OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

Numerical Solution of Differential Equations

Worked solutions 13 questions · 94 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · AQA 7367

Instructions

AQA (7367): numerical solution of second-order differential equations is not on the specification. Questions 11 to 13 can be left out.

AQA (7367): the questions give each step as an approximation to the derivative. Rearranged, these are the formulae booklet's Euler's method and midpoint formula, where $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h} \quad \text{is Euler's method,} \quad y_{r+1} = y_r + hf(x_r, y_r)$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h} \quad \text{is the midpoint formula,} \quad y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

Questions

Question 1	*6 marks*	2
Question 2	*9 marks*	4
Question 3	*6 marks*	6
Question 4	*10 marks*	8
Question 5	*9 marks*	10
Question 6	*6 marks*	12
Question 7	*10 marks*	14
Question 8	*4 marks*	16
Question 9	*7 marks*	17
Question 10	*6 marks*	18
Question 11	*6 marks*	20
Question 12	*8 marks*	22
Question 13	*7 marks*	24

1. It is given that $y(1) = 3$ and that y satisfies the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 2$$

(a) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. Hence use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

with $h = 0.1$ to obtain an estimate for $y(1.1)$. **[3]**

(b) Use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

with your answer to part **(a)** to obtain an estimate for $y(1.2)$, giving your answer to 3 decimal places. **[3]**

Solution

(a) Dividing both sides by $x + 1$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy + 2}{x + 1}$$

The starting values are $x_0 = 1$ and $y_0 = 3$, so

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = \frac{1 \times 3 + 2}{1 + 1} = \frac{5}{2} = 2.5$$

Rearranging the approximation gives

$$y_{n+1} \approx y_n + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_r + hf(x_r, y_r)$.

Taking $n = 0$ and $h = 0.1$:

$$\begin{aligned} y_1 &= y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \\ &= 3 + 0.1 \times 2.5 \\ &= 3.25 \end{aligned}$$

So $y(1.1) \approx 3.25$.

(b) Now $x_1 = 1.1$ and, from part **(a)**, $y_1 = 3.25$, so

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 = \frac{1.1 \times 3.25 + 2}{1.1 + 1} = \frac{5.575}{2.1} = \frac{223}{84}$$

Rearranging the approximation gives

$$y_{n+1} \approx y_{n-1} + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$.

Taking $n = 1$ and $h = 0.1$:

$$\begin{aligned} y_2 &= y_0 + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 \\ &= 3 + 2 \times 0.1 \times \frac{223}{84} \\ &= 3 + \frac{223}{420} \\ &= \frac{1483}{420} \end{aligned}$$

So

$$y(1.2) \approx 3.531 \qquad (3 \text{ d.p.})$$

2. A parachutist is falling vertically. At time $t = 0$ her parachute opens, when her speed is $20\,\mathrm{m\,s^{-1}}$. Her speed $v\,\mathrm{m\,s^{-1}}$, t seconds after the parachute opens, is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - 0.05v^2 \qquad t \geq 0$$

(a) Use three iterations of the approximation formula

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_n \approx \frac{v_{n+1} - v_n}{h}$$

with a step length of 0.2 to estimate the speed of the parachutist 0.6 seconds after her parachute opens. Give your answer to 3 significant figures. **[7]**

(b) The constant 9.8 in the model represents the acceleration due to gravity, which is known accurately. Observations show that the parachutist's speed settles at a steady value of $12\,\mathrm{m\,s^{-1}}$, which is less than the steady speed predicted by the current model.
Refine the model by changing the value of one constant, giving its new value to 3 significant figures. **[2]**

Solution

(a) With $h = 0.2$, rearranging the approximation formula gives

$$v_{n+1} \approx v_n + h\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_n = v_n + 0.2\left(9.8 - 0.05v_n^2\right)$$

with $t_0 = 0$ and $v_0 = 20$.

