OPTIONAL PURE / FURTHER CALCULUS

Reduction Formulae for Integrals

Mark scheme 14 questions · 152 marks

EXAM BOARDS Edexcel FP2 9FM0/4A · OCR A Y545 · AQA 7367 · CIE 9231

Questions

Question	Marks by part	Marks	Page
Question 1	(a) 3 (b) 4	7	4
Question 2	(a) 4 (b) 4	8	5
Question 3	(a) 5 (b) 4	9	6
Question 4	(a) 5 (b) 4	9	7
Question 5	(a) 3 (b)(i) 4 (b)(ii) 2	9	8
Question 6	(a) 5 (b) 5	10	9
Question 7	(a) 2 (b) 5 (c) 4	11	10
Question 8	(a) 5 (b) 5	10	11
Question 9	(a) 2 (b) 5 (c) 4	11	12
Question 10	(a) 6 (b) 6	12	13
Question 11	(a) 3 (b) 2 (c) 7	12	14
Question 12	(a) 7 (b) 6	13	16
Question 13	(a) 5 (b) 4 (c) 6	15	17
Question 14	(a) 6 (b) 5 (c) 5	16	19
	Total	**152**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$I_n = [(1+2x)^n e^x]_0^1$ $- \int_0^1 2n(1+2x)^{n-1}e^x \, \mathrm{d}x$	**M1 A1**	**M1** Integrates by parts with $u = (1+2x)^n$, reaching $(1+2x)^n e^x$ minus a multiple of $\int (1+2x)^{n-1}e^x \, \mathrm{d}x$. Condone coefficient slips. **A1** Correct, oe. Limits need not appear until evaluated.
	$I_n = \left(3^n e - e^0\right) - 2n\int_0^1 (1+2x)^{n-1}e^x \, \mathrm{d}x$ $= 3^n e - 1 - 2nI_{n-1}$	**A1 AG**	Boundary term evaluated at both limits and the integral written as I_{n-1}, leading without error to the printed result.
		[3]	
1(b)	$I_0 = [e^x]_0^1 = e - 1$	**B1**	
	$I_1 = 3e - 1 - 2\text{“}(e-1)\text{”} = \ldots$ $I_2 = 9e - 1 - 4I_1 = \ldots$ $I_3 = 27e - 1 - 6I_2 = \ldots$	**M1**	Applies the reduction formula at least twice in succession, from their I_0. Condone arithmetic slips.
	$I_1 = e + 1$ or $I_2 = 5e - 5$	**A1**	Either value correct.
	$I_3 = 29 - 3e$	**A1**	Any exact equivalent.
		[4]	
	Total for Question 1	**7**	

Question	Scheme	Marks	Notes
2(a)	$I_n = \left[x\left(8-x^3\right)^n\right]_0^2 + 3n\int_0^2 x^3\left(8-x^3\right)^{n-1}\,\mathrm{d}x$	**M1**	Integrates by parts, reaching a boundary term and an integral of $x^3\left(8-x^3\right)^{n-1}$ or equivalent. Condone sign and coefficient slips.
	$I_n = 3n\int_0^2 x^3\left(8-x^3\right)^{n-1}\,\mathrm{d}x$	**A1**	Correct, with the boundary term shown or stated to be zero at both limits.
	$I_n = 3n\int_0^2 \left(8-\left(8-x^3\right)\right)\left(8-x^3\right)^{n-1}\,\mathrm{d}x$ $= 24nI_{n-1} - 3nI_n$	**M1**	Writes x^3 in terms of $8-x^3$ and splits the integral into multiples of I_{n-1} and I_n. Condone sign and coefficient slips.
	$(3n+1)I_n = 24nI_{n-1} \Rightarrow I_n = \frac{24n}{3n+1}I_{n-1}$	**A1 AG**	Correct equation in I_n and I_{n-1} leading without error to the printed result.
		[4]	
2(b)	$I_0 = \int_0^2 1\,\mathrm{d}x = 2$	**B1**	
	$I_1 = \frac{24}{4} \times$ "2", $\quad I_2 = \frac{48}{7} \times$ "I_1", $\quad \ldots$	**M1**	Applies the result of (a) in succession from their I_0, at least as far as I_2, or forms the equivalent product.
	$I_1 = 12, \quad I_2 = \frac{576}{7}, \quad I_3 = \frac{20\,736}{35}$	**A1**	All three correct, oe.
	$I_4 = \frac{96}{13} \times \frac{20\,736}{35} = \frac{1\,990\,656}{455} \Rightarrow n = 4$	**A1**	cso. Needs I_4 found exactly and matched to the given value.
		[4]	
	Total for Question 2	**8**	

