OPTIONAL PURE / FURTHER CALCULUS

Surface Area of Revolution

Mark scheme 13 questions · 131 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

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	Total	**131**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{x}{2}\right)$ $S = 2\pi \int 2\cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)\mathrm{d}x$	**M1**	Uses $2\pi \int y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ with their $\frac{\mathrm{d}y}{\mathrm{d}x} = k\sinh\left(\frac{x}{2}\right)$ substituted, oe in exponentials. Condone a slip in k.
	$S = 2\pi \int_0^{2\ln 2} 2\cosh^2\left(\frac{x}{2}\right)\mathrm{d}x$	**A1**	Correct integral, $2\cosh^2\left(\frac{x}{2}\right)$ oe in exponentials, with the root simplified and limits 0 and $2\ln 2$.
	$S = 2\pi \int_0^{2\ln 2} (1+\cosh x)\,\mathrm{d}x$ $= 2\pi\left[x+\sinh x\right]_0^{2\ln 2}$	**M1 A1**	**M1** Uses a double-angle identity, or exponentials, and integrates to the form $px + q\sinh x$ or its exponential equivalent. Condone sign and coefficient slips. **A1** Correct integration, oe in exponentials.
	$S = 2\pi\left(2\ln 2 + \sinh(2\ln 2) - 0\right)$	**dM1**	Substitutes both limits into their integral and subtracts. Need not be simplified.
	$\sinh(2\ln 2) = \frac{1}{2}\left(4-\frac{1}{4}\right) = \frac{15}{8}$	**B1**	$\sinh(2\ln 2) = \frac{15}{8}$, oe such as 1.875, seen or used; working in exponentials need not be shown.
	$S = \pi\left(\frac{15}{4} + 4\ln 2\right)$	**A1**	cso. In the form $\pi(a + b\ln 2)$, oe such as $\pi(3.75 + 4\ln 2)$; a and b stated alone do not score.
	Total for Question 1	**7**	

Question	Scheme	Marks	Notes
2(a)	$A = 2\pi \int_0^{\sqrt{3}} 2t\sqrt{2^2 + (2t)^2}\,\mathrm{d}t$ $= 8\pi \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,\mathrm{d}t$	**M1 A1**	**M1** Uses $2\pi \int x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ with $x = 2t$ and both their derivatives, or a Cartesian equivalent. Not for y in place of x. **A1** Correct integral in any form, with limits 0 and $\sqrt{3}$ or their Cartesian equivalents.
	$A = 8\pi\left[\frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{3}}$	**M1 A1**	**M1** Integrates to the form $k\left(1+t^2\right)^{\frac{3}{2}}$, by inspection or substitution, or its Cartesian equivalent. Condone a wrong k. **A1** Correct integral, oe.
	$A = \frac{8\pi}{3}\left(4^{\frac{3}{2}} - 1\right)$ $= \frac{8\pi}{3}(8-1) = \frac{56\pi}{3}$	**dM1** **A1 AG**	**dM1** Substitutes both limits into their $k\left(1+t^2\right)^{\frac{3}{2}}$, or its equivalent after a substitution, and subtracts. **A1 AG** cso. Correct substituted limits leading without error to $\frac{56\pi}{3}$; combined steps accepted.
		[6]	
2(b)	$0.98 \times 58 = \ldots,\ 1.02 \times 58 = \ldots$ $0.995 \times \frac{56\pi}{3} = \ldots,\ 1.005 \times \frac{56\pi}{3} = \ldots$	**M1**	Finds bounds from $58(1 \pm 0.02)$ and $\frac{56\pi}{3}(1 \pm 0.005)$, or equivalent percentages, and compares them. Condone arithmetic slips.
	$56.84 \leq$ area ≤ 59.16 $58.35 \leq$ area ≤ 58.94 Inside, so all reflectors are suitable	**A1**	Both ranges correct, awrt 3 s.f., or an equivalent correct percentage comparison, with the conclusion that all reflectors are suitable.
		[2]	
	Total for Question 2	**8**	

