OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

Series Solutions of Differential Equations

Worked solutions 12 questions · 105 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*6 marks*	2
Question 2	*8 marks*	4
Question 3	*7 marks*	6
Question 4	*8 marks*	8
Question 5	*6 marks*	10
Question 6	*10 marks*	11
Question 7	*10 marks*	13
Question 8	*9 marks*	15
Question 9	*10 marks*	17
Question 10	*10 marks*	19
Question 11	*11 marks*	21
Question 12	*10 marks*	24

1. The variable y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos 2x$$

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 0$, find a series expansion for y in terms of x, up to and including the term in x^4. **[6]**

Solution

Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots$$

With $f(x) = y$, the values of the derivatives of y up to the fourth are needed at $x = 0$. Rearranging the differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \cos 2x - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - y$$

When $x = 0$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, so

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 - 0 - 2 = -1$$

Differentiating, using the product rule on $2x\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -2\sin 2x - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} \\ &= -2\sin 2x - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\end{aligned}$$

When $x = 0$:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0 - 3 \times (-1) - 0 = 3$$

Differentiating again:

$$\begin{aligned}\frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= -4\cos 2x - 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \\ &= -4\cos 2x - 5\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\end{aligned}$$

When $x = 0$:

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -4 - 5 \times (-1) - 0 = 1$$

Substituting these values into Maclaurin's series:

$$\begin{aligned}y &\approx 2 + (-1)x + \left(-\frac{1}{2!}\right)x^2 + \frac{3}{3!}x^3 + \frac{1}{4!}x^4 \\ &= 2 - x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{24}x^4\end{aligned}$$

Alternative Solution:

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Differentiating term by term:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

When $x = 0$, $y = a_0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = a_1$, so $a_0 = 2$ and $a_1 = -1$.

Replacing n by $n + 2$, so that the power is x^n, gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Also $2x\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} 2n a_n x^n$, which may start at $n = 0$, since the term added is zero. Substituting into the differential equation:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \cos 2x$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (2n+1) a_n] x^n = \cos 2x$$

The right-hand side is written out, not as a sum, since its series holds only even powers of x. The series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \cdots$$

so

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots = 1 - 2x^2 + \cdots$$

Equating the coefficients of x^0, x^1 and x^2 in turn, that is $n = 0, 1, 2$:

$$2a_2 + a_0 = 1 \quad \Longrightarrow \quad a_2 = \frac{1-2}{2} = -\frac{1}{2}$$

$$6a_3 + 3a_1 = 0 \quad \Longrightarrow \quad a_3 = \frac{3}{6} = \frac{1}{2}$$

$$12a_4 + 5a_2 = -2 \quad \Longrightarrow \quad a_4 = \frac{1}{12}\left(-2 + \frac{5}{2}\right) = \frac{1}{24}$$

Hence

$$y \approx 2 - x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{24}x^4$$

2. The variable y satisfies the differential equation

$$(2+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y^2 - 3x\frac{\mathrm{d}y}{\mathrm{d}x}$$

(a) Show that

$$(2+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (3x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2y-3)\frac{\mathrm{d}y}{\mathrm{d}x} \quad \textbf{[3]}$$

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 0$,

(b) find a series solution for y in ascending powers of x, up to and including the term in x^3. **[3]**

(c) Hence estimate the value of y when $x = 0.2$, giving your answer to 3 decimal places. **[2]**

Solution

(a) Differentiate both sides of the equation with respect to x, using the product rule on the left-hand side and the chain rule on y^2:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

$$(2+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(2+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (3x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2y-3)\frac{\mathrm{d}y}{\mathrm{d}x}$$

as required.

(b) When $x = 0$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$. Substituting into the original differential equation:

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2^2 - 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

Substituting $x = 0$, $y = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$ into the result of part (a):

$$2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 1 \times 2 = (2 \times 2 - 3) \times (-1)$$

$$2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -3$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\frac{3}{2}$$

Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Applying it with $f(x) = y$:

$$y \approx 2 + x(-1) + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times \left(-\frac{3}{2}\right)$$

$$= 2 - x + x^2 - \frac{1}{4}x^3$$

(c) Substituting $x = 0.2$ into the series from part (b):

$$\begin{aligned} y &\approx 2 - 0.2 + 0.2^2 - \frac{1}{4} \times 0.2^3 \\ &= 2 - 0.2 + 0.04 - 0.002 \\ &= 1.838 \qquad (3 \text{ d.p.}) \end{aligned}$$

3. A function y satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = e^{2x-2}$$

It is given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 1$.