Question	Scheme	Marks	Notes
3(a)	$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{6}}$ $+ (n-1)\int_0^{\frac{\pi}{6}} \sin^{n-2} x \cos^2 x \, \mathrm{d}x$	**M1 A1**	**M1** Integrates $\sin^{n-1} x \sin x$ by parts, reaching the form $p \sin^{n-1} x \cos x + q \int \sin^{n-2} x \cos^2 x \, \mathrm{d}x$. Condone sign and coefficient slips. **A1** Correct expression; limits may be applied later.
	$\left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{6}} = -\left(\frac{1}{2}\right)^{n-1} \frac{\sqrt{3}}{2}$ $= -\frac{\sqrt{3}}{2^n}$	**B1**	oe, such as $-\sqrt{3}\left(\frac{1}{2}\right)^n$.
	$I_n = -\frac{\sqrt{3}}{2^n} + (n-1)\int_0^{\frac{\pi}{6}} \left(\sin^{n-2} x - \sin^n x\right) \mathrm{d}x$ $= -\frac{\sqrt{3}}{2^n} + (n-1)I_{n-2} - (n-1)I_n$	**M1**	Replaces $\cos^2 x$ by $1 - \sin^2 x$ and writes the integral in terms of I_{n-2} and I_n. Condone sign slips.
	$nI_n = (n-1)I_{n-2} - \frac{\sqrt{3}}{2^n}$ $\Rightarrow I_n = \left(\frac{n-1}{n}\right) I_{n-2} - \frac{\sqrt{3}}{n(2^n)}$	**A1 AG**	Correct equation in I_n and I_{n-2} leading without error to the printed result; collecting and dividing by n may be combined.
		[5]	
3(b)	$I_0 = \frac{\pi}{6}$	**B1**	
	$I_2 = \frac{1}{2}I_0 - \frac{\sqrt{3}}{8} = \ldots$ $I_4 = \frac{3}{4}I_2 - \frac{\sqrt{3}}{64} = \ldots$ $I_6 = \frac{5}{6}I_4 - \frac{\sqrt{3}}{384} = \ldots$	**M1**	Applies the reduction formula at least twice in succession, from their I_0. Condone arithmetic slips.
	$I_2 = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$ or $I_4 = \frac{\pi}{16} - \frac{7\sqrt{3}}{64}$	**A1**	Either value correct, oe.
	$I_6 = \frac{5\pi - 9\sqrt{3}}{96} \Rightarrow a = 5,\ b = -9$	**A1**	Both values, or $\frac{5\pi-9\sqrt{3}}{96}$ seen.
		[4]	
	Total for Question 3	**9**	

Question	Scheme	Marks	Notes
4(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(4-x^2)^n\right)$ $= (4-x^2)^n + x\cdot n(4-x^2)^{n-1}(-2x)$ $= (4-x^2)^n - 2nx^2(4-x^2)^{n-1}$	**M1 A1**	**M1** Applies the product and chain rules, reaching the form $\alpha(4-x^2)^n + \beta x^2(4-x^2)^{n-1}$, or integrates I_n by parts to that form. Condone sign and coefficient slips. **A1** Correct derivative or by-parts result, oe.
	$x^2 = 4-(4-x^2) \Rightarrow$ $\left[x(4-x^2)^n\right]_0^{\sqrt{2}} = (2n+1)I_n - 8nI_{n-1}$	**M1**	Writes $x^2 = 4-(4-x^2)$ oe and reaches an equation in I_n and I_{n-1}, from both sides integrated from 0 to $\sqrt{2}$ or from a by-parts result. Condone sign and coefficient slips.
	$\left[x(4-x^2)^n\right]_0^{\sqrt{2}} = \sqrt{2}(4-2)^n - 0 = 2^n\sqrt{2}$	**B1**	
	$2^n\sqrt{2} = (2n+1)I_n - 8nI_{n-1}$ $\Rightarrow (2n+1)I_n = 2^n\sqrt{2} + 8nI_{n-1}$	**A1 AG**	Correct equation in I_n, I_{n-1} and $2^n\sqrt{2}$ leading without error to the printed result.
		[5]	
4(b)	$I_0 = \sqrt{2}$	**B1**	
	$3I_1 = 2\sqrt{2} + 8I_0 \Rightarrow I_1 = \ldots$ $5I_2 = 4\sqrt{2} + 16I_1 \Rightarrow I_2 = \ldots$ $7I_3 = 8\sqrt{2} + 24I_2 \Rightarrow I_3 = \ldots$	**M1**	Uses the result at least twice in succession, from their I_0. Condone arithmetic slips.
	$I_1 = \frac{10\sqrt{2}}{3}$ or $I_2 = \frac{172\sqrt{2}}{15}$	**A1**	Either correct, oe.
	$I_3 = \frac{1416}{35}\sqrt{2}$	**A1**	In the form $k\sqrt{2}$, with k any equivalent fraction such as $\frac{4248}{105}$.
		[4]	
	Total for Question 4	**9**	

