OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

Simpson's Rule

Worked solutions 10 questions · 68 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · AQA 7367

Questions

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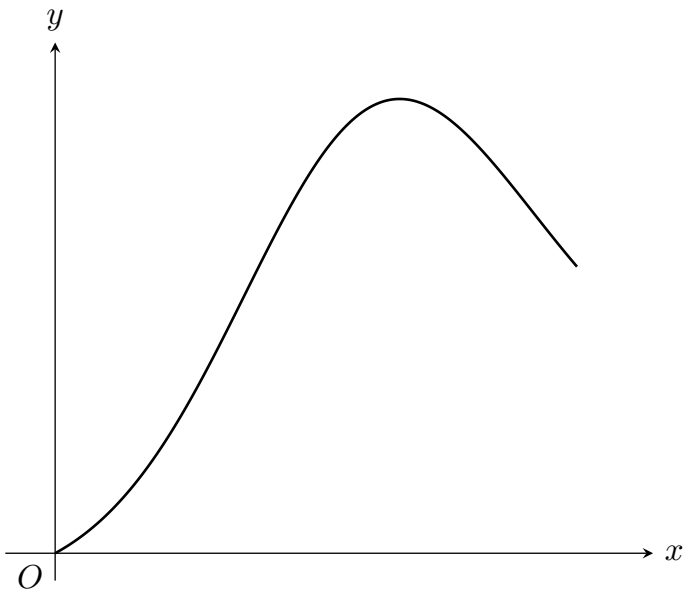
1. The function g is defined by

$$g(x) = x\,e^{\sin x} \qquad (0 \leq x \leq \pi)$$

The diagram below shows the graph of $y = g(x)$.



(a) Use Simpson's rule with 4 strips to estimate

$$\int_0^{\pi} g(x)\,\mathrm{d}x$$

giving your answer to 3 decimal places. **[4]**

(b) Explain how Simpson's rule could be used to find a more accurate estimate of the integral in part (a). **[1]**

Solution

(a) Simpson's rule is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

With $n = 4$ strips,

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

The ordinates, with $y = x\,e^{\sin x}$, are

$$y_0 = g(0) = 0$$

$$y_1 = g\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}e^{\frac{\sqrt{2}}{2}} \approx 1.5929$$

$$y_2 = g\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}e \approx 4.2699$$

$$y_3 = g\left(\tfrac{3\pi}{4}\right) = \tfrac{3\pi}{4}e^{\frac{\sqrt{2}}{2}} \approx 4.7786$$

$$y_4 = g(\pi) = \pi e^0 = \pi \approx 3.1416$$

Substituting into Simpson's rule:

$$\begin{aligned}\int_0^{\pi} g(x)\,\mathrm{d}x &\approx \frac{1}{3}\times\frac{\pi}{4}\{(y_0+y_4)+4(y_1+y_3)+2y_2\}\\ &= \frac{\pi}{12}\{(0+3.1416)+4(1.5929+4.7786)+2\times 4.2699\}\\ &= \frac{\pi}{12}\times 37.1674\\ &= 9.730 \qquad (3\text{ d.p.})\end{aligned}$$

(b) Use Simpson's rule with more strips, keeping the number of strips even, so that the strip width h is smaller: for example, 8 strips with $h = \frac{\pi}{8}$.

2. **(a)** Use Simpson's rule with 6 strips to find an estimate for

$$\int_0^{\pi} \sqrt{\sin x}\,\mathrm{d}x$$

giving your answer to 4 significant figures. **[4]**

You are given that $\int_0^{\pi} \sqrt{\sin x}\,\mathrm{d}x = 2.3963$ to 5 significant figures.

