OPTIONAL PURE / FURTHER CALCULUS

Taylor Series

Mark scheme 15 questions · 132 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question	Marks by part	Marks	Page
Question 1	(a) 2 (b) 3 (c) 2	7	4
Question 2	(a) 2 (b) 4	6	5
Question 3	(a) 4 (b) 2	6	6
Question 4	(a) 5 (b) 3	8	7
Question 5	(a) 4 (b) 3	7	8
Question 6	(a) 4 (b) 2	6	9
Question 7	(a) 4 (b) 2	6	10
Question 8	(a) 6 (b) 2 (c) 2	10	11
Question 9	(a) 4 (b) 3 (c) 1 (d) 3	11	12
Question 10	(a) 4 (b) 3 (c) 3	10	13
Question 11	(a) 5 (b) 3 (c) 2	10	14
Question 12	(a) 7 (b) 2	9	15
Question 13	(a) 3 (b) 5	8	16
Question 14	(a) 4 (b) 3 (c) 3 (d) 4	14	17
Question 15	(a) 3 (b) 3 (c) 3 (d) 1 (e) 4	14	19
	Total	**132**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [***n***] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$	**M1 A1**	**M1** Differentiates $x^{\frac{1}{2}}$ at least twice, the power decreasing by 1 each time. Condone coefficient slips. **A1** All three correct, oe.
		[2]	
1(b)	$f(x) \approx 2 + \text{“}\frac{1}{4}\text{”}(x-4) + \frac{(x-4)^2}{2!}\left(\text{“}-\frac{1}{32}\text{”}\right)$ $+ \frac{(x-4)^3}{3!}\left(\text{“}\frac{3}{256}\text{”}\right)$	**M1 A1**	**M1** Substitutes their derivatives evaluated at $x = 4$ into the Taylor series, with factorials, up to the cubic term. Condone slips in values. **A1** $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$, $f'''(4) = \frac{3}{256}$, soi.
	$f(x) \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$	**A1**	Correct series to the $(x-4)^3$ term, oe.
		[3]	
1(c)	$\sqrt{5} \approx 2 + \text{“}\frac{1}{4}\text{”} - \text{“}\frac{1}{64}\text{”} + \text{“}\frac{1}{512}\text{”}$	**M1**	Substitutes $x = 5$ into their cubic series from (b).
	$\sqrt{5} \approx \frac{1145}{512}$	**A1**	cao, as a fraction in its simplest form.
		[2]	
Total for Question 1		**7**	

Question	Scheme	Marks	Notes
2(a)	$e^{2\ln 2} = 4, \quad e^{-2\ln 2} = \frac{1}{4}$ $\cosh(2\ln 2) = \frac{1}{2}\left(4 + \frac{1}{4}\right)$	**M1**	Substitutes $e^{\pm 2\ln 2}$ as 4 and $\frac{1}{4}$ into the exponential form of cosh or sinh, or uses the double angle formulae with exact values at $\ln 2$.
	$\cosh(2\ln 2) = \frac{17}{8}, \quad \sinh(2\ln 2) = \frac{15}{8}$	**A1**	Printed value reached without error and $\frac{15}{8}$ oe exact.
		[2]	
2(b)	$f'(x) = 2\sinh 2x, \quad f''(x) = 4\cosh 2x$ $f'''(x) = 8\sinh 2x$	**M1 A1**	**M1** Differentiates at least twice, alternating between forms $k\sinh 2x$ and $k\cosh 2x$. Condone coefficient slips. **A1** All three derivatives correct.
	$f(x) \approx \frac{17}{8} + (x - \ln 2) \times \text{“}\frac{15}{4}\text{”}$ $+ \frac{(x-\ln 2)^2}{2!} \times \text{“}\frac{17}{2}\text{”} + \frac{(x-\ln 2)^3}{3!} \times \text{“}15\text{”}$	**M1**	Substitutes their values of f and its derivatives at $\ln 2$ into the Taylor series, with factorials, up to the cubic term. Condone slips in values.
	$f(x) \approx \frac{17}{8} + \frac{15}{4}(x - \ln 2) + \frac{17}{4}(x - \ln 2)^2$ $+ \frac{5}{2}(x - \ln 2)^3$	**A1**	Correct series; coefficients as exact fractions or integers, need not be simplified.
		[4]	
	Total for Question 2	**6**	