Question	Scheme	Marks	Notes
5(a)	$I_n + I_{n-2} = \int_0^{\frac{1}{\sqrt{3}}} \frac{x^{n-2}\left(x^2+1\right)}{1+x^2}\,\mathrm{d}x$ $= \int_0^{\frac{1}{\sqrt{3}}} x^{n-2}\,\mathrm{d}x$	**M1**	Writes $I_n + I_{n-2}$ as one integral and factorises the numerator to cancel $1+x^2$. Condone a slip in the power of x.
	$= \left[\frac{x^{n-1}}{n-1}\right]_0^{\frac{1}{\sqrt{3}}}$	**dM1**	Integrates x^{n-2} to the form kx^{n-1} and substitutes both limits.
	$I_n + I_{n-2} = \frac{1}{n-1}\left(\frac{1}{\sqrt{3}}\right)^{n-1}$	**A1 AG**	Correct integral and limits leading without error to the printed result.
		[3]	
5(b)(i)	$I_0 = [\arctan x]_0^{\frac{1}{\sqrt{3}}} = \frac{\pi}{6}$	**B1**	
	$I_2 + \text{“}\frac{\pi}{6}\text{”} = \frac{1}{\sqrt{3}} \Rightarrow I_2 = \ldots$ $I_4 + I_2 = \frac{1}{9\sqrt{3}} \Rightarrow I_4 = \ldots$ $I_6 + I_4 = \frac{1}{45\sqrt{3}} \Rightarrow I_6 = \ldots$	**M1**	Uses the relation from (a) at least twice in succession, starting from their I_0. Condone slips in the powers of $\frac{1}{\sqrt{3}}$.
	$I_2 = \frac{\sqrt{3}}{3} - \frac{\pi}{6}$ or $I_4 = \frac{\pi}{6} - \frac{8\sqrt{3}}{27}$	**A1**	Either value correct, oe.
	$I_6 = \frac{41}{135}\sqrt{3} - \frac{1}{6}\pi$	**A1**	oe in the form $a\sqrt{3} + b\pi$, or $a = \frac{41}{135}$, $b = -\frac{1}{6}$ stated.
		[4]	
5(b)(ii)	$I_6 > 0 \Rightarrow \text{“}\frac{41\sqrt{3}}{135} - \frac{\pi}{6}\text{”} > 0$	**M1**	Uses $I_6 > 0$ with their I_6 of the form $a\sqrt{3} + b\pi$, $b \neq 0$, to form an inequality in π.
	$\frac{x^6}{1+x^2} > 0$ for $0 < x \leq \frac{1}{\sqrt{3}}$ $\pi < \frac{246\sqrt{3}}{135} = \frac{82\sqrt{3}}{45}$	**A1 AG**	Needs the integrand shown positive on the interval, and correct I_6 leading without error to the printed inequality.
		[2]	
	Total for Question 5	**9**	

Question	Scheme	Marks	Notes
6(a)	$I_n = \left[\frac{1}{2}\cos^{n-1} 2x \sin 2x\right]_0^{\frac{\pi}{6}}$ $+ (n-1)\int_0^{\frac{\pi}{6}} \cos^{n-2} 2x \sin^2 2x \, \mathrm{d}x$	**M1 A1**	**M1** Integrates by parts with $\cos^n 2x$ split as $\cos^{n-1} 2x \times \cos 2x$ or equivalent, reaching a boundary term and an integral in $\cos^{n-2} 2x \sin^2 2x$. Condone sign and coefficient slips. **A1** Correct expression; limits need not yet be applied.
	$\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2^{n+1}}$	**B1**	
	$\sin^2 2x = 1 - \cos^2 2x$ $\Rightarrow I_n = \frac{\sqrt{3}}{2^{n+1}} + (n-1)(I_{n-2} - I_n)$	**M1**	Uses $\sin^2 2x = 1 - \cos^2 2x$ to write the remaining integral in terms of I_n and I_{n-2}. Condone sign slips.
	$nI_n = (n-1)I_{n-2} + \frac{\sqrt{3}}{2^{n+1}}$	**A1 AG**	Correct equation in I_n and I_{n-2} leading without error to the printed result; combined steps accepted.
		[5]	
6(b)	$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$ $= \cos 2x \Rightarrow \text{integral} = 64I_5$	**M1**	Uses valid identities to reduce the integrand to a multiple of $\cos^5 2x$ and writes the integral as a multiple of I_5. Condone a slip in the constant.
	$I_1 = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{4}$	**B1**	
	$3I_3 = 2\text{“}I_1\text{”} + \frac{\sqrt{3}}{16}$ $5I_5 = 4\text{“}I_3\text{”} + \frac{\sqrt{3}}{64}$	**M1**	Uses the result of (a) with $n = 3$ and $n = 5$, starting from their value of I_1. Condone slips in the $\frac{\sqrt{3}}{2^{n+1}}$ terms.
	$I_3 = \frac{3\sqrt{3}}{16}$ or $I_5 = \frac{49\sqrt{3}}{320}$	**A1**	Either value correct.
	$64 \times \frac{49\sqrt{3}}{320} = \frac{49\sqrt{3}}{5}$	**A1**	cao; any exact equivalent.
		[5]	
	Total for Question 6	**10**	

