of $f(x)$ about $x = k$ is given by

$$f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \cdots + \frac{(x-k)^r}{r!}f^{(r)}(k) + \cdots$$

Use Leibnitz's theorem to show that the coefficient of $\left(x - \frac{\pi}{2}\right)^7$ in the Taylor series expansion of $f(x)$ about $x = \frac{\pi}{2}$ can be written in the form $p\pi^2 + q$, where p and q are rational numbers to be determined. **[8]**

5.

$$f(x) = x^3 \ln x, \qquad x > 0$$

The Taylor series expansion of $f(x)$ about $x = k$ is given by

$$f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \cdots + \frac{(x-k)^r}{r!}f^{(r)}(k) + \cdots$$

(a) Use Leibnitz's theorem to show that

$$f^{(6)}(x) = \frac{12}{x^3}$$ **[5]**

(b) Hence find the coefficients of $(x-1)^6$ and $(x-1)^7$ in the Taylor series expansion of $f(x)$ about $x = 1$, giving each as a fraction in its simplest form. **[3]**

6. A curve has equation

$$y = e^{-x}\cos x$$

(a) Use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -4y$$ **[5]**

(b) Hence find $\frac{\mathrm{d}^{20}y}{\mathrm{d}x^{20}}$ in terms of y, and find the exact value of $\frac{\mathrm{d}^{22}y}{\mathrm{d}x^{22}}$ when $x = \frac{\pi}{2}$. **[4]**

7. A curve has equation

$$y = e^{2x} \sin x$$

(a) Use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = e^{2x} \left(41 \cos x - 38 \sin x\right) \quad \textbf{[5]}$$

(b) Hence express $\frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$ in the form

$$Re^{2x} \cos(x + \alpha)$$

where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give R as an exact surd in its simplest form and α to 3 significant figures. **[3]**

8. It is given that

$$y = \left(x^2 + 3\right) e^{-2x}$$

(a) Use Leibnitz's theorem to show that, for all positive integers k,

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-2)^{k-2} e^{-2x} \left(Ax^2 + Bkx + Ck(k-1) + D\right)$$

where A, B, C and D are integers to be determined. **[5]**

(b) Hence find the least value of k for which the equation

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = 0$$

has a real solution, and find this solution. **[3]**

9. It is given that $y = e^{2x}u$, where u is a function of x. The rth derivatives $\frac{\mathrm{d}^r y}{\mathrm{d}x^r}$ and $\frac{\mathrm{d}^r u}{\mathrm{d}x^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively, with $u^{(0)} = u$.
You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^{2x} \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} u^{(r)} \qquad \textbf{[7]}$$

(b) Hence find $\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(x^3 e^{2x}\right)$, giving your answer in the form $e^{2x}p(x)$, where $p(x)$ is a polynomial. **[2]**

10. It is given that $y = x^2 \ln(1+x)$, where $x > -1$.

(a) Show that

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \frac{2\left(x^2+3x+3\right)}{(1+x)^3} \qquad \textbf{[5]}$$

(b) Prove by mathematical induction that, for every integer $n \geq 3$,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1}\frac{(n-3)!\left(2x^2+2nx+n(n-1)\right)}{(1+x)^n} \qquad \textbf{[6]}$$