(b) Find the percentage error in your answer to part (a), giving your answer to 2 significant figures, and comment on the accuracy of your estimate. **[2]**

Solution

(a) With 6 strips on $0 \le x \le \pi$, the strip width is

$$h = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

The ordinates $y = \sqrt{\sin x}$ are:

$$\begin{aligned}
x_0 &= 0, & y_0 &= \sqrt{\sin 0} = 0 \\
x_1 &= \tfrac{\pi}{6}, & y_1 &= \sqrt{\tfrac{1}{2}} \approx 0.707107 \\
x_2 &= \tfrac{\pi}{3}, & y_2 &= \sqrt{\tfrac{\sqrt{3}}{2}} \approx 0.930605 \\
x_3 &= \tfrac{\pi}{2}, & y_3 &= \sqrt{1} = 1 \\
x_4 &= \tfrac{2\pi}{3}, & y_4 &= \sqrt{\tfrac{\sqrt{3}}{2}} \approx 0.930605 \\
x_5 &= \tfrac{5\pi}{6}, & y_5 &= \sqrt{\tfrac{1}{2}} \approx 0.707107 \\
x_6 &= \pi, & y_6 &= \sqrt{\sin \pi} = 0
\end{aligned}$$

Simpson's rule is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

Substituting the ordinates:

$$\begin{aligned}
\int_0^{\pi} \sqrt{\sin x}\,\mathrm{d}x &\approx \frac{\pi}{18}\{(0+0) + 4(0.707107 + 1 + 0.707107) + 2(0.930605 + 0.930605)\} \\
&= \frac{\pi}{18}(9.656854 + 3.722419) \\
&= \frac{\pi}{18} \times 13.379273 \\
&= 2.335 \qquad (4 \text{ s.f.})
\end{aligned}$$

(b) The percentage error is

$$\frac{2.3963 - 2.335124}{2.3963} \times 100 = 2.6\% \qquad (2 \text{ s.f.})$$

The estimate is an underestimate, but it is fairly accurate, within about 3% of the true value. The graph of $y = \sqrt{\sin x}$ is very steep near $x = 0$ and $x = \pi$, which the quadratic arcs of Simpson's rule fit poorly, so using more strips would improve the accuracy.

3. The table below shows values of $y = \ln\left(4 + x^3\right)$ for $0 \leq x \leq 3$, with one value missing.

x	0	0.5	1	1.5	2	2.5	3
y	1.386	1.417	1.609		2.485	2.977	3.434

(a) Complete the table by finding the value of y when $x = 1.5$, giving your answer to 3 decimal places. **[1]**

(b) Use Simpson's rule with all the values of y in the completed table to estimate, to 3 significant figures, the value of

$$\int_0^3 \ln\left(4 + x^3\right)\, \mathrm{d}x$$ **[3]**

(c) Using your answer to part (b) and making your method clear, estimate the value of

$$\int_0^3 \ln\left(\frac{\left(4 + x^3\right)^2}{e^x}\right) \mathrm{d}x$$ **[2]**

Solution

(a) When $x = 1.5$, $x^3 = 3.375$, so

$$y = \ln(4 + 3.375) = \ln 7.375 = 1.998 \qquad (3 \text{ d.p.})$$

(b) Simpson's rule is

$$\int_a^b y\, \mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

Here there are $n = 6$ strips with $h = 0.5$. The three sums are

$$y_0 + y_6 = 1.386 + 3.434 = 4.820$$
$$y_1 + y_3 + y_5 = 1.417 + 1.998 + 2.977 = 6.392$$
$$y_2 + y_4 = 1.609 + 2.485 = 4.094$$

so

$$\begin{aligned}\int_0^3 \ln\left(4 + x^3\right) \mathrm{d}x &\approx \frac{1}{3} \times 0.5 \times \{4.820 + 4 \times 6.392 + 2 \times 4.094\} \\ &= \frac{1}{6} \times (4.820 + 25.568 + 8.188) \\ &= \frac{38.576}{6} \\ &= 6.43 \qquad (3 \text{ s.f.})\end{aligned}$$

(c) Using the laws of logarithms, with $\ln e^x = x$:

$$\ln\left(\frac{\left(4 + x^3\right)^2}{e^x}\right) = 2\ln\left(4 + x^3\right) - x$$

so

$$\int_0^3 \ln\left(\frac{\left(4+x^3\right)^2}{e^x}\right)\mathrm{d}x = 2\int_0^3 \ln\left(4+x^3\right)\mathrm{d}x - \int_0^3 x\,\mathrm{d}x$$