Question	Scheme	Marks	Notes
7(a)	$I_1 = \left[\arcsin\left(\frac{x}{4}\right)\right]_0^{2\sqrt{3}}$	**M1**	Integrates to $k\arcsin\left(\frac{x}{4}\right)$, or to θ after the substitution $x = 4\sin\theta$. Limits need not be applied.
	$I_1 = \frac{\pi}{3}$	**A1**	
		[2]	
7(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(16-x^2\right)^{-\frac{1}{2}n}\right)$ $= \left(16-x^2\right)^{-\frac{1}{2}n} + nx^2\left(16-x^2\right)^{-\frac{1}{2}n-1}$	**M1 A1**	**M1** Differentiates by the product and chain rules to two terms, the second of the form $kx^2\left(16-x^2\right)^{-\frac{1}{2}n-1}$, or integrates I_n by parts. Condone sign and coefficient slips. **A1** Correct derivative, or correct by-parts expression, oe.
	Uses $x^2 = 16-\left(16-x^2\right)$: $nx^2\left(16-x^2\right)^{-\frac{1}{2}(n+2)}$ $= 16n\left(16-x^2\right)^{-\frac{1}{2}(n+2)} - n\left(16-x^2\right)^{-\frac{1}{2}n}$ $\Rightarrow \left[x\left(16-x^2\right)^{-\frac{1}{2}n}\right]_0^{2\sqrt{3}} = (1-n)I_n + 16nI_{n+2}$	**M1**	Uses $x^2 = 16-\left(16-x^2\right)$ oe and reaches an equation in I_n and I_{n+2}, from both sides integrated or from a by-parts result. Condone sign and coefficient slips.
	$2\sqrt{3}\times 4^{-\frac{1}{2}n} - 0 = 2^{1-n}\sqrt{3}$	**B1**	Any correct form, such as $2\sqrt{3}\times 4^{-\frac{1}{2}n}$.
	$2^{1-n}\sqrt{3} = (1-n)I_n + 16nI_{n+2}$ $\Rightarrow 16nI_{n+2} = 2^{1-n}\sqrt{3} + (n-1)I_n$	**A1 AG**	Correct equation with the boundary value substituted, leading without error to the printed result.
		[5]	
7(c)	$n=1$: $16I_3 = 2^0\sqrt{3} + 0\times I_1 \Rightarrow I_3 = \ldots$	**M1**	Substitutes $n=1$ into the result of (b), with the I_1 term vanishing.
	$I_3 = \frac{\sqrt{3}}{16}$	**A1**	
	$48I_5 = 2^{-2}\sqrt{3} + 2\times$ "$\frac{\sqrt{3}}{16}$" $\Rightarrow I_5 = \ldots \left[= \frac{\sqrt{3}}{128}\right]$ $80I_7 = 2^{-4}\sqrt{3} + 4\times$ "$\frac{\sqrt{3}}{128}$" $\Rightarrow I_7 = \ldots$	**dM1**	Uses the result with $n=3$ and their I_3, then $n=5$ and their I_5, each a multiple of $\sqrt{3}$. Condone arithmetic slips.
	$I_7 = \frac{3\sqrt{3}}{2560}$, $k = \frac{3}{2560}$	**A1**	$\frac{3\sqrt{3}}{2560}$ or $k = \frac{3}{2560}$ stated. Not for $\frac{9}{2560\sqrt{3}}$.
		[4]	
	Total for Question 7	**11**	

Question	Scheme	Marks	Notes
8(a)	$I_n = \left[\operatorname{sech}^{n-2} x \tanh x\right]_0^{\ln(2+\sqrt{3})}$ $+ (n-2)\int_0^{\ln(2+\sqrt{3})} \operatorname{sech}^{n-2} x \tanh^2 x \, \mathrm{d}x$	**M1 A1**	**M1** Integrates by parts, such as with $\operatorname{sech}^2 x$ integrated and $\operatorname{sech}^{n-2} x$ differentiated, reaching a boundary term and an integral. Condone sign and coefficient slips. **A1** Correct expression, oe. Limits need not yet be substituted.
	$\operatorname{sech}(\ln(2+\sqrt{3})) = \frac{1}{2},\ \tanh(\ln(2+\sqrt{3})) = \frac{\sqrt{3}}{2}$ $\Rightarrow \left(\frac{1}{2}\right)^{n-2} \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2^{n-1}}$	**B1**	Boundary term reaches $\frac{\sqrt{3}}{2^{n-1}}$ with the values of sech and tanh, or cosh and sinh, at the upper limit shown.
	$\tanh^2 x = 1 - \operatorname{sech}^2 x$ $\Rightarrow I_n = \frac{\sqrt{3}}{2^{n-1}} + (n-2)I_{n-2} - (n-2)I_n$	**M1**	Uses $\tanh^2 x = 1 - \operatorname{sech}^2 x$ oe to write the integral in terms of I_n and I_{n-2}. Condone sign slips.
	$(n-1)I_n = \frac{\sqrt{3}}{2^{n-1}} + (n-2)I_{n-2}$	**A1 AG**	Correct equation in I_n and I_{n-2} leading without error to the printed result; collected steps accepted.
		[5]	
8(b)	$I_1 = \int_0^{\ln(2+\sqrt{3})} \frac{\cosh x}{1+\sinh^2 x} \, \mathrm{d}x$ $= \left[\arctan(\sinh x)\right]_0^{\ln(2+\sqrt{3})}$	**M1**	Integrates $\operatorname{sech} x$ to an arctan form, such as $k\arctan(\sinh x)$ or $k\arctan(e^x)$. Condone a wrong constant k.
	$= \arctan\sqrt{3} - \arctan 0 = \frac{\pi}{3}$	**A1**	Exact, cso.
	$2I_3 = \frac{\sqrt{3}}{4} + \text{“}\frac{\pi}{3}\text{”}$ $4I_5 = \frac{\sqrt{3}}{16} + 3I_3$	**M1**	Uses the reduction formula with $n = 3$ and with $n = 5$, with their I_1. Condone slips in the power of 2.
	$I_3 = \frac{\sqrt{3}}{8} + \frac{\pi}{6}$	**A1**	Exact, oe.
	$I_5 = \frac{7}{64}\sqrt{3} + \frac{1}{8}\pi$	**A1**	Exact, oe, or $a = \frac{7}{64}$, $b = \frac{1}{8}$ stated.
		[5]	
	Total for Question 8	**10**	

