OPTIONAL PURE / FURTHER CALCULUS

L'Hôpital's Rule

Mark scheme 12 questions · 53 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · AQA 7367

Questions

Question	Marks by part	Marks	Page
Question 1		1	4
Question 2		1	5
Question 3		1	6
Question 4		5	7
Question 5		5	8
Question 6		5	9
Question 7		5	10
Question 8		6	11
Question 9		6	12
Question 10		3	13
Question 11	(a) 1 (b) 5	6	14
Question 12	(a)(i) 1 (a)(ii) 3 (b) 5	9	15
	Total	**53**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [***n***] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$\lim_{x\to\infty}\left(\frac{\ln x}{x^2}\right)$	**B1**	This option alone circled or clearly identified.
	Total for Question 1	**1**	

Question	Scheme	Marks	Notes
2	$\frac{3}{2}$	**B1**	$\frac{3}{2}$ circled, or otherwise clearly indicated, and no other option chosen.
	Total for Question 2	**1**	

Question	Scheme	Marks	Notes
3	$\lim_{x \to 0^+} (x^x)$	**B1**	Circled or otherwise clearly identified, with no other option chosen.
	Total for Question 3	**1**	

Question	Scheme	Marks	Notes
4	$\lim_{x\to 0} \frac{2\sec(2x)\tan(2x)}{\sin(3x)+3x\cos(3x)}$	**M1 A1**	**M1** Differentiates the numerator and denominator of the quotient separately, the product rule giving two terms, and forms their quotient. Condone coefficient slips. **A1** Correct derivatives of numerator and denominator, oe, of the original quotient or of a correctly rewritten one.
	$\lim_{x\to 0} \frac{4\sec(2x)\tan^2(2x)+4\sec^3(2x)}{6\cos(3x)-9x\sin(3x)}$	**dM1 A1**	**dM1** Differentiates their new numerator and denominator separately again, using the product rule where needed, and forms their quotient. Condone coefficient slips. **A1** Correct second numerator and denominator, oe.
	$= \frac{0+4}{6-0} = \frac{2}{3}$	**A1**	$\frac{2}{3}$, or awrt 0.667, cso.
	Total for Question 4	**5**	

Question	Scheme	Marks	Notes
5	$1\ln 1 - 1 + 1 = 0,\ \sin^2\pi = 0$ $\lim_{x\to 1}\frac{\ln x + x\cdot\frac{1}{x} - 1}{2\pi\sin(\pi x)\cos(\pi x)}$	**M1**	Differentiates numerator and denominator separately. Condone slips in the product or chain rule. Not for the quotient rule.
	$= \lim_{x\to 1}\frac{\ln x}{\pi\sin(2\pi x)}$	**A1**	Correct quotient oe, such as $\frac{\ln x}{2\pi\sin(\pi x)\cos(\pi x)}$; need not be simplified.
	$\ln 1 = 0,\ \pi\sin 2\pi = 0$ $\lim_{x\to 1}\frac{\frac{1}{x}}{2\pi^2\cos(2\pi x)}$	**dM1 A1**	**dM1** Differentiates their numerator and denominator again separately. Condone slips. **A1** Correct $\frac{1}{x}$ and $2\pi^2\cos(2\pi x)$, oe.
	$= \frac{1}{2\pi^2\cos 2\pi} = \frac{1}{2\pi^2}$	**A1 AG**	cso. $\frac{0}{0}$ shown at each application, and a correct substituted quotient, such as $\frac{1}{2\pi^2\cos 2\pi}$, leading to $\frac{1}{2\pi^2}$.
	Total for Question 5	**5**	

Question	Scheme	Marks	Notes
6	$\ln 1 + e^0 - 1 = 0,\ \sin \pi = 0$ $\cot x - 3\sin(3x)\, e^{\cos(3x)}$	**M1 A1**	**M1** Differentiates numerator and denominator separately, not as a quotient. Condone slips in the derivatives. **A1** Correct derivative of the numerator, oe such as $\frac{2\cos x}{2\sin x}$ for the first term.
	$6\cos(6x)$	**A1**	
	$\frac{\cot \frac{\pi}{6} - 3\sin \frac{\pi}{2}\, e^{\cos \frac{\pi}{2}}}{6\cos \pi} = \ldots$	**dM1**	Substitutes $x = \frac{\pi}{6}$ into both their derivatives and forms their quotient.
	$= \frac{\sqrt{3}-3}{-6} = \frac{3-\sqrt{3}}{6}$	**A1 AG**	cso. $\frac{0}{0}$ shown at $x = \frac{\pi}{6}$, and a correct substituted quotient leading without error to the printed result.
	Total for Question 6	**5**	