First iteration, from $t_0 = 0$ to $t_1 = 0.2$:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_0 = 9.8 - 0.05 \times 20^2 = -10.2$$

$$v_1 = 20 + 0.2 \times (-10.2) = 17.96$$

Second iteration, from $t_1 = 0.2$ to $t_2 = 0.4$:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_1 = 9.8 - 0.05 \times 17.96^2 = -6.32808$$

$$v_2 = 17.96 + 0.2 \times (-6.32808) = 16.694384$$

Third iteration, from $t_2 = 0.4$ to $t_3 = 0.6$:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_2 = 9.8 - 0.05 \times 16.694384^2 \approx -4.13512$$

$$v_3 \approx 16.694384 + 0.2 \times (-4.13512) \approx 15.8674$$

So the estimated speed 0.6 seconds after the parachute opens is

$$15.9\,\mathrm{m\,s^{-1}} \qquad (3\text{ s.f.})$$

(b) The speed is steady when $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$. In the current model this gives $0.05v^2 = 9.8$, so $v = \sqrt{196} = 14$, which is too large.

Keep 9.8 and replace 0.05 by a constant k, so the model is $\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - kv^2$. For a steady speed of $12\,\mathrm{m\,s^{-1}}$:

$$9.8 - k \times 12^2 = 0 \quad \Longrightarrow \quad k = \frac{9.8}{144} \approx 0.0681 \qquad (3 \text{ s.f.})$$

The refined model changes 0.05 to 0.0681:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - 0.0681v^2$$

3. Water flows into a tank through a pipe and, at the same time, leaks out through a hole in the base of the tank. The depth of water, H metres, in the tank is modelled by the differential equation

$$2(H+3)\frac{\mathrm{d}H}{\mathrm{d}t} = 12 - 5t\sqrt{H} \qquad H > 0$$

where t is the time in hours after the pipe was opened.
The depth of water in the tank 15 minutes after the pipe was opened was 1 m.

(a) Use two applications of the approximation formula

$$\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_n \approx \frac{H_{n+1} - H_n}{h}$$

to estimate the depth of water in the tank 1 hour 45 minutes after the pipe was opened. Give your answer to 3 significant figures. **[5]**

(b) Explain how the same approximation formula could be used to obtain a more accurate estimate of this depth. **[1]**

Solution

(a) Rearranging the differential equation:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{12 - 5t\sqrt{H}}{2(H+3)}$$

Rearranging the approximation formula gives

$$H_{n+1} \approx H_n + h\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_n$$

15 minutes is $t = 0.25$ and 1 hour 45 minutes is $t = 1.75$, so two applications need a step length of $h = \frac{1.75-0.25}{2} = 0.75$. Take $t_0 = 0.25$ and $H_0 = 1$.

First application, at $t_0 = 0.25$:

$$\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_0 = \frac{12 - 5 \times 0.25 \times \sqrt{1}}{2(1+3)} = \frac{10.75}{8} = 1.34375$$

$$H_1 \approx 1 + 0.75 \times 1.34375 = 2.0078125$$

This estimates the depth at $t_1 = 1$.

Second application, at $t_1 = 1$:

$$\begin{aligned}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_1 &= \frac{12 - 5 \times 1 \times \sqrt{2.0078125}}{2(2.0078125+3)} \\ &\approx \frac{12 - 7.084865}{10.015625} \\ &\approx 0.490747 \\ H_2 &\approx 2.0078125 + 0.75 \times 0.490747 \\ &\approx 2.375873 \\ &\approx 2.38 \qquad \text{(3 s.f.)}\end{aligned}$$

The estimated depth of water 1 hour 45 minutes after the pipe was opened is 2.38 m.

(b) Use a smaller step length h, and so more applications of the formula between $t = 0.25$ and $t = 1.75$: for example $h = 0.25$ with six applications. A smaller step length gives a more accurate estimate.