Question	Scheme	Marks	Notes
9(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^3} = -\frac{3x^2}{2\sqrt{9-x^3}}$ $\Rightarrow I_2 = \left[-\frac{2}{3}\sqrt{9-x^3}\right]_0^2$	**M1**	Obtains $k\sqrt{9-x^3}$ as the integral, for any nonzero constant k, using the derivative.
	$I_2 = -\frac{2}{3} + 2 = \frac{4}{3}$	**A1**	
		[2]	
9(b)	$I_n = \left[-\frac{2}{3}x^{n-2}\sqrt{9-x^3}\right]_0^2$ $\quad + \frac{2}{3}(n-2)\int_0^2 x^{n-3}\sqrt{9-x^3}\,\mathrm{d}x$	**M1 A1**	**M1** Integrates by parts with $u = x^{n-2}$ and $v = k\sqrt{9-x^3}$, reaching $Ax^{n-2}\sqrt{9-x^3} \pm B\int x^{n-3}\sqrt{9-x^3}\,\mathrm{d}x$. Condone coefficient slips.
	$\left[-\frac{2}{3}x^{n-2}\sqrt{9-x^3}\right]_0^2 = -\frac{2^{n-1}}{3}$	**B1**	oe, such as $-\frac{2}{3}\times 2^{n-2}$.
	$\int_0^2 \frac{x^{n-3}(9-x^3)}{\sqrt{9-x^3}}\,\mathrm{d}x = 9I_{n-3} - I_n$	**M1**	Writes $\sqrt{9-x^3}$ as $\frac{9-x^3}{\sqrt{9-x^3}}$, or equivalent, to express the remaining integral in terms of I_{n-3} and I_n.
	$I_n = -\frac{2^{n-1}}{3} + \frac{2}{3}(n-2)(9I_{n-3} - I_n)$ $\Rightarrow (2n-1)I_n = 18(n-2)I_{n-3} - 2^{n-1}$	**A1 AG**	Correct substituted equation in I_n and I_{n-3} leading without error to the printed result; combined steps accepted.
		[5]	
9(c)	$\int_0^2 \frac{x^5(9-x^3)}{\sqrt{9-x^3}}\,\mathrm{d}x = 9I_5 - I_8$	**M1**	Writes the integral as a combination of I_5 and I_8. Condone sign or coefficient slips.
	$9I_5 = 54 \times \text{“}\frac{4}{3}\text{”} - 2^4$ $15I_8 = 108I_5 - 2^7$	**M1**	Uses the reduction formula with $n = 5$ and $n = 8$, starting from their I_2 from (a).
	$I_5 = \frac{56}{9}, \quad I_8 = \frac{544}{15}$	**A1**	
	$9I_5 - I_8 = 56 - \frac{544}{15} = \frac{296}{15}$	**A1**	$\frac{296}{15}$ or any exact equivalent.
		[4]	
Total for Question 9		**11**	

Question	Scheme	Marks	Notes
10(a)	$\sin\big((n+1)x \pm x\big)$ $= \sin\big((n+1)x\big)\cos x \pm \cos\big((n+1)x\big)\sin x$	**M1**	Expands $\sin\big((n+1)x \pm x\big)$ by the addition formulae, or uses a sum-to-product formula, for $\sin\big((n+2)x\big) + \sin(nx)$. Condone sign slips.
	$\sin\big((n+2)x\big) + \sin(nx) = 2\sin\big((n+1)x\big)\cos x$	**A1**	oe, such as $\sin\big((n+2)x\big) = 2\sin\big((n+1)x\big)\cos x - \sin(nx)$.
	$I_{n+2} + I_n = \int \frac{\text{“}2\sin((n+1)x)\cos x\text{”}}{\cos x}\,\mathrm{d}x$ $= \int 2\sin\big((n+1)x\big)\,\mathrm{d}x$	**M1 A1**	**M1** Writes $I_{n+2} + I_n$, or I_{n+2} in terms of I_n, as one integral using their identity, and cancels $\cos x$.
	$= -\frac{2\cos\big((n+1)x\big)}{n+1}$ $(+c)$	**dM1**	Integrates $\sin\big((n+1)x\big)$ to $\pm k\cos\big((n+1)x\big)$. Condone a wrong coefficient.
	$I_{n+2} = -\frac{2\cos\big((n+1)x\big)}{n+1} - I_n$	**A1 AG**	Correct integral leading without error to the printed result. A constant of integration may be shown or absorbed.
		[6]	
10(b)	$I_1 = \int_0^{\frac{\pi}{4}} \tan x\,\mathrm{d}x = \big[-\ln(\cos x)\big]_0^{\frac{\pi}{4}}$	**M1**	Integrates $\tan x$ to $\pm\ln(\cos x)$ or $\pm\ln(\sec x)$, with or without a modulus. Condone I_n for $\int_0^{\frac{\pi}{4}} \frac{\sin(nx)}{\cos x}\,\mathrm{d}x$ throughout.
	$I_1 = \frac{1}{2}\ln 2$	**A1**	oe, such as $\ln\sqrt{2}$.
	Uses (a) with limits 0, $\frac{\pi}{4}$ and $n = 1$: $I_3 = \big[-\cos 2x\big]_0^{\frac{\pi}{4}} - \text{“}\frac{1}{2}\ln 2\text{”}$	**M1**	Applies (a) between 0 and $\frac{\pi}{4}$ for one value of n, evaluating the bracket at both limits. Condone sign slips.
	$I_3 = 1 - \frac{1}{2}\ln 2$	**A1**	
	$I_5 = \frac{1}{2}(1 - \cos\pi) - \text{“}I_3\text{”}$ $[= \frac{1}{2}\ln 2]$ $I_7 = \frac{1}{3}\big(1 - \cos\frac{3\pi}{2}\big) - \text{“}I_5\text{”} = \ldots$	**dM1**	Applies the formula again with $n = 3$ and $n = 5$, using their earlier values. Condone sign slips.
	$I_7 = \frac{1}{3} - \frac{1}{2}\ln 2$ $a = \frac{1}{3},\ b = -\frac{1}{2},\ c = 2$	**A1**	cso. Accept any rational a, b, c giving this value, such as $\frac{1}{3} + \frac{1}{2}\ln\frac{1}{2}$.
		[6]	
	Total for Question 10	**12**	