Question	Scheme	Marks	Notes
3(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 6t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 3 - 3t^2$ $(6t)^2 + \left(3 - 3t^2\right)^2 = 36t^2 + 9 - 18t^2 + 9t^4$	**M1**	Differentiates both, at least one power decreased by 1, then squares and adds their derivatives. Condone slips in expanding.
	$= 9t^4 + 18t^2 + 9 = 9\left(1 + t^2\right)^2$	**A1 AG**	Correct derivatives and expansion leading without error to the printed result; collected steps accepted.
		[2]	
3(b)	$3t - t^3 = 0 \Rightarrow t = 0, \ \sqrt{3}$ $S = 2\pi \int_0^{\sqrt{3}} \left(3t - t^3\right) \times 3\left(1 + t^2\right) \mathrm{d}t$	**B1**	Correct integral in t with limits 0 and $\sqrt{3}$; the root may be left unsimplified. Needs the 2π.
	$6\pi \int_0^{\sqrt{3}} \left(3t + 2t^3 - t^5\right) \mathrm{d}t$ $= 6\pi \left[\frac{3t^2}{2} + \frac{t^4}{2} - \frac{t^6}{6}\right]_0^{\sqrt{3}}$	**M1 A1**	**M1** Expands their integrand to a polynomial in t and integrates, at least one power increased by 1. Condone slips in expanding. **A1** Correct integral, oe; the factor 6π need not appear here.
	$6\pi\left(\frac{9}{2} + \frac{9}{2} - \frac{27}{6}\right)$	**dM1**	Substitutes their limits from $y = 0$, the non-zero root and 0, into their integrated polynomial and subtracts.
	$= 27\pi$	**A1**	
		[5]	
	Total for Question 3	**7**	

Question	Scheme	Marks	Notes
4	$S = 2\pi \int_0^{\ln 2} x\sqrt{1+\sinh^2 x}\,\mathrm{d}x$ $= 2\pi \int_0^{\ln 2} x\cosh x\,\mathrm{d}x$	**M1 A1**	**M1** Uses $2\pi\int x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ with their derivative of $\cosh x$. Not for y before the root. **A1** $2\pi\int x\cosh x\,\mathrm{d}x$, the root simplified by $\cosh^2 x - \sinh^2 x = 1$; the limits 0 and $\ln 2$ may first appear when applied.
	$\int x\cosh x\,\mathrm{d}x = x\sinh x - \int \sinh x\,\mathrm{d}x$	**M1**	Integrates by parts, reaching the form $\alpha x\sinh x - \beta\int \sinh x\,\mathrm{d}x$. Condone sign slips.
	$= x\sinh x - \cosh x$	**A1**	
	$\sinh(\ln 2) = \frac{3}{4}, \quad \cosh(\ln 2) = \frac{5}{4}$	**B1**	Both values, soi.
	$S = 2\pi\left[\left(\text{“}\frac{3}{4}\text{”}\ln 2 - \text{“}\frac{5}{4}\text{”}\right) - (0-1)\right]$ $= \ldots$	**dM1**	Substitutes both limits into their integral of the form $\alpha x\sinh x + \beta\cosh x$ and subtracts. Condone a missing 2π.
	$S = \frac{\pi}{2}(3\ln 2 - 1)$	**A1**	cso. In the form asked for, or $p=3$, $q=-1$ stated; $2\pi\left(\frac{3}{4}\ln 2 - \frac{1}{4}\right)$ alone scores A0.
	Total for Question 4	**7**	

Question	Scheme	Marks	Notes
5(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 12t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 12 - 3t^2$	**B1**	
	$(12t)^2 + \left(12 - 3t^2\right)^2 = \ldots$	**M1**	Squares and adds their derivatives. Condone slips in expanding.
	$9t^4 + 72t^2 + 144 = 9\left(4 + t^2\right)^2$ $\Rightarrow \sqrt{\ldots} = 3\left(4 + t^2\right)$	**A1**	Correct root as $3\left(4 + t^2\right)$ oe, a polynomial in t.
	$S = 2\pi \int_0^1 \left(6 + 6t^2\right) \times 3\left(4 + t^2\right) \mathrm{d}t$	**M1**	Uses $2\pi \int x$ times their root, with $x = 6 + 6t^2$ and limits 0 and 1. Not for y in place of x.
	$= 36\pi \int_0^1 \left(1 + t^2\right)\left(4 + t^2\right) \mathrm{d}t$ $= 36\pi \int_0^1 \left(4 + 5t^2 + t^4\right) \mathrm{d}t$	**A1 AG**	Correct substituted integral leading without error to the printed result; combined steps accepted.
		[5]	
5(b)	$36\pi \left[4t + \frac{5t^3}{3} + \frac{t^5}{5}\right]_0^1 = \ldots$	**M1**	Integrates, the power of at least one term increased by 1, and substitutes the limits 0 and 1.
	$= \frac{1056\pi}{5}$	**A1**	
	$\pi \times 6^2 = 36\pi$	**B1**	
	$\frac{1056\pi}{5} + 36\pi = \frac{1236\pi}{5}$	**A1**	Exact multiple of π, oe such as 247.2π.
		[4]	
5(c)	$2 \times$ "$\frac{1056\pi}{5}$" $+ 36\pi = \ldots$ $\left[= \frac{2292\pi}{5} \approx 1440.1\right]$	**M1**	Finds the area glazed per bowl, doubling their curved area and adding the base area once, or adding their curved area to their total from (b).
	$\frac{20\,000}{\text{"}1440.1\text{"}} = \ldots \ [\approx 13.89]$	**M1**	Divides the area one pot covers by their area per bowl, both in cm^2 or both in m^2.
	13	**A1ft**	ft their area from (b), their quotient rounded down to a whole number.
		[3]	