Question	Scheme	Marks	Notes
3(a)	$f'(x) = -2\sin 2x, \quad f''(x) = -4\cos 2x$ $f'''(x) = 8\sin 2x$	**M1 A1**	**M1** Differentiates at least twice, reaching derivatives alternating between forms $\pm k\sin 2x$ and $\pm k\cos 2x$. Condone sign and coefficient slips. **A1** All three derivatives correct.
	$\cos 2x \approx$ "$\frac{1}{2}$" $+ \left(x - \frac{\pi}{6}\right)$ "$\left(-\sqrt{3}\right)$" $+ \frac{1}{2!}\left(x - \frac{\pi}{6}\right)^2$ "(-2)" $+ \ldots$	**M1**	Uses their values at $x = \frac{\pi}{6}$ in the Taylor formula with factorials, up to the cubic term. Condone slips in values.
	$\cos 2x \approx \frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{6}\right) - \left(x - \frac{\pi}{6}\right)^2$ $+ \frac{2\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^3$	**A1**	Correct series, any exact coefficients, such as $\frac{2}{\sqrt{3}}$.
		[4]	
3(b)	$\cos 1 = f\left(\frac{1}{2}\right) \approx \frac{1}{2} - \sqrt{3}\left(\frac{1}{2} - \frac{\pi}{6}\right)$ $- \left(\frac{1}{2} - \frac{\pi}{6}\right)^2 + \ldots$	**M1**	Substitutes $x = \frac{1}{2}$ into their series from (a) in powers of $\left(x - \frac{\pi}{6}\right)$.
	$\cos 1 \approx 0.540302$	**A1**	awrt 0.540302, cso. A calculator value of $\cos 1$ alone scores nothing.
		[2]	
	Total for Question 3	**6**	

Question	Scheme	Marks	Notes
4(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sin x} \times \cos x = \cot x$	**M1 A1**	**M1** Uses the chain rule to reach $\frac{k\cos x}{\sin x}$ or $k\cot x$. **A1** oe, such as $\frac{\cos x}{\sin x}$.
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\operatorname{cosec}^2 x$	**B1**	oe, such as $-\frac{1}{\sin^2 x}$ or $-(1+\cot^2 x)$.
	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -2\operatorname{cosec} x \times (-\operatorname{cosec} x \cot x)$	**M1**	Differentiates their second derivative by any valid rule, such as the chain rule on $(\operatorname{cosec} x)^2$ or $(\sin x)^{-2}$. Condone sign slips.
	$= 2\operatorname{cosec}^2 x \cot x$	**A1 AG**	Correct derivative of $-\operatorname{cosec}^2 x$ leading without error to the printed result; combined steps accepted.
		[5]	
4(b)	$y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\ln 2, \quad y'\left(\frac{\pi}{4}\right) = 1$ $y''\left(\frac{\pi}{4}\right) = -2, \quad y'''\left(\frac{\pi}{4}\right) = 4$ $\ln(\sin x) \approx -\frac{1}{2}\ln 2 + \left(x-\frac{\pi}{4}\right) - \frac{2}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{4}{3!}\left(x-\frac{\pi}{4}\right)^3$	**M1 A1**	**M1** Substitutes their values at $x=\frac{\pi}{4}$ into a Taylor series with factorial denominators, up to the cubic term. Condone slips in values. **A1** All four values correct; $y\left(\frac{\pi}{4}\right)$ oe, such as $\ln\frac{1}{\sqrt{2}}$.
	$\ln(\sin x) \approx -\frac{1}{2}\ln 2 + \left(x-\frac{\pi}{4}\right) - \left(x-\frac{\pi}{4}\right)^2 + \frac{2}{3}\left(x-\frac{\pi}{4}\right)^3$	**A1**	oe, with exact coefficients, such as $\ln\frac{1}{\sqrt{2}}$.
		[3]	
	Total for Question 4	**8**	