4. Priya stocks a new pond with 50 carp, which she plans to breed and sell.
After t months the number of carp, P, is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.2P - 0.0008P^2 \qquad t \geq 0$$

Priya will begin selling carp only once the pond holds at least 75 of them.

(a) Use three iterations of the approximation formula

$$\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_n \approx \frac{P_{n+1} - P_n}{h}$$

with a step length of 1 month to determine whether, according to the model, Priya will be able to begin selling carp after 3 months. **[8]**

(b) By considering how $\frac{\mathrm{d}P}{\mathrm{d}t}$ changes as P increases from 50 to 80, explain whether your estimate in part (a) is likely to be an underestimate or an overestimate of the number of carp predicted by the model, and state what this means for your conclusion in part (a). **[2]**

Solution

(a) With $h = 1$, rearranging the approximation formula gives

$$P_{n+1} \approx P_n + h\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_n = P_n + 1 \times \left(0.2P_n - 0.0008P_n^2\right)$$

Starting from $t_0 = 0$, $P_0 = 50$:

$$\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_0 = 0.2 \times 50 - 0.0008 \times 50^2 = 10 - 2 = 8$$

$$P_1 = 50 + 8 = 58$$

Next, at $t_1 = 1$:

$$\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_1 = 0.2 \times 58 - 0.0008 \times 58^2 = 11.6 - 2.6912 = 8.9088$$

$$P_2 = 58 + 8.9088 = 66.9088$$

Then, at $t_2 = 2$:

$$\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_2 = 0.2 \times 66.9088 - 0.0008 \times 66.9088^2 \approx 13.38176 - 3.58143 = 9.80033$$

$$P_3 \approx 66.9088 + 9.80033 \approx 76.7$$

So after 3 months the model predicts about 76.7 carp. Since $76.7 \geq 75$, according to the model Priya will be able to begin selling carp after 3 months.

(b) Completing the square:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.2P - 0.0008P^2 = 12.5 - 0.0008(P - 125)^2$$

so for $50 \leq P \leq 80$ (where $P < 125$), $\frac{\mathrm{d}P}{\mathrm{d}t}$ is positive and increases as P increases. Since $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$, P increases with t, so the gradient increases during each step. The approximation uses the gradient at the start of each step, which is the smallest gradient on that step, so each step underestimates

the increase in P. Hence 76.7 is likely to be an underestimate of the number predicted by the model. The model's value is therefore greater than 76.7, and so still at least 75: the conclusion in part (a) remains valid.

5. A particle of mass 2 kg moves in a straight line on a smooth horizontal surface.
At time t seconds, the velocity of the particle is $v\,\mathrm{m\,s^{-1}}$.
The following horizontal forces act on the particle:

- a driving force of magnitude $(8 + 2t)$ newtons, acting in the direction of motion
- a resistive force of magnitude $0.3v^2$ newtons

When $t = 0$, $v = 1$.

(a) Show that

$$20\frac{\mathrm{d}v}{\mathrm{d}t} = 80 + 20t - 3v^2$$ **[2]**

(b) Use the approximation formula

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_n \approx \frac{v_{n+1} - v_n}{h}$$

with a step length of 0.2 to estimate the velocity of the particle when $t = 0.6$, giving your answer to 4 significant figures. **[7]**

Solution

(a) Take the direction of motion as positive. The surface is smooth and horizontal, so the only horizontal forces are the driving force and the resistance. Using Newton's second law, with acceleration $a = \frac{\mathrm{d}v}{\mathrm{d}t}$:

$$2\frac{\mathrm{d}v}{\mathrm{d}t} = (8 + 2t) - 0.3v^2$$

Multiplying both sides by 10:

$$20\frac{\mathrm{d}v}{\mathrm{d}t} = 80 + 20t - 3v^2$$

as required.

(b) Dividing the equation from part (a) by 20 gives

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 4 + t - 0.15v^2$$

Rearranging the approximation formula gives

$$v_{n+1} \approx v_n + h\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_r + hf(x_r, y_r)$.