Question 5 continued

Question	Scheme	Marks	Notes
5(d)	Glaze is wasted: spilt, left in the pot or absorbed by the clay.	**E1**	States a reason less glaze reaches the bowls or each needs more, such as waste, absorption, thicker coats or a larger true area. Not for "the model is inaccurate" alone.
		[1]	
	Total for Question 5	**13**	

Question	Scheme	Marks	Notes
6(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$ $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2$	**M1**	Differentiates, at least one power decreased by 1, squares and adds 1, and attempts a perfect square. Condone coefficient slips.
	$A = 2\pi\int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2}\right)$ $\times \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)\mathrm{d}x$	**A1**	Correct integrand with limits 1 and 2; the product need not be expanded.
	$A = 2\pi\int_1^2 \left(\frac{1}{16}x^7 + \frac{3}{16}x + \frac{1}{8}x^{-5}\right)\mathrm{d}x$ $= 2\pi\left[\frac{x^8}{128} + \frac{3x^2}{32} - \frac{1}{32x^4}\right]_1^2$	**M1 A1**	**M1** Expands their product and integrates, at least one power increased by 1. **A1** Correct integral, need not be simplified.
	$A = 2\pi\left(2 + \frac{3}{8} - \frac{1}{512}\right)$ $- 2\pi\left(\frac{1}{128} + \frac{3}{32} - \frac{1}{32}\right)$	**dM1**	Substitutes 2 and 1 into their integral and subtracts.
	$A = \frac{1179\pi}{256}$	**A1**	Exact, oe such as $\frac{1179}{256}\pi$. A decimal alone scores A0.
		[6]	
6(b)	$y = \frac{3}{8} + \frac{27}{16}(x-1)$	**B1**	Any correct equation of PQ, or the gradient $\frac{27}{16}$ or the length $PQ = \frac{\sqrt{985}}{16}$ seen.
	$2\pi\int_1^2 \left(\frac{3}{8} + \frac{27}{16}(x-1)\right)\sqrt{1 + \left(\text{“}\frac{27}{16}\text{”}\right)^2}\,\mathrm{d}x$	**M1**	Uses the surface-area integral with their linear y and constant gradient, or the frustum formula $\pi l(r_1 + r_2)$ with radii $\frac{3}{8}$ and $\frac{33}{16}$.
	$= \frac{\pi\sqrt{985}}{8}\left[\frac{3x}{8} + \frac{27}{32}(x-1)^2\right]_1^2$ $= \ldots$	**dM1**	Integrates and substitutes 1 and 2, or evaluates their frustum expression with their slant length.
	$\frac{39}{256}\pi\sqrt{985}$	**A1**	Any correct answer in the form $k\pi\sqrt{n}$. A decimal alone, awrt 15.0, scores A0.
		[4]	
6(c)	$M = \frac{1}{2-1}\int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2}\right)\mathrm{d}x$ $= \left[\frac{x^5}{40} - \frac{1}{4x}\right]_1^2 = \ldots$	**M1**	Integrates f, at least one power increased by 1, and substitutes 2 and 1. Division by the interval length 1 may be implicit.
	$M = \frac{9}{10}$	**A1**	
	$2\pi \times \text{“}\frac{9}{10}\text{”} \times 1 = \frac{9\pi}{5}$ $\frac{\left\lvert\frac{9\pi}{5} - \text{“}\frac{1179\pi}{256}\text{”}\right\rvert}{\text{“}\frac{1179\pi}{256}\text{”}} \times 100 = \ldots$	**M1**	Finds $2\pi M$ with their M, then the difference from their part (a) area as a percentage of that area. Not for dividing by the approximation.