(a) Find a Taylor series for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$. **[5]**

(b) Hence find an estimate of the value of y when $x = 1.1$, giving your answer to 3 significant figures. **[2]**

Solution

(a) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Here $f(x) = y$ and $a = 1$, so the values of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 1$ are needed.

Substituting $x = 1$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ into the differential equation gives

$$1 \times \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2 \times 1 \times 2 - 1^2 = e^0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3 = 1$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2$$

Differentiating the differential equation with respect to x, using the product rule on $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $2y\frac{\mathrm{d}y}{\mathrm{d}x}$, and the chain rule on y^2:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2e^{2x-2}$$

When $x = 1$, substituting $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2$ gives

$$-2 + \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2 \times 2^2 + 2 \times 1 \times (-2) - 2 \times 1 \times 2 = 2e^0$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2 = 2$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 4$$

Substituting into the Taylor series:

$$y \approx 1 + 2(x-1) + \left(-\frac{2}{2!}\right)(x-1)^2 + \frac{4}{3!}(x-1)^3$$

$$= 1 + 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3$$

(b) When $x = 1.1$, $x - 1 = 0.1$, so using part (a):

$$y \approx 1 + 2 \times 0.1 - 0.1^2 + \frac{2}{3} \times 0.1^3$$

$$\approx 1.19 \qquad (3 \text{ s.f.})$$

4. A function y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

(a) Show that

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \textbf{[3]}$$

When $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$.

(b) Find the Taylor series solution for y, in ascending powers of $(x-1)$ up to and including the term in $(x-1)^4$, giving each coefficient in simplest form. **[5]**

Solution

(a) Differentiate the differential equation with respect to x, using the product rule on $y\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

So differentiating both sides of the equation gives

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

Rearranging gives

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

as required.

(b) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(4)}(a) + \cdots$$

which is used here with $f(x) = y$ and $a = 1$.

When $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$. Rearranging the differential equation and substituting:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x - y\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \times 1 - 2 \times (-1) = 4$$

Using part (a):

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2 - (-1)^2 - 2 \times 4 = -7$$

Now differentiate the result of part (a), using the product rule on $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$:

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = -2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)$$

$$= -3\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

Substituting the values when $x = 1$ gives

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -3 \times (-1) \times 4 - 2 \times (-7) = 12 + 14 = 26$$

Substituting these values into the Taylor series:

$$\begin{aligned} y &= 2 + (x-1)(-1) + \frac{(x-1)^2}{2!} \times 4 + \frac{(x-1)^3}{3!} \times (-7) + \frac{(x-1)^4}{4!} \times 26 + \cdots \\ &= 2 - (x-1) + 2(x-1)^2 - \frac{7}{6}(x-1)^3 + \frac{13}{12}(x-1)^4 + \cdots \end{aligned}$$

5. The displacement x metres of a particle at time t seconds satisfies the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x\frac{\mathrm{d}x}{\mathrm{d}t} + 2\cos x = 0$$

At time $t = 0$, the particle is at $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 3$.
Find a Taylor series solution for x in ascending powers of t, up to and including the term in t^4. **[6]**

Solution

Maclaurin's series, with t as the variable, is

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \frac{t^3}{3!}f'''(0) + \frac{t^4}{4!}f^{(4)}(0) + \cdots$$

Here $f(t) = x$, so the values of x and its first four derivatives are needed at $t = 0$. When $t = 0$, $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 3$.

Rearranging the differential equation gives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x\frac{\mathrm{d}x}{\mathrm{d}t} - 2\cos x$$

so when $t = 0$:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -0 \times 3 - 2\cos 0 = -2$$

Differentiate with respect to t, using the product rule on $x\frac{\mathrm{d}x}{\mathrm{d}t}$ and the chain rule on $\cos x$:

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = -\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\sin x\frac{\mathrm{d}x}{\mathrm{d}t}$$

so when $t = 0$:

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = -3^2 - 0 \times (-2) + 2\sin 0 \times 3 = -9$$