Question	Scheme	Marks	Notes
11(a)	$I_n = \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \left(\sec^2 x - 1\right) \mathrm{d}x$ $= \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \sec^2 x \,\mathrm{d}x - I_{n-2}$	**M1**	Uses $\tan^2 x = \sec^2 x - 1$ on $\tan^{n-2} x \tan^2 x$ and splits off I_{n-2}. Condone a sign slip in the identity.
	$\int_0^{\frac{1}{4}\pi} \tan^{n-2} x \sec^2 x \,\mathrm{d}x = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{1}{4}\pi}$ $= \ldots$	**dM1**	Integrates $\tan^{n-2} x \sec^2 x$ to the form $k \tan^{n-1} x$, by inspection or substitution, and substitutes both limits.
	$I_n = \frac{1}{n-1} - I_{n-2}$ $\Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$	**A1 AG**	Correct integral and limits, with $\tan \frac{1}{4}\pi = 1$, leading without error to the printed result.
		[3]	
11(b)	$0 < \tan x < 1$ for $0 < x < \frac{1}{4}\pi$ $\Rightarrow \tan^n x < \tan^{n-2} x$	**B1**	Comparison with its reason, $0 < \tan x < 1$ or $\tan^2 x < 1$ on the interval. Condone $\leq$ at the end-points.
	Integrand of I_n smaller throughout the interval $\Rightarrow I_n < I_{n-2}$	**E1**	States that the smaller integrand over the same interval gives the smaller integral. Not for $I_n < I_{n-2}$ asserted without reference to the integrands.
		[2]	
11(c)	$2I_n < I_n + I_{n-2} = \frac{1}{n-1}$	**M1**	Combines $I_n < I_{n-2}$ with part (a) to form an inequality in I_n alone.
	$\frac{1}{n+1} = I_{n+2} + I_n < 2I_n$	**M1**	Replaces n by $n+2$ in part (a) and uses $I_{n+2} < I_n$ to form a second inequality in I_n alone.
	$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$	**A1**	Both bounds reached without error and the printed inequality stated.
	$I_0 = \frac{\pi}{4} \Rightarrow I_2 = 1 - I_0 = 1 - \frac{\pi}{4}$ $I_4 = \frac{1}{3} - I_2,\ I_6 = \frac{1}{5} - I_4,\ I_8 = \frac{1}{7} - I_6$	**M1**	Uses $I_0 = \frac{\pi}{4}$ and part (a) with $n = 2, 4, 6, 8$ in turn to reach I_8. Condone arithmetic slips.
	$I_8 = \frac{\pi}{4} - \frac{76}{105}$	**A1**	oe, such as $\frac{\pi}{4} - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7}$.
	$\frac{1}{18} < \text{“}\frac{\pi}{4} - \frac{76}{105}\text{”} < \frac{1}{14}$ $\Rightarrow \ldots < \pi < \ldots$	**dM1**	Substitutes $n = 8$ and their I_8 of the form $\pm\frac{\pi}{4} + k$ into the inequality and rearranges for π. Condone arithmetic slips.