Question	Scheme	Marks	Notes
7	Uses L'Hôpital's rule, $\frac{0}{0}$ at $x = 0$: $\lim_{x\to 0} \frac{\tanh x - \tan x \sec^2 x}{4x^3}$	**M1 A1**	**M1** Differentiates numerator and denominator separately, not as a quotient, or expands $\ln(\cosh x)$ as a series to x^4. Condone sign and coefficient slips. **A1** Both correct, oe such as $\frac{\sinh x}{\cosh x}$ or $\tan x\left(1+\tan^2 x\right)$, or a correct series for $\ln(\cosh x)$ to x^4.
	Still $\frac{0}{0}$, so differentiates again: $\lim_{x\to 0} \frac{\operatorname{sech}^2 x - \sec^4 x - 2\tan^2 x \sec^2 x}{12x^2}$	**dM1**	Differentiates both again, attempting the product or chain rule on their $\tan x \sec^2 x$ term, or expands $\tan^2 x$ as a series to x^4. Condone sign and coefficient slips.
	$= \lim_{x\to 0} \frac{-\tanh^2 x - 4\tan^2 x - 3\tan^4 x}{12x^2}$ $\frac{\tanh x}{x} \to 1, \quad \frac{\tan x}{x} \to 1$ $\lim = -\frac{1}{12}(1+4+0) = -\frac{5}{12}$	**M1** **A1 AG**	**M1** Resolves their still-indeterminate quotient, by $\frac{\tan x}{x} \to 1$ and $\frac{\tanh x}{x} \to 1$, further applications of the rule, or subtracting the series and dividing by x^4. **A1 AG** cso. Correct second derivatives, or series, to $-\frac{5}{12}$, with $\frac{0}{0}$ shown at each application of the rule. Standard limits may be stated.
	Total for Question 7	**5**	

Question	Scheme	Marks	Notes
8	$x(\ln x)^2 = \frac{(\ln x)^2}{1/x}$	**M1**	Writes the product as a quotient whose numerator and denominator both tend to ∞, or both to 0.
	$(\ln x)^2 \to \infty,\ \frac{1}{x} \to \infty$ $\lim_{x\to 0^+} \frac{(\ln x)^2}{1/x} = \lim_{x\to 0^+} \frac{2\ln x/x}{-1/x^2}$	**M1 A1**	**M1** Differentiates numerator and denominator of their quotient separately. Condone slips. Not for the quotient rule. **A1** Correct quotient of derivatives, oe.
	$= \lim_{x\to 0^+}(-2x\ln x) = \lim_{x\to 0^+} \frac{-2\ln x}{1/x}$ $-2\ln x \to \infty,\ \frac{1}{x} \to \infty$ $= \lim_{x\to 0^+} \frac{-2/x}{-1/x^2}$	**dM1 A1**	**dM1** Simplifies their result to a multiple of $x\ln x$, rewrites it as a quotient and differentiates numerator and denominator again. Not for quoting $x\ln x \to 0$ as a standard result. **A1** Correct second quotient of derivatives, oe.
	$= \lim_{x\to 0^+} 2x = 0$	**A1 AG**	Both indeterminate forms justified by the limits of numerator and denominator, with correct working throughout to 0. A0 if either form is unjustified.
	Total for Question 8	**6**	

Question	Scheme	Marks	Notes
9	$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{x-\ln(1+x)}{x\ln(1+x)}$	**M1**	Combines the terms over a common denominator, such as $x\ln(1+x)$. Condone a sign slip in the numerator.
	$\frac{0}{0}$: $\lim_{x\to 0} \frac{1-\frac{1}{1+x}}{\ln(1+x)+\frac{x}{1+x}}$	**M1 A1**	**M1** Differentiates numerator and denominator separately, attempting the product rule on the denominator. Condone slips. Not for the quotient rule on the whole fraction. **A1** Correct quotient in any form.
	Still $\frac{0}{0}$: $\lim_{x\to 0} \frac{1}{\ln(1+x)+2}$	**dM1 A1**	**dM1** Differentiates their new numerator and denominator separately, simplified first or not. **A1** Correct second quotient in any form, such as $\frac{(1+x)^{-2}}{(1+x)^{-1}+(1+x)^{-2}}$.
	$= \frac{1}{2}$	**A1**	$\frac{1}{2}$ or 0.5, cso.
	Total for Question 9	**6**	