Differentiate again:

$$\begin{aligned}\frac{\mathrm{d}^4x}{\mathrm{d}t^4} &= -2\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - x\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + 2\cos x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 2\sin x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \\ &= -3\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - x\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + 2\cos x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 2\sin x\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\end{aligned}$$

so when $t = 0$:

$$\frac{\mathrm{d}^4x}{\mathrm{d}t^4} = -3 \times 3 \times (-2) - 0 \times (-9) + 2\cos 0 \times 3^2 + 2\sin 0 \times (-2) = 18 + 18 = 36$$

Substituting these values into Maclaurin's series:

$$\begin{aligned}x &= 0 + 3t + \frac{t^2}{2!} \times (-2) + \frac{t^3}{3!} \times (-9) + \frac{t^4}{4!} \times 36 + \cdots \\ &= 3t - t^2 - \frac{3}{2}t^3 + \frac{3}{2}t^4 + \cdots\end{aligned}$$

6. The variable y satisfies the differential equation

$$(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2xy=4$$

At $x=0$, $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$.

(a) Show that, at $x=0$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{7}{2}$. **[6]**

(b) Find a series solution for y in ascending powers of x, up to and including the term in x^3, and use it to estimate the value of y when $x=0.2$, giving your answer to 3 decimal places. **[4]**

Solution

(a) Substituting $x=0$, $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ into the differential equation:

$$(0+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(-1)^2-2\times0\times2=4$$

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3}{2}$$

Now differentiate the differential equation with respect to x, term by term, using the product rule and the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]=\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]=2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[2xy]=2y+2x\frac{\mathrm{d}y}{\mathrm{d}x}$$

Since the right-hand side is constant, the differentiated equation is

$$\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2y-2x\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

At $x=0$, substituting $y=2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{3}{2}$:

$$(1-1)\times\frac{3}{2}+2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\times(-1)\times\frac{3}{2}-2\times2-0=0$$

$$2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-3-4=0$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{7}{2}$$

as required.

(b) Maclaurin's series is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Applying it to y, with the values at $x = 0$ from part (a):

$$\begin{aligned} y &\approx 2 + x \times (-1) + \frac{x^2}{2!} \times \frac{3}{2} + \frac{x^3}{3!} \times \frac{7}{2} \\ &= 2 - x + \frac{3}{4}x^2 + \frac{7}{12}x^3 \end{aligned}$$

When $x = 0.2$:

$$\begin{aligned} y &\approx 2 - 0.2 + \frac{3}{4} \times 0.04 + \frac{7}{12} \times 0.008 \\ &= 2 - 0.2 + 0.03 + \frac{7}{1500} \\ &= \frac{688}{375} \\ &\approx 1.835 \qquad \text{(3 d.p.)} \end{aligned}$$

7. The function y satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x^2y = 0$$

It is given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 1$.

(a) Determine the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 1$. **[6]**

(b) Hence find the series expansion for y about $x = 1$, in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$, giving each coefficient in simplest form, and use your expansion to estimate the value of y at $x = 1.1$, giving your answer to 3 decimal places. **[4]**

Solution

(a) Substituting $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ into the differential equation:

$$1 \times \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2 \times 2 \times (-1)^2 + 1^2 \times 2 = 0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4 + 2 = 0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

Now differentiate each term of the differential equation with respect to x, using the product rule and the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right) = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + 4y\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y) = 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

So the differentiated equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 - 4y\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Substituting $x = 1$, $y = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$:

$$2 + \frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2 \times (-1)^3 - 4 \times 2 \times (-1) \times 2 + 2 \times 1 \times 2 + 1^2 \times (-1) = 0$$

$$2 + \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2 + 16 + 4 - 1 = 0$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -23$$

(b) The Taylor series about $x = a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Take $f(x) = y$ and $a = 1$. At $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, and from part (a), $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$ and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -23$, so

$$\begin{aligned} y &\approx 2 + (x-1) \times (-1) + \frac{(x-1)^2}{2} \times 2 + \frac{(x-1)^3}{6} \times (-23) \\ &= 2 - (x-1) + (x-1)^2 - \frac{23}{6}(x-1)^3 \end{aligned}$$

At $x = 1.1$, $x - 1 = 0.1$, so

$$\begin{aligned} y &\approx 2 - 0.1 + 0.1^2 - \frac{23}{6} \times 0.1^3 \\ &\approx 1.906 \qquad (3 \text{ d.p.}) \end{aligned}$$