Question 11 continued

Question	Scheme	Marks	Notes
	$\frac{982}{315} < \pi < \frac{334}{105}$	**A1**	Both end-points as fractions, not necessarily in lowest terms, with strict inequalities or as an open interval.
		[7]	
	Total for Question 11	**12**	

Question	Scheme	Marks	Notes
12(a)	$C_n = \left[\frac{1}{2}x^n \sinh 2x\right]_0^a$ $- \int_0^a \frac{1}{2}nx^{n-1} \sinh 2x \, \mathrm{d}x$	**M1 A1**	**M1** Integrates by parts with $u = x^n$, reaching the form $px^n \sinh 2x -$ $q\int x^{n-1} \sinh 2x \, \mathrm{d}x$. Condone coefficient slips. **A1** Correct; the boundary term need not be evaluated.
	$\int_0^a x^{n-1} \sinh 2x \, \mathrm{d}x$ $= \left[\frac{1}{2}x^{n-1} \cosh 2x\right]_0^a - \frac{n-1}{2}C_{n-2}$	**dM1 A1**	**dM1** Integrates by parts again, differentiating the power of x, reaching the form $px^{n-1} \cosh 2x -$ $q\int x^{n-2} \cosh 2x \, \mathrm{d}x$. Condone coefficient slips. **A1** Correct, in terms of C_{n-2}.
	$\left[\frac{1}{2}x^n \sinh 2x\right]_0^a = \frac{1}{2}a^n \sinh 2a$ $\left[\frac{1}{2}x^{n-1} \cosh 2x\right]_0^a = \frac{1}{2}a^{n-1} \cosh 2a$	**B1**	Both boundary terms correct, each lower limit giving zero; may be seen in any later line.
	$C_n = \frac{1}{2}a^n \sinh 2a$ $- \frac{n}{2}\left(\text{“}\frac{1}{2}a^{n-1} \cosh 2a - \frac{n-1}{2}C_{n-2}\text{”}\right)$	**M1**	Substitutes their second result into their first, giving an equation in C_n and C_{n-2} only. Condone sign slips.
	$4C_n = n(n-1)C_{n-2} + 2a^n \sinh 2a - na^{n-1} \cosh 2a$	**A1 AG**	Correct substituted equation leading without error to the printed result; collected steps accepted.
		[7]	
12(b)	$C_1 = \left[\frac{1}{2}x \sinh 2x\right]_0^1 - \left[\frac{1}{4} \cosh 2x\right]_0^1$	**M1**	Integrates $x \cosh 2x$ by parts, or uses their first result in (a) with $n = 1$, reaching the form $px \sinh 2x - q \cosh 2x$ with limits used.
	$C_1 = \frac{1}{2} \sinh 2 - \frac{1}{4} \cosh 2 + \frac{1}{4}$	**A1**	oe, such as $\frac{1}{8}\left(e^2 + 2 - 3e^{-2}\right)$.
	$4C_3 = 6\text{“}C_1\text{”} + 2 \sinh 2 - 3 \cosh 2$	**M1**	Uses the result of (a) with $n = 3$ and $a = 1$, and their C_1. Condone one coefficient slip.
	$4C_3 = 5 \sinh 2 - \frac{9}{2} \cosh 2 + \frac{3}{2}$	**A1**	oe, including a correct form in exponentials.
	$4C_3 = \frac{5}{2}\left(e^2 - e^{-2}\right) - \frac{9}{4}\left(e^2 + e^{-2}\right) + \frac{3}{2}$	**M1**	Uses the exponential definitions of $\sinh 2$ and $\cosh 2$ in their expression; may be seen earlier, as in C_1.
	$\int_0^1 x^3 \cosh 2x \, \mathrm{d}x = \frac{1}{16}\left(e^2 + 6 - 19e^{-2}\right)$	**A1 AG**	Correct expression in exponentials leading without error to the printed result.
		[6]	
	Total for Question 12	**13**	

Question	Scheme	Marks	Notes
13(a)	$I_n = \left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}$ $-(n-2)\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^{n-2} x \cot^2 x \, \mathrm{d}x$	**M1 A1**	**M1** Integrates by parts, differentiating $\operatorname{cosec}^{n-2} x$ and integrating $\operatorname{cosec}^2 x$ to $\pm\cot x$, reaching the form $\pm\cot x \operatorname{cosec}^{n-2} x \pm k\int \operatorname{cosec}^{n-2} x \cot^2 x \, \mathrm{d}x$. Condone sign and coefficient slips. **A1** Correct boundary term and remaining integral, oe.
	$\left[-\cot x \operatorname{cosec}^{n-2} x\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} = 2^{n-2}\sqrt{3}$	**B1**	
	$\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^{n-2} x \cot^2 x \, \mathrm{d}x$ $= \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \left(\operatorname{cosec}^n x - \operatorname{cosec}^{n-2} x\right) \mathrm{d}x = I_n - I_{n-2}$	**M1**	Uses $\cot^2 x = \operatorname{cosec}^2 x - 1$ to write the remaining integral in terms of I_n and I_{n-2}. Condone sign slips.
	$I_n = 2^{n-2}\sqrt{3} - (n-2)(I_n - I_{n-2})$ $\Rightarrow (n-1)I_n = 2^{n-2}\sqrt{3} + (n-2)I_{n-2}$	**A1 AG**	Correct substituted equation in I_n and I_{n-2} leading without error to the printed result; collected steps accepted.
		[5]	
13(b)	$I_2 = [-\cot x]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} = \sqrt{3}$	**B1**	
	$3I_4 = 2^2\sqrt{3} + 2I_2, \quad 5I_6 = 2^4\sqrt{3} + 4I_4$ $7I_8 = 2^6\sqrt{3} + 6I_6$	**M1**	Applies the reduction formula at least twice in succession, starting from their I_2. Condone arithmetic slips.
	$I_4 = 2\sqrt{3}, \quad I_6 = \frac{24}{5}\sqrt{3}$	**A1**	Either value correct.
	$7I_8 = 64\sqrt{3} + \frac{144}{5}\sqrt{3} \Rightarrow I_8 = \frac{464}{35}\sqrt{3}$	**A1 AG**	Correct chain from $I_2 = \sqrt{3}$ leading without error to the printed I_8; combined steps accepted.
		[4]	
13(c)	$n = 1$: $I_2 = \sqrt{3} = 1 \times \sqrt{3}$, and 1 is rational	**B1**	Needs $I_2 = \sqrt{3}$, from part (b) or evaluated, and the multiplier 1 noted as rational.
	Assumes $I_{2k} = q\sqrt{3}$, q rational: $(2k+1)I_{2k+2} = 2^{2k}\sqrt{3} + 2kI_{2k}$	**M1**	Assumes the statement for $n = k$ and uses part (a) with $n = 2k+2$. Condone slips in the coefficients.
	$(2k+1)I_{2k+2} = 4^k\sqrt{3} + 2kq\sqrt{3} \Rightarrow I_{2k+2} = \ldots$	**M1**	Substitutes their $I_{2k} = q\sqrt{3}$ into the recurrence and divides by the coefficient of I_{2k+2}. Condone coefficient slips.
	$I_{2k+2} = \frac{4^k + 2kq}{2k+1}\sqrt{3}$	**A1**	oe, such as $\frac{2^{2k} + 2kq}{2k+1}\sqrt{3}$.