Question	Scheme	Marks	Notes
10	$p\ln\left(\frac{p+k}{p-k}\right) = \frac{1}{a}\ln\left(\frac{1+ka}{1-ka}\right)$ $= \frac{\ln(1+ka)-\ln(1-ka)}{a}$, $a \to 0^+$	**M1**	Substitutes $a = \frac{1}{p}$ and writes the expression as a single quotient in a whose numerator and denominator both tend to 0. Condone slips simplifying the fraction.
	$\lim_{a\to 0^+} \frac{\frac{k}{1+ka}+\frac{k}{1-ka}}{1} = \ldots$	**dM1**	Differentiates numerator and denominator separately with respect to a, the logarithm by the chain rule as one term or two. Condone sign slips. Not for the quotient rule on the whole fraction.
	$= k + k = 2k$	**A1**	cso.
	Total for Question 10	**3**	

Question	Scheme	Marks	Notes
11(a)	$\cos 3x = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$	**B1**	Accept unsimplified coefficients.
		[1]	
11(b)	$\lim \frac{x}{\ln(1+x)} = \lim(1+x) = 1$ $\lim \frac{ax^2+2\cos 3x-2}{x^4} = \lim \frac{2ax-6\sin 3x}{4x^3}$	**M1**	Applies L'Hôpital's rule at least once to the quotient, or a factor of it, or substitutes series in the numerator; the $\ln(1+x)$ factor need not be handled yet. Condone slips.
	$= \lim \frac{2a-18\cos 3x}{12x^2}$	**A1**	oe, such as a numerator $(a-9)x^2 + \frac{27}{4}x^4 + \cdots$ from series, or the correct second quotient of the whole expression.
	$2a - 18 = 0 \Rightarrow a = 9$	**M1 A1**	**M1** Sets the limit of their numerator, or their coefficient of x^2, equal to zero for a finite limit. **A1** $a = 9$ cao.
	$\lim \frac{18-18\cos 3x}{12x^2} = \lim \frac{54\sin 3x}{24x}$ $= \lim \frac{162\cos 3x}{24} = \frac{27}{4}$	**A1**	$\frac{27}{4}$ oe, such as 6.75, from $a = 9$ and correct work, the $\ln(1+x)$ factor resolved by $\frac{x}{\ln(1+x)} \to 1$ shown, its series used, or the rule applied to the whole quotient.
		[5]	
	Total for Question 11	**6**	

Question	Scheme	Marks	Notes
12(a)(i)	$\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4$	**B1**	All four terms correct; coefficients need not be simplified.
		[1]	
12(a)(ii)	$\ln(1+3x) - \ln(1-3x)$ $= \left(3x - \frac{9}{2}x^2 + \cdots\right) - \left(-3x - \frac{9}{2}x^2 - \cdots\right)$	**M1**	Uses $\ln(1+3x) - \ln(1-3x)$ with their series for $\ln(1-3x)$ and subtracts, or any valid series method. Condone sign slips.
	$= 6x + 18x^3$	**A1**	
	$-\frac{1}{3} < x < \frac{1}{3}$	**B1**	Both inequalities strict.
		[3]	
12(b)	$2 - \sqrt{4-x^2} = \frac{4-(4-x^2)}{2+\sqrt{4-x^2}}$	**M1**	Multiplies by $2+\sqrt{4-x^2}$ to remove the surd, or expands $\sqrt{4-x^2}$ binomially, or differentiates $x\left(2-\sqrt{4-x^2}\right)$ by the product rule in applying the rule. Condone sign slips.
	$= \frac{x^2}{2+\sqrt{4-x^2}}$ $\Rightarrow \text{limit} = 4\lim_{x\to 0}\frac{\ln\left(\frac{1+3x}{1-3x}\right)-6x}{x^3}$	**A1**	Correct bracket and the factor 4 from $2+\sqrt{4-x^2} \to 4$, oe, or correct $2-\sqrt{4-x^2}+\frac{x^2}{\sqrt{4-x^2}}$.
	Form $\frac{0}{0}$, so differentiates: $\lim_{x\to 0}\frac{\frac{3}{1+3x}+\frac{3}{1-3x}-6}{3x^2}$	**M1**	Differentiates numerator and denominator of their quotient separately, or uses their series from (a)(ii) in the numerator; on the direct route, applies the rule again. Condone sign slips.
	$= \lim_{x\to 0}\frac{54x^2}{3x^2(1-9x^2)} = \lim_{x\to 0}\frac{18}{1-9x^2}$	**A1**	Any correct equivalent ratio, or numerator $18x^3+\cdots$ by the series route, or a correct further quotient on the direct route.
	$\text{limit} = 4\times 18 = 72$	**A1**	cao
		[5]	
	Total for Question 12	**9**	