The second integral is exact:

$$\int_0^3 x\,\mathrm{d}x = \left[\tfrac{1}{2}x^2\right]_0^3 = \frac{9}{2}$$

Using part (b), the estimate is

$$2\times\frac{38.576}{6} - \frac{9}{2} \approx 12.859 - 4.5 \approx 8.36$$

4. The function f is defined by $f(x) = \sqrt{1+x^3}$.
The table below shows some values of $f(x)$, given to 4 decimal places.

x	0	0.2	0.4	0.6	0.8	1.0	1.2
$f(x)$	1.0000	1.0040		1.1027		1.4142	1.6517

(a) Complete the table, and use Simpson's rule with all the values in the table to find an estimate for

$$\int_0^{1.2} \sqrt{1+x^3}\,\mathrm{d}x$$

giving your answer to 3 decimal places. **[4]**

(b) Hence find an estimate for

$$\int_0^{1.2} \left(x^2 + 2\sqrt{1+x^3}\right)\mathrm{d}x$$ **[2]**

Solution

(a) The missing values are

$$f(0.4) = \sqrt{1+0.4^3} = \sqrt{1.064} = 1.0315, \qquad f(0.8) = \sqrt{1+0.8^3} = \sqrt{1.512} = 1.2296$$

The table has 7 ordinates, so there are 6 strips of width $h = 0.2$. Simpson's rule is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

Substituting the values from the table:

$$\begin{aligned}\int_0^{1.2} \sqrt{1+x^3}\,\mathrm{d}x &\approx \frac{0.2}{3}\Big\{(1.0000+1.6517) + 4(1.0040+1.1027+1.4142) \\ &\qquad + 2(1.0315+1.2296)\Big\} \\ &= \frac{0.2}{3}(2.6517+14.0836+4.5222) \\ &= \frac{0.2}{3}\times 21.2575 \\ &= 1.417 \qquad (3\text{ d.p.})\end{aligned}$$

(b) Splitting the integral:

$$\begin{aligned}\int_0^{1.2}\left(x^2+2\sqrt{1+x^3}\right)\mathrm{d}x &= \int_0^{1.2} x^2\,\mathrm{d}x + 2\int_0^{1.2}\sqrt{1+x^3}\,\mathrm{d}x \\ &\approx \left[\frac{x^3}{3}\right]_0^{1.2} + 2\times 1.417 \\ &= 0.576 + 2.834 \\ &= 3.410\end{aligned}$$

5. The table below shows some values of y, given to 2 decimal places, where $y = \ln(3 - \cos 2x)$.

x	0.3	0.5	0.7	0.9	1.1	1.3	1.5
y	0.78	0.90		1.17		1.35	1.38

(a) Complete the table, giving the values of y corresponding to $x = 0.7$ and $x = 1.1$ to 2 decimal places. **[2]**

(b) Use Simpson's rule, with all the values of y in the completed table, to estimate, to 3 decimal places, the value of

$$\int_{0.3}^{1.5} \ln(3 - \cos 2x)\,\mathrm{d}x \qquad \textbf{[3]}$$

(c) Using your answer to part (b), and making your method clear, estimate, to 2 decimal places, the value of

$$\int_{0.3}^{1.5} \ln\left(1 + \sin^2 x\right)\,\mathrm{d}x \qquad \textbf{[2]}$$

Solution

(a) With the calculator in radians:

$$x = 0.7: \quad y = \ln(3 - \cos 1.4) \approx 1.04 \qquad (2 \text{ d.p.})$$
$$x = 1.1: \quad y = \ln(3 - \cos 2.2) \approx 1.28 \qquad (2 \text{ d.p.})$$

(b) The strip width is $h = 0.2$, and there are 6 strips. Simpson's rule is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