8. The variable y satisfies the differential equation

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2 \qquad (*)$$

(a) Show that

$$(1+x)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = (ay+b)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + c\,\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

where a, b and c are integers to be determined. **[4]**

(b) Given that $y = 1$ when $x = 0$, find a series solution of $(*)$ in ascending powers of x, up to and including the term in x^4. Hence find an approximate value of y when $x = 0.2$, giving your answer to 3 decimal places. **[5]**

Solution

(a) Differentiating $(*)$ with respect to x, using the product rule on the left and the chain rule on the right:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + (1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + (2y-1)\frac{\mathrm{d}y}{\mathrm{d}x}$$

Differentiating again:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (2y-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

$$(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (2y-2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

Differentiating a third time:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (1+x)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 4\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2y-2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

$$(1+x)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = (2y-3)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 6\,\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

This is in the required form with $a = 2$, $b = -3$, $c = 6$.

(b) When $x = 0$, $y = 1$. Substituting into $(*)$, the second-derivative equation, the third-derivative equation and the result of part (a) in turn, each with $1 + x = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 + 1^2 = 1$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + (2-1)\times 1 = 2$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\times 1^2 + (2-2)\times 2 = 2$$

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = (2-3)\times 2 + 6\times 1\times 2 = 10$$

Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots$$

Applying it to y with the values above:

$$\begin{aligned} y &\approx 1 + 1 \times x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \frac{10}{4!}x^4 \\ &= 1 + x + x^2 + \frac{1}{3}x^3 + \frac{5}{12}x^4 \end{aligned}$$

Hence, when $x = 0.2$:

$$\begin{aligned} y &\approx 1 + 0.2 + 0.2^2 + \frac{1}{3} \times 0.2^3 + \frac{5}{12} \times 0.2^4 \\ &= 1.24 + \frac{0.008}{3} + \frac{0.008}{12} \\ &= \frac{373}{300} \\ &\approx 1.243 \qquad \text{(3 d.p.)} \end{aligned}$$

9. The function y satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 2$$

(a) Show that

$$x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + a\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

where a and b are constants to be found. **[4]**

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 1$,

(b) find a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^4$. Write each term in its simplest form. **[3]**

(c) Use your answer to part (b) to find an approximate value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0.9$, giving your answer to 3 decimal places. **[3]**

Solution

(a) Differentiate the differential equation with respect to x, using the product rule on $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and xy:

$$x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

$$x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

Differentiate again:

$$x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + 4\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

This is in the required form with $a = 4$ and $b = 2$.

(b) The Taylor series about $x = \alpha$ is

$$f(x) = f(\alpha) + (x-\alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2!}f''(\alpha) + \frac{(x-\alpha)^3}{3!}f'''(\alpha) + \frac{(x-\alpha)^4}{4!}f^{(4)}(\alpha) + \cdots$$

Take $f(x) = y$ and $\alpha = 1$. When $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$. Substituting these into the original equation:

$$1 \times \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(-1) + 1 \times 2 = 2 \quad \Longrightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

Substituting into the first differentiated equation from part (a):

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3 \times 2 + 1 \times (-1) + 2 = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -7$$

Substituting into the result of part (a):

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + 4 \times (-7) + 1 \times 2 + 2 \times (-1) = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 28$$

Hence

$$y \approx 2 + (x-1)(-1) + \frac{(x-1)^2}{2} \times 2 + \frac{(x-1)^3}{6} \times (-7) + \frac{(x-1)^4}{24} \times 28$$
$$= 2 - (x-1) + (x-1)^2 - \frac{7}{6}(x-1)^3 + \frac{7}{6}(x-1)^4$$

(c) Differentiating the series from part (b):

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx -1 + 2(x-1) - \frac{7}{2}(x-1)^2 + \frac{14}{3}(x-1)^3$$

When $x = 0.9$, $x - 1 = -0.1$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx -1 + 2(-0.1) - \frac{7}{2}(-0.1)^2 + \frac{14}{3}(-0.1)^3$$
$$\approx -1 - 0.2 - 0.035 - 0.004\,667$$
$$\approx -1.240 \qquad (3 \text{ d.p.})$$

10. Given that

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=6x$$

(a) show that

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{6}{y}+\frac{10}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3-\frac{30x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} \quad \textbf{[5]}$$

It is also given that $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ at $x=1$.