Question	Scheme	Marks	Notes
5(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2+\ln x} \times \frac{1}{x} = \frac{1}{x(2+\ln x)}$	**M1 A1**	**M1** Uses the chain rule to reach the form $\frac{k}{x(2+\ln x)}$ oe.
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\big(x(2+\ln x)\big)^{-2} \times \big((2+\ln x) + x \times \frac{1}{x}\big)$	**M1**	Differentiates their $\frac{k}{x(2+\ln x)}$ by quotient, product or chain rule, with the derivatives of x and of $2+\ln x$ both attempted. Condone sign slips.
	$= -\frac{3+\ln x}{x^2(2+\ln x)^2}$	**A1 AG**	Correct derivative leading without error to the printed result; combined steps accepted.
		[4]	
5(b)	$y \approx \ln 3 + (x-e) \times \frac{1}{3e} + \frac{(x-e)^2}{2!} \times \big(-\frac{4}{9e^2}\big)$	**M1 A1**	**M1** Evaluates y, y' and y'' at $x = e$ from their derivatives and substitutes into the Taylor series, including the $\frac{1}{2!}$. Condone slips in values. **A1** Correct values $\ln 3$, $\frac{1}{3e}$, $-\frac{4}{9e^2}$, soi.
	$y \approx \ln 3 + \frac{1}{3e}(x-e) - \frac{2}{9e^2}(x-e)^2$	**A1**	cao. In powers of $(x-e)$, coefficients exact and simplified; $\frac{1}{3}e^{-1}$ and $-\frac{2}{9}e^{-2}$ accepted.
		[3]	
	Total for Question 5	**7**	

Question	Scheme	Marks	Notes
6(a)	$f'(x) = \ln x + x \times \frac{1}{x}$ $f''(x) = \ldots, \quad f'''(x) = \ldots$	**M1**	Uses the product rule on $x \ln x$ and differentiates their result twice more. Condone sign and coefficient slips.
	$f'(x) = \ln x + 1, \quad f''(x) = \frac{1}{x}, \quad f'''(x) = -\frac{1}{x^2}$	**A1**	
	$f(1) = 0,\ f'(1) = 1,\ f''(1) = 1,\ f'''(1) = -1$ $x \ln x \approx (x-1) \times$ "1" $+ \frac{(x-1)^2}{2!} \times$ "1" $+ \frac{(x-1)^3}{3!} \times$ "-1"	**M1**	Evaluates their derivatives at $x = 1$ and substitutes into the Taylor series with $a = 1$ and factorial denominators. Condone slips in values.
	$x \ln x \approx (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$	**A1**	Correct series in powers of $(x-1)$, oe.
		[4]	
6(b)	$1 - x^2 = -2(x-1) - (x-1)^2$ $\frac{2x \ln x + 1 - x^2}{(x-1)^3} = \frac{\text{"}-\frac{1}{3}(x-1)^3\text{"} + \cdots}{(x-1)^3}$	**M1**	Writes $1 - x^2$ in powers of $(x-1)$, or of $t = x - 1$, combines with twice their series from (a) and divides by $(x-1)^3$. Condone sign slips.
	$\lim_{x \to 1} \left(\frac{2x \ln x + 1 - x^2}{(x-1)^3} \right) = -\frac{1}{3}$	**A1**	$-\frac{1}{3}$ cso.
		[2]	
	Total for Question 6	**6**	

Question	Scheme	Marks	Notes
7(a)	$\ln x \approx \ln 3 + \frac{x-3}{3} - \frac{1}{2}\left(\frac{x-3}{3}\right)^2 + \frac{1}{3}\left(\frac{x-3}{3}\right)^3$	**M1**	Substitutes $u = \frac{x-3}{3}$ into the series for $\ln(1+u)$ up to the term in u^3. Need not be simplified; condone sign slips.
	$\ln x \approx \ln 3 + \frac{x-3}{3} - \frac{(x-3)^2}{18} + \frac{(x-3)^3}{81}$	**A1 A1**	**A1** Two of the terms in $(x-3)$, $(x-3)^2$ and $(x-3)^3$ correct, oe. **A1** Correct series including $\ln 3$, oe; need not be simplified.
	$-1 < \frac{x-3}{3} \leq 1 \Rightarrow 0 < x \leq 6$	**B1**	$0 < x \leq 6$, both ends with correct signs; $x > 0$ and $x \leq 6$ given together accepted.
		[4]	
7(b)	$\ln 2.7 \approx \ln 3 + \frac{-0.3}{3} - \frac{(-0.3)^2}{18} + \frac{(-0.3)^3}{81}$	**M1**	Substitutes $x = 2.7$, or $x - 3 = -0.3$, into their series from (a) with at least three terms.
	$[\ln 2.7 \approx \ln 3 - \frac{1}{10} - \frac{1}{200} - \frac{1}{3000}]$ $\ln 2.7 \approx \ln 3 - \frac{79}{750}$	**A1**	cao, in the form $\ln 3 - p$ with p in lowest terms. $\ln 3 - \frac{316}{3000}$ or a decimal scores A0.
		[2]	
	Total for Question 7	**6**	