Substituting the values from the completed table:

$$\begin{aligned}\int_{0.3}^{1.5} \ln(3 - \cos 2x)\,\mathrm{d}x &\approx \frac{0.2}{3}\{(0.78 + 1.38) + 4(0.90 + 1.17 + 1.35) + 2(1.04 + 1.28)\} \\ &= \frac{0.2}{3}(2.16 + 13.68 + 4.64) \\ &= \frac{0.2}{3} \times 20.48 \\ &\approx 1.365 \qquad (3 \text{ d.p.})\end{aligned}$$

(c) Using the double angle formula $\cos 2x = 1 - 2\sin^2 x$:

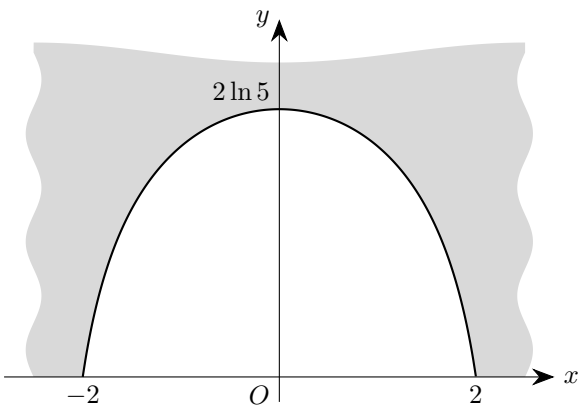
$$3 - \cos 2x = 3 - \left(1 - 2\sin^2 x\right) = 2\left(1 + \sin^2 x\right)$$

so

$$\ln(3 - \cos 2x) = \ln 2 + \ln\left(1 + \sin^2 x\right) \quad \Longrightarrow \quad \ln\left(1 + \sin^2 x\right) = \ln(3 - \cos 2x) - \ln 2$$

Integrating from 0.3 to 1.5 and using part (b):

$$\begin{aligned}\int_{0.3}^{1.5} \ln\left(1+\sin^2 x\right) \mathrm{d}x &= \int_{0.3}^{1.5} \ln(3-\cos 2x)\,\mathrm{d}x - \int_{0.3}^{1.5} \ln 2\,\mathrm{d}x \\ &\approx 1.365 - (1.5-0.3)\ln 2 \\ &= 1.365 - 1.2\ln 2 \\ &\approx 0.53 \qquad (2 \text{ d.p.})\end{aligned}$$



6. The diagram above shows a sketch of the vertical cross-section of the entrance to a tunnel. The entrance is 4 metres wide at its base.
The cross-section can be modelled by the region between the x-axis and the curve with equation $y = f(x)$, where x and y are measured in metres and

$$f(x) = 2\ln\left(5 - x^2\right), \qquad -2 \leq x \leq 2$$

(a) Use Simpson's rule with 8 strips to estimate the area of the cross-section, giving your answer in m^2 to 4 significant figures. **[4]**

(b) A wooden door of uniform thickness 75 mm is to be made to seal the tunnel entrance. The door contains a rectangular glass panel, 0.6 m wide and 1.2 m tall, which is not made of wood.
Using your answer to part (a), estimate the volume of wood required for the door, giving your answer in m^3 to 3 significant figures. **[2]**

Solution

(a) Simpson's rule is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

With $n = 8$ strips from $x = -2$ to $x = 2$, the strip width is

$$h = \frac{2 - (-2)}{8} = 0.5$$

The area of the cross-section is $\int_{-2}^{2} 2\ln\left(5 - x^2\right)\mathrm{d}x$. Since $f(-x) = f(x)$, the ordinates are symmetric about $x = 0$:

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	0	2.02320	2.77259	3.11629	3.21888	3.11629	2.77259	2.02320	0

Here, for example, $y_1 = f(-1.5) = 2\ln 2.75$, $y_2 = f(-1) = 2\ln 4$, $y_3 = f(-0.5) = 2\ln 4.75$ and $y_4 = f(0) = 2\ln 5$, while $y_0 = y_8 = 2\ln 1 = 0$.