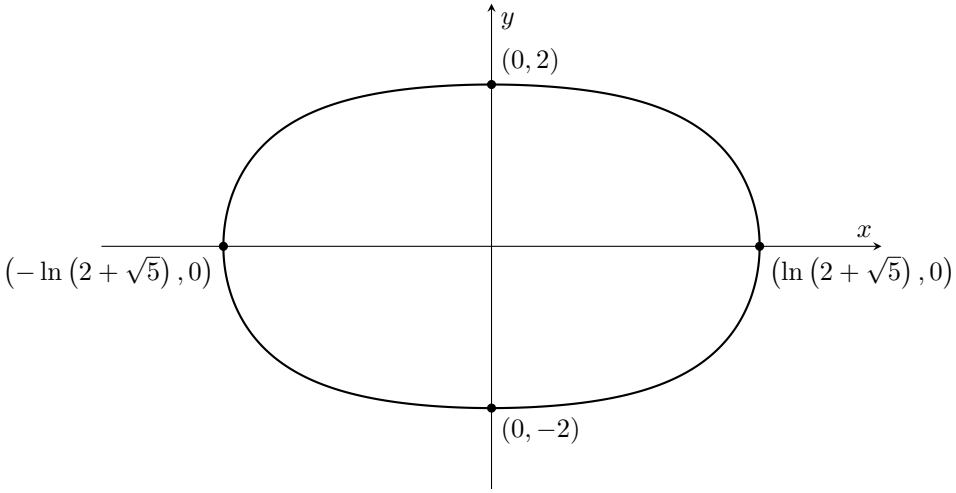
Question 6 continued

Question	Scheme	Marks	Notes
	60.9%	**A1**	awrt 60.9, or −60.9.
		[4]	
	Total for Question 6	**14**	

Question	Scheme	Marks	Notes
7(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ $S = 2\pi \int_0^1 x^2 \sqrt{1+(2x)^2}\,\mathrm{d}x$	**M1**	Uses the surface area formula with $y = x^2$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ of the form kx in the root.
	$S = 2\pi \int_0^1 x^2 \sqrt{1+4x^2}\,\mathrm{d}x$	**A1 AG**	Correct substituted formula leading without error to the printed integral, limits included.
		[2]	
7(b)	$x = \frac{1}{2}\sinh\theta,\ \mathrm{d}x = \frac{1}{2}\cosh\theta\,\mathrm{d}\theta$ $\sqrt{1+4x^2} = \sqrt{1+\sinh^2\theta} = \cosh\theta$ $S = 2\pi \int \frac{1}{4}\sinh^2\theta\cosh\theta \times \frac{1}{2}\cosh\theta\,\mathrm{d}\theta$	**M1**	Substitutes for x^2, $\sqrt{1+4x^2}$ and $\mathrm{d}x$, with $\mathrm{d}x$ of the form $k\cosh\theta\,\mathrm{d}\theta$. Condone coefficient slips.
	$S = \frac{\pi}{4}\int \sinh^2\theta\cosh^2\theta\,\mathrm{d}\theta$	**A1**	Correct integral, oe. Limits need not be seen.
	$\sinh^2\theta\cosh^2\theta = \frac{1}{4}\sinh^2 2\theta = \frac{1}{8}(\cosh 4\theta - 1)$ $\int(\cosh 4\theta - 1)\,\mathrm{d}\theta = \ldots$	**M1**	Uses double-angle identities or exponentials to write the integrand as $p\cosh 4\theta + q$ oe, and integrates. Condone sign and coefficient slips.
	$S = \frac{\pi}{32}\left(\frac{1}{4}\sinh 4\theta - \theta\right)$	**A1**	oe, including an exponential form.
	$\theta = \operatorname{arsinh} 2 = \ln\left(2+\sqrt{5}\right)$	**B1**	Upper limit in logarithmic form, seen or used.
	$\sinh\theta = 2,\ \cosh\theta = \sqrt{5}$ $\Rightarrow \sinh 4\theta = 2\sinh 2\theta\cosh 2\theta = 72\sqrt{5}$	**B1**	$\sinh 4\theta = 72\sqrt{5}$ at the upper limit, or $e^{\pm 4\theta} = 161 \pm 72\sqrt{5}$ in an exponential form, by any method; seen or used.
	$S = \frac{\pi}{32}\left[\frac{1}{4} \times \text{“}72\sqrt{5}\text{”} - \ln\left(2+\sqrt{5}\right) - 0\right]$	**dM1**	Substitutes their upper limit into their integral of the form $r\sinh 4\theta + s\theta$ and subtracts its value at 0, which may be implied.
	$S = \frac{\pi}{32}\left(18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right)$ $a = 18,\ b = 2$	**A1**	cso. In the printed form; a and b need not be stated separately.
		[8]	
	Total for Question 7	**10**	