Question	Scheme	Marks	Notes
8(a)	$f'(x) = (1+x^2)^{-1}$ $f''(x) = -2x(1+x^2)^{-2}$	**M1 A1**	**M1** Differentiates twice by any valid method, reaching $f''(x)$ of the form $kx(1+x^2)^{-2}$. Condone a wrong k. **A1** Correct $f'(x)$ and $f''(x)$, any form.
	$f'''(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$	**M1**	Differentiates their $f''(x)$ by product or quotient rule, or any valid method, with two terms of the right form. Condone sign and coefficient slips.
	$= \frac{6x^2-2}{(1+x^2)^3}$	**A1**	Correct $f'''(x)$, any form, simplified or not.
	$f(1) = \frac{\pi}{4},\ f'(1) = \frac{1}{2}$ $f''(1) = -\frac{1}{2},\ f'''(1) = \frac{1}{2}$ $\arctan x \approx \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right) + \ldots$	**M1**	Evaluates their derivatives at $x = 1$ and substitutes into the Taylor series with 2! and 3!, up to the $(x-1)^3$ term. Condone slips in values.
	$\arctan x \approx \frac{\pi}{4} + \frac{1}{2}(x-1)$ $-\frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$	**A1**	Correct series, oe; coefficients need not be simplified.
		[6]	
8(b)	$\arctan 1.2 \approx \frac{\pi}{4} + \frac{1}{2}(0.2) - \frac{1}{4}(0.2)^2 + \frac{1}{12}(0.2)^3$	**M1**	Substitutes $x = 1.2$, or $x - 1 = 0.2$, into their series.
	$= \frac{\pi}{4} + \frac{34}{375}$	**A1**	cao, with the fraction in lowest terms. A decimal such as 0.0907 scores A0.
		[2]	
8(c)	$\pi \approx 4\left(0.876058 - \text{“}\frac{34}{375}\text{”}\right)$	**M1**	Subtracts their fraction from 0.876058 and multiplies by 4.
	≈ 3.1416	**A1**	awrt 3.1416
		[2]	
	Total for Question 8	**10**	

Question	Scheme	Marks	Notes
9(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2e^{2x}}{3+e^{2x}}$	**M1 A1**	**M1** Differentiates to the form $\frac{ke^{2x}}{3+e^{2x}}$.
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{4e^{2x}(3+e^{2x})-2e^{2x}\times 2e^{2x}}{(3+e^{2x})^2}$	**M1**	Applies the quotient or product rule to their first derivative of the form $\frac{ke^{2x}}{3+e^{2x}}$. Condone sign slips.
	$= \frac{12e^{2x}+4e^{4x}-4e^{4x}}{(3+e^{2x})^2}$ $= \frac{12e^{2x}}{(3+e^{2x})^2}$	**A1 AG**	Correct second derivative leading without error to the printed result; combined steps accepted.
		[4]	
9(b)	$y \approx \ln 4 + x \times \frac{1}{2} + \frac{x^2}{2!} \times \frac{3}{4}$	**M1 A1**	**M1** Substitutes their $y(0)$, $y'(0)$, $y''(0)$ into Maclaurin's series with $\frac{x^2}{2!}$ or $\frac{x^2}{2}$. Not for quoting the series alone. **A1** Correct values $\ln 4$ or $2\ln 2$, $\frac{1}{2}$ and $\frac{3}{4}$, soi.
	$y \approx 2\ln 2 + \frac{1}{2}x + \frac{3}{8}x^2$	**A1**	Accept $\ln 4$ for $2\ln 2$; other coefficients simplified.
		[3]	
9(c)	$\cos 2x \approx 1 - 2x^2$	**B1**	Both terms, simplified.
		[1]	
9(d)	$\frac{\text{“}2\ln 2+\frac{1}{2}x+\frac{3}{8}x^2\text{”}-2\ln 2-\frac{1}{2}x}{1-\text{“}(1-2x^2)\text{”}}$ $= \frac{\frac{3}{8}x^2+\cdots}{2x^2+\cdots}$	**M1**	Replaces numerator and denominator by their series from (b) and (c), reducing each to leading terms in x^2. Condone coefficient slips.
	$\lim_{x\to 0}\frac{\frac{3}{8}+\cdots}{2+\cdots} = \ldots$	**dM1**	Divides by, or cancels, x^2 and lets $x \to 0$ to give the ratio of their x^2 coefficients.
	$\frac{3}{16}$	**A1**	Accept 0.1875.
		[3]	
	Total for Question 9	**11**	