The sums needed are

$$y_0 + y_8 = 0$$

$$y_1 + y_3 + y_5 + y_7 = 2(2.02320 + 3.11629) = 10.27898$$

$$y_2 + y_4 + y_6 = 2(2.77259) + 3.21888 = 8.76405$$

Substituting into Simpson's rule:

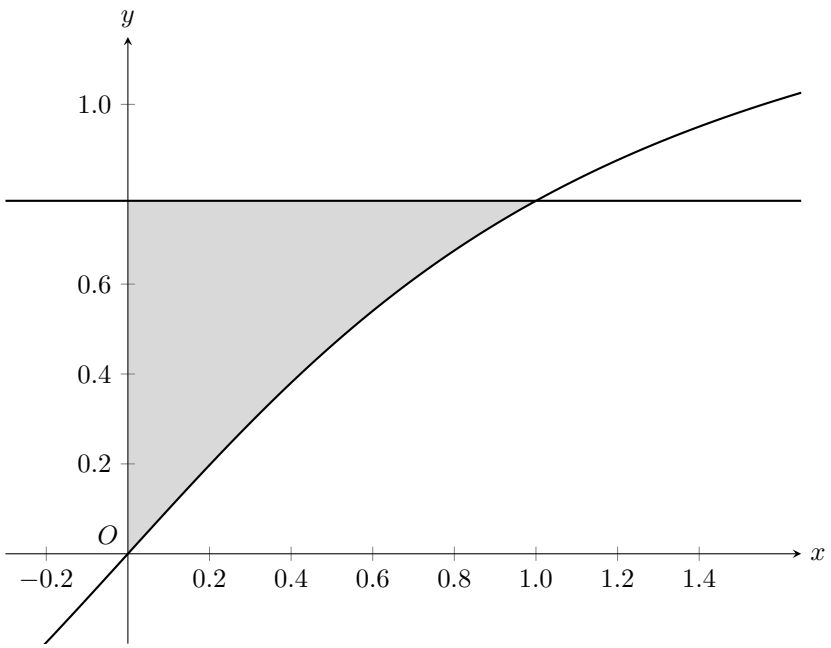
$$\begin{aligned}\text{Area} &\approx \frac{1}{3} \times 0.5 \times \{0 + 4 \times 10.27898 + 2 \times 8.76405\} \\ &= \frac{1}{6} \times (41.11593 + 17.52811) \\ &= 9.774\,\text{m}^2 \qquad (4\text{ s.f.})\end{aligned}$$

(b) The door seals the entrance, so its face has the same area as the cross-section. Removing the glass panel, the area of wood is

$$9.774 - 0.6 \times 1.2 = 9.054\,\text{m}^2$$

The thickness is $75\,\text{mm} = 0.075\,\text{m}$, so the volume of wood is

$$9.054 \times 0.075 = 0.679\,\text{m}^3 \qquad (3\text{ s.f.})$$



7. The diagram above shows part of the curve $y = \arctan x$ and the line $y = \frac{\pi}{4}$, which meet at the point where $x = 1$.
The shaded region, bounded by the curve, the line $y = \frac{\pi}{4}$ and the y-axis, is rotated through 2π radians about the x-axis to form a solid.
Use Simpson's rule with 6 strips to estimate the volume of the solid. Give your answer to 4 decimal places. **[6]**

Solution

Rotating the region between the line $y = \frac{\pi}{4}$ and the curve $y = \arctan x$ for $0 \leq x \leq 1$ about the x-axis gives the solid formed by the line minus the solid formed by the curve. Using $V = \pi \int_a^b y^2 \, \mathrm{d}x$ for each:

$$V = \pi \int_0^1 \left(\frac{\pi}{4}\right)^2 \mathrm{d}x - \pi \int_0^1 (\arctan x)^2 \, \mathrm{d}x = \pi \int_0^1 \left(\frac{\pi^2}{16} - (\arctan x)^2\right) \mathrm{d}x$$