(b) Find a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$, simplifying the coefficients where possible. **[3]**

(c) Use your series to estimate the value of y when $x=1.2$, giving your answer to 3 decimal places. **[2]**

Solution

(a) Differentiate the equation with respect to x, using the product rule on $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and the chain rule on $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+4\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6$$

so

$$y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=6-5\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

From the original equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{y}\left(6x-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)$$

Substituting this:

$$\begin{aligned}y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}&=6-\frac{5}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\left(6x-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)\\&=6-\frac{30x}{y}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{10}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3\end{aligned}$$

Dividing by y:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{6}{y}+\frac{10}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3-\frac{30x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

as required.

(b) The Taylor series about $x=a$ is

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots$$

Take $f(x)=y$ and $a=1$.

When $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$. From the original equation:

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(-1)^2 = 6 \quad \Longrightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2$$

From part (a):

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{6}{2} + \frac{10}{4}(-1)^3 - \frac{30 \times 1}{4} \times (-1) = 3 - \frac{5}{2} + \frac{15}{2} = 8$$

Substituting into the series:

$$\begin{aligned} y &\approx 2 + (x-1)(-1) + \frac{(x-1)^2}{2} \times 2 + \frac{(x-1)^3}{6} \times 8 \\ &= 2 - (x-1) + (x-1)^2 + \frac{4}{3}(x-1)^3 \end{aligned}$$

(c) When $x = 1.2$, $x - 1 = 0.2$, so

$$\begin{aligned} y &\approx 2 - 0.2 + 0.2^2 + \frac{4}{3} \times 0.2^3 \\ &= \frac{694}{375} \\ &\approx 1.851 \qquad (3 \text{ d.p.}) \end{aligned}$$

11. The variable y satisfies the differential equation

$$(1+x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}-6y=0$$

(a) Show that

$$(1+x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+ax\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+b\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

where a and b are constants to be found. **[4]**

Given that $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ at $x=0$,

(b) find a series solution for y in ascending powers of x up to and including the term in x^5, **[5]**

(c) use your series to estimate the value of y at $x=0.1$, giving your answer to four decimal places. **[2]**

Solution

(a) Differentiate the equation with respect to x, using the product rule on the first two terms:

$$2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-6\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Collecting like terms gives

$$(1+x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-5\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Differentiate again:

$$2x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+(1+x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-5\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

Collecting like terms gives

$$(1+x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+5x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

This is in the required form with $a=5$ and $b=-2$.

(b) Maclaurin's series is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\frac{x^5}{5!}f^{(5)}(0)+\cdots$$

so the derivatives of y up to the fifth are needed at $x=0$, where $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$.

Setting $x=0$ in the original equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-6y=0 \quad\Longrightarrow\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6\times 2=12$$

Setting $x=0$ in the third-order equation from part (a):

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-5\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad\Longrightarrow\quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=5\times(-1)=-5$$

Setting $x = 0$ in the result of part (a):

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 2 \times 12 = 24$$

For the fifth derivative, differentiate the result of part (a):

$$2x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + (1+x^2)\frac{\mathrm{d}^5y}{\mathrm{d}x^5} + 5\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 5x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0$$

which simplifies to

$$(1+x^2)\frac{\mathrm{d}^5y}{\mathrm{d}x^5} + 7x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + 3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0$$

Setting $x = 0$:

$$\frac{\mathrm{d}^5y}{\mathrm{d}x^5} = -3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -3 \times (-5) = 15$$

Substituting into Maclaurin's series:

$$y = 2 + (-1)x + \frac{12}{2!}x^2 + \left(-\frac{5}{3!}\right)x^3 + \frac{24}{4!}x^4 + \frac{15}{5!}x^5 + \cdots$$

$$= 2 - x + 6x^2 - \frac{5}{6}x^3 + x^4 + \frac{1}{8}x^5 + \cdots$$

Alternative Solution:

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Differentiating term by term:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

When $x = 0$, $y = a_0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = a_1$, so $a_0 = 2$ and $a_1 = -1$. Expanding the first term of the differential equation into two sums:

$$(1+x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

Replacing n by $n+2$ in the first sum, so that the power is x^n, gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

The second sum, and $x\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=1}^{\infty} n a_n x^n$, may start at $n = 0$, since the terms added are zero. Substituting into the differential equation:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} [n(n-1) + n - 6] a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n^2 - 6) a_n\right] x^n = 0$$

Every coefficient is zero, so for $n \geq 0$ the recurrence is

$$a_{n+2} = \frac{(6-n^2)a_n}{(n+1)(n+2)}$$

With $n = 2$ the recurrence gives $a_4 = \frac{a_2}{6}$, which is part (a)'s relation $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at $x = 0$, since a_n is the value of $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ at $x = 0$ divided by $n!$.