Question	Scheme	Marks	Notes
10(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x = 1 + \tan^2 x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\tan x \sec^2 x$	**M1**	Obtains $\frac{\mathrm{d}y}{\mathrm{d}x}$ as $\sec^2 x$ or $1 + \tan^2 x$ and differentiates again to a form $k\tan x \sec^2 x$ or $k\tan x(1 + \tan^2 x)$. Condone coefficient slips.
	$= 2\tan x + 2\tan^3 x$	**A1**	Correct second derivative, oe, such as $2\tan x \sec^2 x$ or $2\tan x(1 + \tan^2 x)$.
	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \left(2 + 6\tan^2 x\right)\sec^2 x$ $= \left(2 + 6\tan^2 x\right)\left(1 + \tan^2 x\right) = \ldots$	**M1**	Differentiates their second derivative, writes $\sec^2 x$ as $1 + \tan^2 x$ and expands in powers of $\tan x$. Condone coefficient slips.
	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 6\tan^4 x + 8\tan^2 x + 2$	**A1**	cso. Correct expression, or $a = 6$, $b = 8$, $c = 2$ stated.
		[4]	
10(b)	$y\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3},\ y'\left(\frac{\pi}{6}\right) = \frac{4}{3}$ $y''\left(\frac{\pi}{6}\right) = \frac{8\sqrt{3}}{9},\ y'''\left(\frac{\pi}{6}\right) = \frac{16}{3}$	**M1 A1**	**M1** Substitutes $x = \frac{\pi}{6}$ into their derivatives and uses the values in a Taylor series with 2! and 3!. Condone slips in values. **A1** All four values correct, oe such as $\frac{1}{\sqrt{3}}$.
	$\tan x \approx \frac{\sqrt{3}}{3} + \frac{4}{3}\left(x - \frac{\pi}{6}\right)$ $+ \frac{4\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3$	**A1**	cao. Exact simplified coefficients; $\frac{1}{\sqrt{3}}$ and $\frac{4}{3\sqrt{3}}$ accepted.
		[3]	
10(c)	$\tan 0.5 \approx \frac{\sqrt{3}}{3} + \frac{4}{3}\left(0.5 - \frac{\pi}{6}\right) + \ldots$	**M1**	Substitutes $x = 0.5$ into their cubic series from (b) and evaluates.
	$\tan 0.5 \approx 0.5463$	**A1**	awrt 0.5463. A calculator value of $\tan 0.5$ without the series scores nothing.
	0.5 is close to $\frac{\pi}{6}$, so higher powers of $\left(x - \frac{\pi}{6}\right)$ are negligible	**E1**	States 0.5 is close to $\frac{\pi}{6}$, or $\left\lvert x - \frac{\pi}{6}\right\rvert$ small, and that omitted higher-power terms are negligible. Both points needed.
		[3]	
	Total for Question 10	**10**	