Let $f(x) = \frac{\pi^2}{16} - (\arctan x)^2$. Simpson's rule is

$$\int_a^b f(x) \, \mathrm{d}x \approx \frac{1}{3} h\{(f_0 + f_n) + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2})\}, \quad h = \frac{b-a}{n}, \ n \text{ even}$$

With 6 strips, $n = 6$ and $h = \frac{1-0}{6} = \frac{1}{6}$. The ordinates are

$$f_0 = f(0) = \frac{\pi^2}{16} \approx 0.616850$$

$$f_1 = f\left(\tfrac{1}{6}\right) \approx 0.589576$$

$$f_2 = f\left(\tfrac{1}{3}\right) \approx 0.513327$$

$$f_3 = f\left(\tfrac{1}{2}\right) \approx 0.401881$$

$$f_4 = f\left(\tfrac{2}{3}\right) \approx 0.271103$$

$$f_5 = f\left(\tfrac{5}{6}\right) \approx 0.134189$$

$$f_6 = f(1) = \frac{\pi^2}{16} - \left(\frac{\pi}{4}\right)^2 = 0$$

The sums needed are

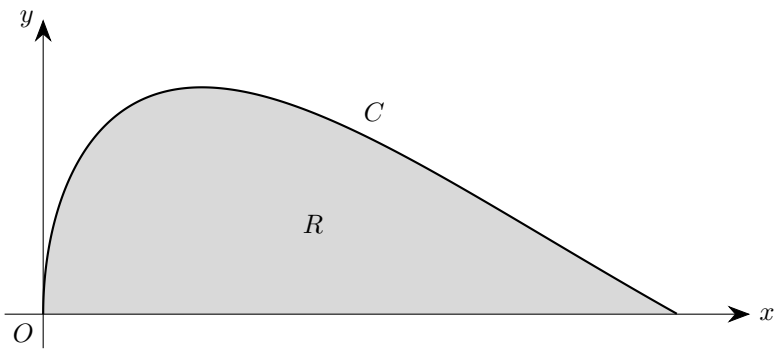
$$f_0 + f_6 \approx 0.616850$$
$$f_1 + f_3 + f_5 \approx 1.125646$$
$$f_2 + f_4 \approx 0.784430$$

Applying Simpson's rule:

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &\approx \frac{1}{3} \times \frac{1}{6}\{0.616850 + 4 \times 1.125646 + 2 \times 0.784430\} \\ &= \frac{1}{18} \times 6.688294 \\ &\approx 0.371572\end{aligned}$$

Multiplying by π:

$$V \approx \pi \times 0.371572 \approx 1.1673 \qquad (4 \text{ d.p.})$$



8. The diagram above shows the curve C with parametric equations

$$x = t^2, \qquad y = \sin t, \qquad 0 \leq t \leq \pi$$

The region R, shown shaded, is bounded by C and the x-axis.

(a) Show that the area of R is given by

$$\int_0^{\pi} 2t \sin t \, \mathrm{d}t \qquad \textbf{[2]}$$

(b) Use Simpson's rule with 4 strips to estimate the area of R, giving your answer to 3 decimal places. **[4]**

(c) Use integration by parts to show that the exact area of R is 2π. **[3]**

Solution

(a) The diagram shows that C lies on or above the x-axis, so the area of R is

$$A = \int y \, \mathrm{d}x = \int y \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$$

Differentiating $x = t^2$ gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$$

The curve meets the x-axis at $x = 0$, where $t = 0$, and at $x = \pi^2$, where $t = \pi$. So

$$A = \int_0^{\pi} \sin t \times 2t \, \mathrm{d}t = \int_0^{\pi} 2t \sin t \, \mathrm{d}t$$

as required.