Using the recurrence with $n = 0, 1, 2, 3$:

$$a_2 = \frac{(6-0)a_0}{1 \times 2} = \frac{6 \times 2}{2} = 6$$

$$a_3 = \frac{(6-1)a_1}{2 \times 3} = \frac{5 \times (-1)}{6} = -\frac{5}{6}$$

$$a_4 = \frac{(6-4)a_2}{3 \times 4} = \frac{2 \times 6}{12} = 1$$

$$a_5 = \frac{(6-9)a_3}{4 \times 5} = \frac{-3 \times \left(-\frac{5}{6}\right)}{20} = \frac{1}{8}$$

Hence

$$y = 2 - x + 6x^2 - \frac{5}{6}x^3 + x^4 + \frac{1}{8}x^5 + \cdots$$

(c) Substituting $x = 0.1$ into the series:

$$\begin{aligned} y &\approx 2 - 0.1 + 6(0.1)^2 - \frac{5}{6}(0.1)^3 + (0.1)^4 + \frac{1}{8}(0.1)^5 \\ &\approx 2 - 0.1 + 0.06 - 0.000\,833\,3 + 0.0001 + 0.000\,001\,25 \\ &\approx 1.959\,267\,9 \\ &\approx 1.9593 \qquad (4 \text{ d.p.}) \end{aligned}$$

12. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

The curve with equation $y = f(x)$ satisfies the differential equation

$$\sin x \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \cos x$$

The point $\left(\frac{\pi}{6}, 1\right)$ is a stationary point of the curve.

(a) Determine the nature of this stationary point, giving a reason for your answer. **[2]**

(b) Find the exact value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at this stationary point. **[4]**

(c) Hence find a series solution for y, in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$, giving each coefficient in exact form. Use your series to estimate the value of y when $x = 0.6$, giving your answer to 4 decimal places. **[4]**

Solution

(a) At the stationary point, $x = \frac{\pi}{6}$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Substituting into the differential equation, with $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$:

$$\frac{1}{2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2(1)(0) = 1^2 \times \frac{\sqrt{3}}{2} \quad \Longrightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{3}$$

Since $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{3} > 0$, the stationary point is a minimum point.

(b) Differentiate the differential equation with respect to x, using the product rule on each term:

$$\cos x \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \sin x \frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}\cos x - y^2 \sin x$$

At the stationary point, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{3}$, so

$$\frac{\sqrt{3}}{2} \times \sqrt{3} + \frac{1}{2}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 0 - 2(1)\sqrt{3} = 0 - 1^2 \times \frac{1}{2}$$

$$\frac{3}{2} + \frac{1}{2}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2\sqrt{3} = -\frac{1}{2}$$

$$\frac{1}{2}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\sqrt{3} - 2$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 4\sqrt{3} - 4$$

(c) Using the Taylor series about $x = \frac{\pi}{6}$, with $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{3}$ and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 4\sqrt{3} - 4$ there:

$$y \approx 1 + 0 \times \left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{4\sqrt{3}-4}{3!}\left(x - \frac{\pi}{6}\right)^3$$

$$= 1 + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)^2 + \frac{2\sqrt{3}-2}{3}\left(x - \frac{\pi}{6}\right)^3$$

When $x = 0.6$, $x - \frac{\pi}{6} = 0.6 - \frac{\pi}{6} \approx 0.076\,401$, so

$$\begin{aligned} y &\approx 1 + \frac{\sqrt{3}}{2}\left(0.6 - \frac{\pi}{6}\right)^2 + \frac{2\sqrt{3} - 2}{3}\left(0.6 - \frac{\pi}{6}\right)^3 \\ &\approx 1 + 0.005\,055\,1 + 0.000\,217\,6 \\ &\approx 1.0053 \qquad (4 \text{ d.p.}) \end{aligned}$$