Question	Scheme	Marks	Notes
11(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sec^2 x \times \sec x \tan x = 3\sec^3 x \tan x$	**M1 A1**	**M1** Uses the chain rule to reach the form $k\sec^3 x \tan x$, or an equivalent from $(\cos x)^{-3}$. **A1** Correct first derivative, oe.
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\sec^3 x \tan^2 x + 3\sec^5 x$ $[= 12\sec^5 x - 9\sec^3 x]$	**M1**	Uses the product rule on their first derivative, giving two terms, each factor differentiated. Condone coefficient slips.
	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 60\sec^5 x \tan x - 27\sec^3 x \tan x$ $= 3\sec^3 x \tan x(\ldots)$	**dM1**	Differentiates their second derivative, every term attempted, and takes out a factor $\sec^3 x \tan x$, using $\tan^2 x = \sec^2 x - 1$ where needed.
	$3\sec^3 x \tan x\left(20\sec^2 x - 9\right) \Rightarrow p = 20,\ q = -9$	**A1**	cso. Both values, or the correct expression with them implied.
		[5]	
11(b)	$y\left(\frac{\pi}{4}\right) = 2\sqrt{2},\ y'\left(\frac{\pi}{4}\right) = 6\sqrt{2}$ $y''\left(\frac{\pi}{4}\right) = 30\sqrt{2},\ y'''\left(\frac{\pi}{4}\right) = 186\sqrt{2}$	**M1 A1**	**M1** Evaluates their y and three derivatives at $x = \frac{\pi}{4}$ and uses them in the Taylor series with divisors 2! and 3!. Condone slips in values. **A1** All four values correct, soi by the series.
	$\sec^3 x \approx 2\sqrt{2} + 6\sqrt{2}\left(x - \frac{\pi}{4}\right) + 15\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$ $+ 31\sqrt{2}\left(x - \frac{\pi}{4}\right)^3$	**A1**	Exact simplified coefficients; $\sqrt{2}$ may be factored out. Not for decimals or unsimplified factorial divisions.
		[3]	
11(c)	$\sec^3\left(\frac{\pi}{4} + 0.1\right) \approx$ "$2\sqrt{2} + 6\sqrt{2}(0.1) + 15\sqrt{2}(0.1)^2 + 31\sqrt{2}(0.1)^3$"	**M1**	Substitutes 0.1 for $x - \frac{\pi}{4}$ in their series from (b). Not for a direct calculator evaluation of $\sec^3\left(\frac{\pi}{4} + 0.1\right)$.
	≈ 3.933	**A1**	awrt 3.933
		[2]	
	Total for Question 11	**10**	

Question	Scheme	Marks	Notes
12(a)	$f'(x) = \frac{\sec^2 x}{1+\tan x}$	**M1**	Differentiates by the chain rule to the form $\frac{k\sec^2 x}{1+\tan x}$. Condone a slip in k.
	$(1+\tan x)f''(x) + \sec^2 x\, f'(x)$ $= 2\sec^2 x \tan x$	**M1**	Differentiates their f' again by the product, quotient or implicit route, with both terms of the rule present. Condone sign slips.
	$f''\left(\frac{\pi}{4}\right) = 1$	**A1**	
	$(1+\tan x)f'''(x) + 2\sec^2 x\, f''(x)$ $+ 2\sec^2 x \tan x\, f'(x)$ $= 4\sec^2 x \tan^2 x + 2\sec^4 x$	**dM1**	Differentiates their f'' relation or expression again, with at least three terms. Condone sign slips.
	$f'''\left(\frac{\pi}{4}\right) = 4$	**A1**	
	$f(x) \approx \ln 2 + \left(x-\frac{\pi}{4}\right) + \frac{\text{“1”}}{2!}\left(x-\frac{\pi}{4}\right)^2$ $+ \frac{\text{“4”}}{3!}\left(x-\frac{\pi}{4}\right)^3$	**M1**	Uses $f\left(\frac{\pi}{4}\right)$ and their f', f'', f''' values at $\frac{\pi}{4}$ in the Taylor series, with 2! and 3! seen or implied.
	$f(x) \approx \ln 2 + \left(x-\frac{\pi}{4}\right) + \frac{1}{2}\left(x-\frac{\pi}{4}\right)^2$ $+ \frac{2}{3}\left(x-\frac{\pi}{4}\right)^3$	**A1**	Coefficients simplified, so $\frac{4}{6}$ does not score.
		[7]	
12(b)	$x = \frac{\pi}{3} \Rightarrow x - \frac{\pi}{4} = \frac{\pi}{12}$ $\ln\left(1+\sqrt{3}\right) \approx \ln 2 + \frac{\pi}{12} + \frac{1}{2}\left(\frac{\pi}{12}\right)^2 + \frac{2}{3}\left(\frac{\pi}{12}\right)^3$	**M1**	Substitutes $x = \frac{\pi}{3}$, or $x - \frac{\pi}{4} = \frac{\pi}{12}$, into their four-term series from (a).
	$A = 288, \quad B = 2592$	**A1**	Both values.
		[2]	
	Total for Question 12	**9**	