Question 13 continued

Question	Scheme	Marks	Notes
	$\frac{4^k+2kq}{2k+1}$ is rational, so true for $n = k + 1$	**A1**	Coefficient stated rational, with a reason (integers and q rational), and the statement given for $n = k + 1$.
	True for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, so true for every positive integer n	**E1**	States true for $n = 1$, that true for $n = k$ implies true for $n = k + 1$, and so true for every positive integer n.
		[6]	
	Total for Question 13	**15**	

Question	Scheme	Marks	Notes
14(a)	$v = \frac{1}{48}\left(9+16t^2\right)^{\frac{3}{2}}$	**B1**	oe, soi in the integration by parts.
	$I_n = \left[\frac{1}{48}t^{n-1}\left(9+16t^2\right)^{\frac{3}{2}}\right]_0^1$ $-\frac{n-1}{48}\int_0^1 t^{n-2}\left(9+16t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$	**M1 A1**	**M1** Integrates by parts, reaching the form $\alpha t^{n-1}\left(9+16t^2\right)^{\frac{3}{2}} - \beta(n-1)\int t^{n-2}\left(9+16t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$. Condone coefficient slips. **A1** Correct expression; limits may be implied.
	$= \frac{125}{48} - \frac{n-1}{48}\int_0^1 t^{n-2}\left(9+16t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$	**B1**	Boundary term $\frac{125}{48}$, with zero at the lower limit.
	$\int_0^1 t^{n-2}\left(9+16t^2\right)\sqrt{9+16t^2}\,\mathrm{d}t = 9I_{n-2}+16I_n$	**M1**	Writes $\left(9+16t^2\right)^{\frac{3}{2}}$ as $\left(9+16t^2\right)\sqrt{9+16t^2}$ and expresses the integral in terms of I_n and I_{n-2}. Condone coefficient slips.
	$48I_n = 125-(n-1)\left(9I_{n-2}+16I_n\right)$ $\Rightarrow 16(n+2)I_n = 125-9(n-1)I_{n-2}$	**A1 AG**	Correct substituted equation leading without error to the printed result; collected steps accepted.
		[6]	
14(b)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 4t^3$	**B1**	
	$\left(3t^2\right)^2 + \left(4t^3\right)^2 = 9t^4+16t^6$	**M1**	Squares and adds their derivatives.
	$\sqrt{9t^4+16t^6} = t^2\sqrt{9+16t^2}$	**A1**	
	$S = 2\pi\int_0^1 t^3\times t^2\sqrt{9+16t^2}\,\mathrm{d}t$	**M1**	Uses $2\pi\int x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ with $x=t^3$, their root and limits 0 and 1. Not for y in place of x.
	$= 2\pi\int_0^1 t^5\sqrt{9+16t^2}\,\mathrm{d}t$	**A1 AG**	Correct substituted integral leading without error to the printed result.
		[5]	
14(c)	$I_1 = \left[\frac{1}{48}\left(9+16t^2\right)^{\frac{3}{2}}\right]_0^1 = \ldots$	**M1**	Integrates to the form $k\left(9+16t^2\right)^{\frac{3}{2}}$, by any method, and applies limits 0 and 1.
	$I_1 = \frac{49}{24}$	**A1**	
	$80I_3 = 125-18\text{“}\frac{49}{24}\text{”}, \quad 112I_5 = 125-36I_3$	**M1**	Uses the result of (a) with $n=3$ and $n=5$, and their I_1. Condone arithmetic slips.

Question 14 continued

Question	Scheme	Marks	Notes
	$I_3 = \frac{353}{320}$	**A1**	
	$I_5 = \frac{6823}{8960} \Rightarrow S = 2\pi I_5 = \frac{6823}{4480}\pi$	**A1**	$\frac{6823}{4480}\pi$, or an equivalent in the form $\frac{p}{q}\pi$.
		[5]	
	Total for Question 14	**16**	