Question	Scheme	Marks	Notes
8(a)	$\int t\ln t\,\mathrm{d}t = \frac{t^2}{2}\ln t - \int \frac{t^2}{2}\times\frac{1}{t}\,\mathrm{d}t$	**M1**	Integrates by parts with $u=\ln t$ to the form $\alpha t^2\ln t - \int \beta t\,\mathrm{d}t$ for the $t\ln t$ term. Condone coefficient slips.
	$= \frac{t^2}{2}\ln t - \frac{t^2}{4}$	**A1**	
	$\int \frac{\ln t}{t}\,\mathrm{d}t = \frac{1}{2}(\ln t)^2$	**B1**	oe, or seen only as the value $\frac{1}{2}(\ln 2)^2$.
	$\left[\text{“}\frac{t^2}{2}\ln t - \frac{t^2}{4}\text{”}\right]_1^2$ $+\left[\text{“}\frac{1}{2}(\ln t)^2\text{”}\right]_1^2 = \ldots$	**dM1**	Applies the limits 2 and 1 to both their integrals and adds the two results.
	$-\frac{3}{4}+2\ln 2+\frac{1}{2}(\ln 2)^2$	**A1**	Correct expression, or $a=-\frac{3}{4}$, $b=2$, $c=\frac{1}{2}$ stated.
		[5]	
8(b)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t-\frac{2}{t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=4$ $\left(2t-\frac{2}{t}\right)^2+4^2 = 4t^2+8+\frac{4}{t^2}$	**M1**	Differentiates both, squares and adds, and attempts the sum as a perfect square. Condone coefficient slips.
	$\sqrt{4t^2+8+\frac{4}{t^2}} = 2\left(t+\frac{1}{t}\right)$	**A1**	oe, such as $2t+\frac{2}{t}$.
	$S = 2\pi\int_1^2 \left(t^2-2\ln t\right)\times\text{“}2\left(t+\frac{1}{t}\right)\text{”}\mathrm{d}t$ $= \ldots - \ldots\int_1^2\left(t+\frac{1}{t}\right)\ln t\,\mathrm{d}t$	**M1**	Uses $2\pi\int x$ times their arc-length factor, limits 1 and 2, and expands to separate an integral of $\left(t+\frac{1}{t}\right)\ln t$. Not for y in place of x.
	$S = 4\pi\int_1^2\left(t^3+t\right)\mathrm{d}t$ $-8\pi\int_1^2\left(t+\frac{1}{t}\right)\ln t\,\mathrm{d}t$	**A1**	oe
	$\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_1^2 = \frac{21}{4}$ $S = 4\pi\times\text{“}\frac{21}{4}\text{”}$ $-8\pi\left(\text{“}-\frac{3}{4}+2\ln 2+\frac{1}{2}(\ln 2)^2\text{”}\right)$	**dM1**	Integrates t^3+t, raising at least one power by 1, applies both limits, and substitutes their part (a) result of the form $a+b\ln 2+c(\ln 2)^2$.
	$S = \pi\left(27-16\ln 2-4(\ln 2)^2\right)$	**A1**	Correct expression in the given form, or $p=27$, $q=-16$, $r=-4$ stated.
		[6]	
	Total for Question 8	**11**	