Question	Scheme	Marks	Notes
13(a)	$\sin x = \sin\left(t + \frac{\pi}{4}\right)$ $= \frac{\sqrt{2}}{2}(\sin t + \cos t)$	**M1**	Uses the addition formula to write $\sin x$ in terms of $\sin t$ and $\cos t$. Condone sign slips.
	$\approx \frac{\sqrt{2}}{2}\left[\left(t - \frac{t^3}{6}\right) + \left(1 - \frac{t^2}{2}\right)\right]$	**dM1**	Substitutes series for $\sin t$ and $\cos t$, each with at least two terms, into their expression. Condone factorial slips.
	$= \frac{\sqrt{2}}{2}\left[1 + t - \frac{1}{2}t^2 - \frac{1}{6}t^3\right]$ $t = x - \frac{\pi}{4} \Rightarrow$ printed series	**A1 AG**	Correct substituted series leading without error to the printed result in terms of $x - \frac{\pi}{4}$; collected steps accepted.
		[3]	
13(b)	$2\sqrt{2}\sin x = 2 + 2t - t^2 - \frac{1}{3}t^3 + \cdots$ $\cos 2x = -\sin 2t = -2t + \frac{4}{3}t^3 + \cdots$	**M1**	Uses part (a) for $2\sqrt{2}\sin x$ and any valid series for $\cos 2x$ in powers of t, up to t^3. Condone sign and coefficient slips.
	Numerator $= (k-1)t^2 + t^3 + \cdots$	**A1**	Or equivalent in $x - \frac{\pi}{4}$.
	$\frac{k-1}{t} + 1 + \cdots \Rightarrow k - 1 = 0$ $k = 1$	**dM1 A1**	**dM1** Divides their numerator by t^3 and sets the coefficient of their $\frac{1}{t}$ term to zero.
	Limit $= 1$	**A1**	
		[5]	
	Total for Question 13	**8**	

Question	Scheme	Marks	Notes
14(a)	$\ln y = x\ln x \Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + 1$	**M1**	Takes logs and differentiates implicitly, with $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$ and the product rule on $x\ln x$, or writes $y = e^{x\ln x}$ and uses the chain rule. Condone sign and coefficient slips.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = x^x(1+\ln x)$	**A1**	
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^x(1+\ln x)(1+\ln x) + x^x \times \frac{1}{x}$	**M1**	Applies the product rule to their $x^x(1+\ln x)$, using their first derivative for $\frac{\mathrm{d}}{\mathrm{d}x}x^x$, or differentiates $\frac{\mathrm{d}y}{\mathrm{d}x} = y(1+\ln x)$ implicitly. Condone sign slips.
	$= x^x\left[(1+\ln x)^2 + \frac{1}{x}\right]$	**A1 AG**	Correct product-rule expression leading without error to the printed result; factorising in one step accepted.
		[4]	
14(b)	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = x^x(1+\ln x)\left[(1+\ln x)^2 + \frac{1}{x}\right]$ $+ x^x\left[\frac{2(1+\ln x)}{x} - \frac{1}{x^2}\right]$	**M1 A1**	**M1** Applies the product rule to the printed second derivative, with $(1+\ln x)^2$ differentiated by the chain rule. Condone sign and coefficient slips. **A1** Correct expression, oe; need not be simplified.
	$x=1:\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 1\times 2 + 1\times 1 = 3$	**A1**	cso.
		[3]	
14(c)	$f(1)=1,\ f'(1)=1,\ f''(1)=2,\ f'''(1) =$ "3" $x^x \approx 1 + (x-1)\times 1 + \frac{(x-1)^2}{2!}\times 2$ $+ \frac{(x-1)^3}{3!}\times$ "3"	**M1 A1ft**	**M1** Substitutes their values of $f(1)$ to $f'''(1)$, or $y(1)$ to $y'''(1)$, into the series up to the $(x-1)^3$ term, with the factorials. **A1ft** $f(1)=1$, $f'(1)=1$, $f''(1)=2$ and their $f'''(1)$ from (b), soi.
	$x^x \approx 1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3$	**A1**	Coefficients in simplest form; $\frac{3}{6}$ not accepted.
		[3]	
14(d)	$x^2 - x + 1 = 1 + (x-1) + (x-1)^2$ $x^x - x^2 + x - 1 =$ "$1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3$" $- \left[1 + (x-1) + (x-1)^2\right]$	**M1**	Writes $x^2 - x + 1$ in powers of $(x-1)$, by any method, and subtracts it from their series in (c). Condone slips in coefficients.
	$= \frac{1}{2}(x-1)^3 + \cdots$	**A1**	Numerator $\frac{1}{2}(x-1)^3$, oe; condone higher-order terms omitted.