(b) Simpson's rule is

$$\int_a^b f(t)\mathrm{d}t \approx \frac{1}{3}h\{(f_0 + f_n) + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2})\}, \quad h = \frac{b-a}{n}, \; n \text{ even}$$

Here $f(t) = 2t \sin t$, $n = 4$ and

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

The ordinates are

$$f_0 = f(0) = 0$$

$$f_1 = f\left(\tfrac{\pi}{4}\right) = \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4} \approx 1.11072$$

$$f_2 = f\left(\tfrac{\pi}{2}\right) = \pi \times 1 = \pi \approx 3.14159$$

$$f_3 = f\left(\tfrac{3\pi}{4}\right) = \frac{3\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{3\pi\sqrt{2}}{4} \approx 3.33216$$

$$f_4 = f(\pi) = 2\pi \times 0 = 0$$

Substituting into Simpson's rule:

$$A \approx \frac{1}{3} \times \frac{\pi}{4}\left\{(0+0) + 4\left(\frac{\pi\sqrt{2}}{4} + \frac{3\pi\sqrt{2}}{4}\right) + 2\pi\right\}$$

$$= \frac{\pi}{12}\left(4\sqrt{2}\,\pi + 2\pi\right)$$

$$= \frac{\pi^2\left(1+2\sqrt{2}\right)}{6}$$

$$\approx 6.298 \qquad (3 \text{ d.p.})$$

(c) The integration by parts formula is

$$\int u\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}t$$

Take $u = 2t$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \sin t$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = 2$ and $v = -\cos t$. Then

$$\int 2t\sin t\,\mathrm{d}t = -2t\cos t - \int(-\cos t) \times 2\,\mathrm{d}t$$

$$= -2t\cos t + \int 2\cos t\,\mathrm{d}t$$

$$= -2t\cos t + 2\sin t + c$$

Using part (a), the area of R is

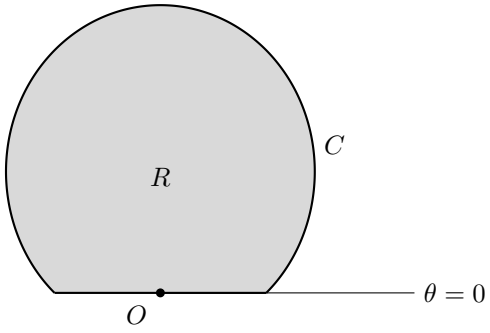
$$\int_0^{\pi} 2t\sin t\,\mathrm{d}t = \left[-2t\cos t + 2\sin t\right]_0^{\pi}$$

$$= (-2\pi\cos\pi + 2\sin\pi) - (0 + 2\sin 0)$$

$$= (2\pi + 0) - 0$$

$$= 2\pi$$

as required.



9. The diagram above shows the curve C with polar equation

$$r = e^{\sin\theta}, \qquad 0 \leq \theta \leq \pi$$

The region R, shown shaded, is enclosed by C and the line segment joining the points of C where $\theta = 0$ and $\theta = \pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole O. **[3]**

(b) Use Simpson's rule with 6 strips to estimate the area of R, giving your answer to 3 significant figures. **[5]**

Solution

(a) Differentiating with respect to θ gives

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \cos\theta\, e^{\sin\theta}$$

At a stationary value of r, $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$. Since $e^{\sin\theta} > 0$, this needs $\cos\theta = 0$, so for $0 \leq \theta \leq \pi$

$$\theta = \frac{\pi}{2}$$

Here $r = e^{\sin\frac{\pi}{2}} = e$, and the diagram shows that this is the point furthest from O. Its polar coordinates are

$$\left(e, \frac{\pi}{2}\right)$$

(b) The area enclosed by a polar curve is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta$$

As θ runs from 0 to π, the radius sweeps out exactly the region R, and $r^2 = \left(e^{\sin\theta}\right)^2 = e^{2\sin\theta}$, so

$$A = \frac{1}{2}\int_{0}^{\pi} e^{2\sin\theta}\,\mathrm{d}\theta$$

Simpson's rule is

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}, \quad h = \frac{b-a}{n},\ n \text{ even}$$

With $n = 6$ strips, $h = \frac{\pi}{6}$. The ordinates of $y = e^{2\sin\theta}$ are

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	1	e	$e^{\sqrt{3}}$	e^2	$e^{\sqrt{3}}$	e	1