Question	Scheme	Marks	Notes
9	$\frac{\mathrm{d}x}{\mathrm{d}t} = -3\sin t + 3\sin 3t$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 3\cos t - 3\cos 3t$ $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \ldots$	**M1**	Differentiates both, $\sin 3t$ and $\cos 3t$ to multiples of $\cos 3t$ and $\sin 3t$, then squares and adds. Condone sign and coefficient slips.
	$= 18 - 18\cos 2t = 36\sin^2 t$	**A1**	$36\sin^2 t$ oe, or its root $6\sin t$, cso.
	$y = 3\sin t - \left(3\sin t - 4\sin^3 t\right) = 4\sin^3 t$ $S = 2\pi\int_0^{\pi} 4\sin^3 t \times 6\sin t\,\mathrm{d}t$	**M1**	Uses $2\pi\int y\sqrt{\ldots}\,\mathrm{d}t$ with their y in t and their root. Simplifying y is not required.
	$= 48\pi\int_0^{\pi} \sin^4 t\,\mathrm{d}t$	**A1**	Correct integral with limits 0 and π, oe such as $12\pi\int_0^{\pi}(3\sin t - \sin 3t)\sin t\,\mathrm{d}t$.
	$\sin^4 t = \frac{1}{4}(1-\cos 2t)^2$ $= \frac{1}{4}\left(1 - 2\cos 2t + \cos^2 2t\right) = \ldots$	**M1**	Uses double-angle or product-to-sum identities to write their integrand in multiple angles. Condone sign and coefficient slips.
	$\sin^4 t = \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t$	**A1**	Correct, or a correct equivalent integrand in multiple angles.
	$\left[\frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t\right]_0^{\pi}$ $= \ldots \left[= \frac{3\pi}{8}\right]$	**dM1**	Integrates, at least one term correctly, and substitutes π and 0. Condone coefficient slips.
	$S = 48\pi \times \frac{3\pi}{8} = 18\pi^2$, so $k = 18$	**A1**	$18\pi^2$ or $k = 18$, cso.
	Total for Question 9	**8**	

Question	Scheme	Marks	Notes
10(a)			
	Shape	**B1**	Closed curve symmetric in both axes, through $(0, \pm 2)$ and $(\pm \ln(2+\sqrt{5}), 0)$, or $(\pm \operatorname{arsinh} 2, 0)$ or awrt $(\pm 1.44, 0)$.
		[1]	
10(b)	$y = \left(4 - \sinh^2 x\right)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \ldots$	**M1**	Differentiates by the chain rule, or implicitly from $y^2 + \sinh^2 x = 4$, reaching a multiple of $\sinh x \cosh x$. Condone sign and coefficient slips.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sinh x \cosh x}{(4-\sinh^2 x)^{\frac{1}{2}}}$	**A1**	Correct, oe such as $-\frac{\sinh x \cosh x}{y}$.
	$y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ $= \sqrt{4 - \sinh^2 x + \sinh^2 x \cosh^2 x}$ $= \sqrt{4 - \sinh^2 x + \sinh^2 x \left(1 + \sinh^2 x\right)}$	**M1**	Substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into $y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ or its square, combines the terms and uses $\cosh^2 x = 1 + \sinh^2 x$. Condone sign slips.
	$= \sqrt{\sinh^4 x + 4}$	**A1**	cso.
	$A = 2\pi \int_{-\ln(2+\sqrt{5})}^{\ln(2+\sqrt{5})} \sqrt{\sinh^4 x + 4}\,\mathrm{d}x$ $= 2 \times 2\pi \int_{0}^{\ln(2+\sqrt{5})} \sqrt{\sinh^4 x + 4}\,\mathrm{d}x$	**M1**	Uses $2\pi \int y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ with their integrand between the x-intercepts, and doubles the integral from 0 by symmetry.
	$A = 4\pi \int_{0}^{\ln(2+\sqrt{5})} \sqrt{\sinh^4 x + 4}\,\mathrm{d}x,\ k = 4$	**A1**	cso, with $k = 4$ stated or the printed form with 4π.
		[6]	
10(c)	$A =$ "4"$\pi \times 3.510\ldots = \ldots$	**M1**	Evaluates the integral by calculator and multiplies by their $k\pi$.

Question 10 continued

Question	Scheme	Marks	Notes
	$A \approx 44.1 < 45$, so the requirement is met	**A1ft**	Awrt 44.1, compared with 45, and concludes the requirement is met. ft their k, as awrt $11.0k$ with the conclusion that follows.
		[2]	
	Total for Question 10	**9**	