Question 14 continued

Question	Scheme	Marks	Notes
	$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$ $\frac{x^x - x^2 + x - 1}{(x-1)^2 \ln x} = \frac{\frac{1}{2} + \cdots}{1 - \frac{1}{2}(x-1) + \cdots}$	**M1**	Uses the $\ln(1+t)$ series with $t = x-1$, at least its first term, and divides numerator and denominator by $(x-1)^3$, or equivalent.
	$\lim_{x \to 1} \frac{x^x - x^2 + x - 1}{(x-1)^2 \ln x} = \frac{1}{2}$	**A1**	cso. Needs a statement that the terms in positive powers of $(x-1)$ tend to 0 as $x \to 1$.
		[4]	
	Total for Question 14	**14**	

Question	Scheme	Marks	Notes
15(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x - \cos x}{\cos x - \sin x}$	**M1**	Differentiates by the chain rule to the form $\frac{\pm\sin x \pm \cos x}{\cos x - \sin x}$. Condone sign slips.
	$= -\frac{(\cos x + \sin x)^2}{(\cos x - \sin x)(\cos x + \sin x)}$ $= -\frac{1+\sin 2x}{\cos 2x}$	**dM1**	Uses the double angle formulae, or any valid identities, to express their derivative in terms of $\sin 2x$ and $\cos 2x$. Condone sign slips.
	$= -\sec 2x - \tan 2x$	**A1 AG**	Correct derivative leading without error to the printed result; combined steps accepted.
		[3]	
15(b)	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\sec 2x \tan 2x - 2\sec^2 2x$	**M1**	Differentiates $\sec 2x$ and $\tan 2x$ to the forms $p\sec 2x \tan 2x$ and $q\sec^2 2x$. Condone sign and coefficient slips.
	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -4\sec^3 2x - 8\sec^2 2x \tan 2x$ $\quad - 4\sec 2x \tan^2 2x$ $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = \ldots$	**dM1**	Applies product and chain rules to every term of their second, then third, derivative to reach a fourth derivative. Condone coefficient slips.
	$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -16\sec^4 2x - 40\sec^3 2x \tan 2x$ $\quad - 32\sec^2 2x \tan^2 2x - 8\sec 2x \tan^3 2x$	**A1**	oe in $\sec 2x$ and $\tan 2x$ only, such as after using $\sec^2 2x = 1 + \tan^2 2x$.
		[3]	
15(c)	$y \approx 0 + x(-1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(-4)$ $\quad + \frac{x^4}{4!}(-16)$	**M1 A1**	**M1** Evaluates their derivatives from (a) and (b) at $x = 0$ and substitutes into Maclaurin's series as far as x^4, with factorials. **A1** Values $0, -1, -2, -4, -16$, seen or used.
	$\ln(\cos x - \sin x) \approx -x - x^2 - \frac{2}{3}x^3 - \frac{2}{3}x^4$	**A1**	cao, with coefficients simplified.
		[3]	
15(d)	$\ln(1+kx) \approx kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3}$	**B1**	oe, such as $-\frac{(kx)^2}{2}$.
		[1]	
15(e)	$\ln(1+kx) + \ln(\cos x - \sin x)$ $= (k-1)x - \left(\frac{k^2}{2} + 1\right)x^2 + \cdots$	**M1**	Uses $\ln(ab) = \ln a + \ln b$ and adds their series from (c) and (d), at least to x^2. Condone sign slips.
	Requires the term in $\frac{1}{x}$ to vanish: $\frac{k-1}{x} - \left(\frac{k^2}{2} + 1\right) + \cdots \Rightarrow k - 1 = 0$	**dM1**	Divides their sum by x^2 and sets the coefficient of $\frac{1}{x}$ to zero.

Question 15 continued

Question	Scheme	Marks	Notes
	$k = 1$	**A1**	
	$\lim_{x\to 0}\left(\frac{1}{x^2}\ln\left((1+x)(\cos x - \sin x)\right)\right)$ $= -\left(\frac{1}{2}+1\right) = -\frac{3}{2}$	**A1**	oe, such as -1.5.
		[4]	
	Total for Question 15	**14**	