Substituting into Simpson's rule:

$$\begin{aligned}\int_0^{\pi} e^{2\sin\theta}\,\mathrm{d}\theta &\approx \frac{1}{3}\times\frac{\pi}{6}\left\{(1+1)+4\left(e+e^2+e\right)+2\left(e^{\sqrt{3}}+e^{\sqrt{3}}\right)\right\}\\ &= \frac{\pi}{18}\left(2+8e+4e^2+4e^{\sqrt{3}}\right)\\ &\approx 13.249\end{aligned}$$

So the area of R is

$$A \approx \frac{\pi}{36}\left(2+8e+4e^2+4e^{\sqrt{3}}\right) \approx 6.62 \qquad (3\text{ s.f.})$$

10. In this question, $h > 0$ and a, b and c are constants. The curve with equation $y = ax^2 + bx + c$ passes through the points $(-h, y_0)$, $(0, y_1)$ and (h, y_2).

(a) Show that

$$\int_{-h}^{h} \left(ax^2 + bx + c\right) \mathrm{d}x = \frac{2h}{3}\left(ah^2 + 3c\right) \quad \textbf{[2]}$$

(b) By writing y_0, y_1 and y_2 in terms of a, b, c and h, show that

$$\int_{-h}^{h} \left(ax^2 + bx + c\right) \mathrm{d}x = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) \quad \textbf{[3]}$$

(c) Simpson's rule fits a curve of this form through each set of three consecutive ordinates. Explain why Simpson's rule needs an even number of strips. **[1]**

(d) Let T_1 and T_2 be the estimates of $\int_{-h}^{h} \left(ax^2 + bx + c\right) \mathrm{d}x$ given by the trapezium rule with 1 strip and with 2 strips respectively. Show that

$$\frac{4T_2 - T_1}{3} = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) \quad \textbf{[3]}$$

Solution

(a) Integrating term by term:

$$\begin{aligned}
\int_{-h}^{h} \left(ax^2 + bx + c\right) \mathrm{d}x &= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h} \\
&= \left(\frac{ah^3}{3} + \frac{bh^2}{2} + ch\right) - \left(-\frac{ah^3}{3} + \frac{bh^2}{2} - ch\right) \\
&= \frac{2ah^3}{3} + 2ch \\
&= \frac{2h}{3}\left(ah^2 + 3c\right)
\end{aligned}$$

as required.

(b) Substituting $x = -h$, $x = 0$ and $x = h$ into $y = ax^2 + bx + c$ gives

$$y_0 = ah^2 - bh + c, \qquad y_1 = c, \qquad y_2 = ah^2 + bh + c$$

Then

$$\begin{aligned}
y_0 + 4y_1 + y_2 &= \left(ah^2 - bh + c\right) + 4c + \left(ah^2 + bh + c\right) \\
&= 2ah^2 + 6c \\
&= 2\left(ah^2 + 3c\right)
\end{aligned}$$

so

$$\frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{2h}{3}\left(ah^2 + 3c\right)$$

By part (a), this is $\int_{-h}^{h} \big(ax^2 + bx + c\big)\mathrm{d}x$, as required.

(c) Each quadratic curve passes through three consecutive ordinates, so it spans two adjacent strips. The whole interval is covered by these curves, each taking a new pair of strips, so the strips are used two at a time and their number must be even.

(d) With 1 strip, the width is $2h$ and the ordinates are y_0 and y_2, so

$$T_1 = \frac{1}{2} \times 2h(y_0 + y_2) = h(y_0 + y_2)$$

With 2 strips, each of width h, the ordinates are y_0, y_1 and y_2, so

$$T_2 = \frac{h}{2}(y_0 + 2y_1 + y_2)$$

Then

$$\begin{aligned} 4T_2 - T_1 &= 2h(y_0 + 2y_1 + y_2) - h(y_0 + y_2) \\ &= h(2y_0 + 4y_1 + 2y_2 - y_0 - y_2) \\ &= h(y_0 + 4y_1 + y_2) \end{aligned}$$

so

$$\frac{4T_2 - T_1}{3} = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

as required.