Question	Scheme	Marks	Notes
11(a)	$W = 2r\sin\theta$ $W^2 = 128\left(1 - 2\sin^2\theta\right)\sin^2\theta$	**M1**	Forms the width $2r\sin\theta$, its half or its square in terms of θ alone, using $r^2 = 32\cos 2\theta$. Condone coefficient slips.
	$W^2 = 16 - 256\left(\sin^2\theta - \frac{1}{4}\right)^2$ $\Rightarrow \sin\theta = \ldots$	**dM1**	Maximises by completing the square, or by differentiating and setting the derivative to zero, reaching a value of $\sin\theta$, $\cos 2\theta$ or θ.
	$\sin\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{6}$ $CD = 4$	**A1**	Both $\theta = \pm\frac{\pi}{6}$ from correct working, and $CD = 4$.
		[3]	
11(b)	$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -64\sin 2\theta$ $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = r^2 + \frac{1024\sin^2 2\theta}{r^2}$	**M1**	Differentiates r or r^2 and substitutes into $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$, oe $\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2$. Condone sign and coefficient slips.
	$= \frac{1024}{r^2} \Rightarrow \sqrt{\ldots} = \frac{32}{r}$	**A1**	oe, such as $\sqrt{\frac{32}{\cos 2\theta}}$.
	$S = 2\pi\int_0^{\frac{\pi}{6}} r\sin\theta \times \text{“}\frac{32}{r}\text{”}\,\mathrm{d}\theta$	**M1**	Uses $2\pi\int y$ times their arc-length factor, with $y = r\sin\theta$ and limits 0 and $\frac{\pi}{6}$, or $\pi\int\lvert r\sin\theta\rvert\ldots\,\mathrm{d}\theta$ from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$.
	$= 64\pi\int_0^{\frac{\pi}{6}} \sin\theta\,\mathrm{d}\theta$	**A1**	Correct simplified integral with correct limits, oe.
	$= 64\pi\left[-\cos\theta\right]_0^{\frac{\pi}{6}} = \ldots$ $\left[= 64\pi - 32\sqrt{3}\,\pi\right]$	**dM1**	Integrates $\sin\theta$ to $\pm\cos\theta$ and substitutes both limits.
	Base: $\pi \times 2^2 = 4\pi$	**B1ft**	$\pi\left(\frac{1}{2}\text{ their }CD\right)^2$, ft their CD from (a).
	$\pi\left(68 - 32\sqrt{3}\right)$	**A1**	cao, in the given form, or $p = 68$, $q = 32$ stated.
		[7]	
Total for Question 11		**10**	

Question	Scheme	Marks	Notes
12(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 4\sinh t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 3 - 2\cosh 2t$	**M1 A1**	**M1** Differentiates both, reaching the forms $a\sinh t$ and $b + c\cosh 2t$. Condone sign and coefficient slips.
	$16\sinh^2 t + (3 - 2\cosh 2t)^2$ $= 8\cosh 2t - 8 + 9 - 12\cosh 2t + 4\cosh^2 2t$	**M1**	Squares and adds their derivatives, using a double-angle identity to write the sum, or both sides, in one hyperbolic function. Condone sign slips.
	$= 4\cosh^2 2t - 4\cosh 2t + 1 = (2\cosh 2t - 1)^2$	**A1 AG**	Correct derivatives and identity leading without error to the printed result. Both sides reduced to a common expression, with a conclusion, accepted.
		[4]	
12(b)	$A = 2\pi\int_0^{\ln 2}(3t - \sinh 2t)(2\cosh 2t - 1)\,\mathrm{d}t$	**B1**	Correct integral with limits 0 and $\ln 2$, the root replaced using part (a). Needs the factor 2π.
	$\int 6t\cosh 2t\,\mathrm{d}t = 3t\sinh 2t - \int 3\sinh 2t\,\mathrm{d}t$	**M1**	Integrates the $t\cosh 2t$ term by parts to the form $at\sinh 2t - \int b\sinh 2t\,\mathrm{d}t$. Condone sign and coefficient slips.
	$= 3t\sinh 2t - \frac{3}{2}\cosh 2t$	**A1**	
	$\int(-3t - 2\sinh 2t\cosh 2t + \sinh 2t)\,\mathrm{d}t$ $= -\frac{3}{2}t^2 - \frac{1}{2}\sinh^2 2t + \frac{1}{2}\cosh 2t$	**M1 A1**	**M1** Integrates the other three terms, with $\sinh 2t\cosh 2t$ to $k\sinh^2 2t$, $k\cosh^2 2t$ or $k\cosh 4t$. Condone coefficient slips. **A1** All three terms correct, oe such as $-\frac{1}{4}\cosh 4t$ for $-\frac{1}{2}\sinh^2 2t$.
	$\sinh(2\ln 2) = \frac{15}{8}, \quad \cosh(2\ln 2) = \frac{17}{8}$	**B1**	Exact values at $\ln 2$, $\sinh(2\ln 2) = \frac{15}{8}$ and $\cosh(2\ln 2) = \frac{17}{8}$, or $\cosh(4\ln 2) = \frac{257}{32}$ in a $\cosh 4t$ form; soi.
	$A = 2\pi\left[3t\sinh 2t - \cosh 2t - \frac{3}{2}t^2 - \frac{1}{2}\sinh^2 2t\right]_0^{\ln 2}$ $= 2\pi\left(\frac{45}{8}\ln 2 - \frac{17}{8} - \frac{3}{2}(\ln 2)^2 - \frac{225}{128} + 1\right)$	**dM1**	Substitutes $\ln 2$ and 0 into their integrated expression and subtracts.

Question 12 continued

Question	Scheme	Marks	Notes
	$A = \pi\left(-\frac{369}{64} + \frac{45}{4}\ln 2 - 3(\ln 2)^2\right)$	**A1**	Exact, in the form asked for, with equivalent rationals accepted. A decimal value of A scores A0.
		[8]	
	Total for Question 12	**12**	

Question	Scheme	Marks	Notes
13(a)	$I_n = \left[\frac{1}{9}t^{n-1}\left(1+3t^2\right)^{\frac{3}{2}}\right]_0^1$ $-\frac{n-1}{9}\int_0^1 t^{n-2}\left(1+3t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$	**M1 A1**	**M1** Integrates by parts, integrating $t\sqrt{1+3t^2}$ to the form $\alpha\left(1+3t^2\right)^{\frac{3}{2}}$ and differentiating t^{n-1}. Condone coefficient slips. **A1** Correct expression; limits need not be evaluated.
	$[\ldots]_0^1 = \frac{1}{9}\times 4^{\frac{3}{2}} - 0 = \frac{8}{9}$	**B1**	
	$\int_0^1 t^{n-2}\left(1+3t^2\right)\sqrt{1+3t^2}\,\mathrm{d}t = I_{n-2}+3I_n$	**M1**	Writes $\left(1+3t^2\right)^{\frac{3}{2}}$ as $\left(1+3t^2\right)\sqrt{1+3t^2}$ and expresses their integral in terms of I_{n-2} and I_n.
	$I_n = \frac{8}{9} - \frac{n-1}{9}\left(I_{n-2}+3I_n\right)$ $\Rightarrow 3(n+2)I_n = 8-(n-1)I_{n-2}$	**A1 AG**	Correct relation in I_n and I_{n-2} leading without error to the printed result; combined steps accepted.
		[5]	
13(b)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^3, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{3}\,t^2$	**B1**	Both, oe such as $\frac{3t^2}{\sqrt{3}}$.
	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 9t^6+3t^4$	**M1**	Squares and adds their two derivatives.
	$= 3t^4\left(1+3t^2\right)$ $\Rightarrow \sqrt{\ldots} = \sqrt{3}\,t^2\sqrt{1+3t^2}$	**A1**	
	$S = 2\pi\int_0^1 \frac{1}{\sqrt{3}}t^3\times\sqrt{3}\,t^2\sqrt{1+3t^2}\,\mathrm{d}t$	**M1**	Uses $2\pi\int y\sqrt{\ldots}\,\mathrm{d}t$ with $y=\frac{1}{\sqrt{3}}t^3$, their root and limits 0 and 1.
	$= 2\pi\int_0^1 t^5\sqrt{1+3t^2}\,\mathrm{d}t$	**A1 AG**	Correct substituted integral leading without error to the printed result.
		[5]	
13(c)	$I_1 = \left[\frac{1}{9}\left(1+3t^2\right)^{\frac{3}{2}}\right]_0^1$	**M1**	Integrates to the form $\alpha\left(1+3t^2\right)^{\frac{3}{2}}$ and uses both limits. Condone a wrong α.
	$= \frac{8}{9}-\frac{1}{9} = \frac{7}{9}$	**A1**	
	$15I_3 = 8-2\text{“}\frac{7}{9}\text{”} \Rightarrow I_3 = \ldots$ $21I_5 = 8-4I_3 \Rightarrow I_5 = \ldots$	**M1**	Uses the reduction formula at least once, $n=3$ with their I_1 or $n=5$ with their I_3. Condone arithmetic slips.
	$I_3 = \frac{58}{135}$	**A1**	

Question 13 continued

Question	Scheme	Marks	Notes
	$I_5 = \frac{848}{2835} \Rightarrow 2\pi I_5 = \frac{1696}{2835}\pi$	**A1**	cao; any equivalent rational multiple of π.
		[5]	
	Total for Question 13	**15**	