Here $h = 0.2$, $t_0 = 0$ and $v_0 = 1$. Three steps are needed to reach $t = 0.6$.

First step, $t_0 = 0$, $v_0 = 1$:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_0 = 4 + 0 - 0.15 \times 1^2 = 3.85$$

$$v_1 = 1 + 0.2 \times 3.85 = 1.77$$

Second step, $t_1 = 0.2$, $v_1 = 1.77$:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_1 = 4.2 - 0.15 \times 1.77^2 = 3.730065$$

$$v_2 = 1.77 + 0.2 \times 3.730065 = 2.516013$$

Third step, $t_2 = 0.4$, $v_2 = 2.516013$:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_2 = 4.4 - 0.15 \times 2.516013^2 \approx 3.450452$$

$$v_3 \approx 2.516013 + 0.2 \times 3.450452 \approx 3.206 \qquad (4 \text{ s.f.})$$

The estimated velocity of the particle when $t = 0.6$ is $3.206\,\mathrm{m\,s^{-1}}$.

6. A curve passes through the point $(0.5, -1.2)$ and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 2y}{3x - y}$$

(a) Use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

once, and then the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

once, each with a step length of 0.05, to estimate the value of y when $x = 0.6$. Give your answer to 4 decimal places. **[5]**

(b) Explain why the second approximation in part (a) cannot be used for the first step from $x = 0.5$. **[1]**

Solution

(a) Here $x_0 = 0.5$, $y_0 = -1.2$ and $h = 0.05$. Rearranging the first approximation gives

$$y_{n+1} \approx y_n + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_r + hf(x_r, y_r)$.

At the starting point:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = \frac{0.25 - 2.4}{1.5 + 1.2} = -\frac{2.15}{2.7} \approx -0.796296$$

so the first step gives

$$\begin{aligned} y_1 &= -1.2 + 0.05 \times (-0.796296) \\ &\approx -1.239815 \end{aligned}$$

at $x_1 = 0.55$. Now evaluate the gradient at (x_1, y_1):

$$\begin{aligned} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 &= \frac{0.3025 + 2 \times (-1.239815)}{3 \times 0.55 + 1.239815} \\ &= \frac{-2.177130}{2.889815} \\ &\approx -0.753380 \end{aligned}$$

Rearranging the second approximation gives

$$y_{n+1} \approx y_{n-1} + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$.

Taking $n = 1$ gives y_2 at $x_2 = 0.6$:

$$\begin{aligned} y_2 &= y_0 + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 \\ &= -1.2 + 2 \times 0.05 \times (-0.753380) \\ &\approx -1.275338 \end{aligned}$$

So when $x = 0.6$:

$$y \approx -1.2753 \qquad (4 \text{ d.p.})$$

(b) The second approximation finds y_{n+1} from two earlier values, y_{n-1} and y_n. For the first step from $x = 0.5$ we would have $n = 0$, so it would need y_{-1}, the value of y at $x = 0.45$, which is not known: only the single value $y_0 = -1.2$ is given. The first approximation, which needs only one previous value, must be used to find y_1 first.

7. A curve passes through the point $(1, 3)$ and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \ln x + \frac{2y}{x} \qquad x > 0$$

(a) Use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

once, and then the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

twice, each with a step length of 0.2, to estimate the value of y when $x = 1.6$, giving your answer to 5 significant figures. **[7]**

(b) It is given that the curve has equation

$$y = \frac{1}{2}x^2\left((\ln x)^2 + 6\right)$$

Find the percentage error in your estimate from part (a). **[3]**

Solution

(a) Here

$$h = 0.2, \qquad x_0 = 1, \qquad y_0 = 3$$

Rearranging the first approximation gives

$$y_{n+1} \approx y_n + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_r + hf(x_r, y_r)$.

Taking $n = 0$:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 1 \times \ln 1 + \frac{2 \times 3}{1} = 6$$

$$\begin{aligned} y_1 &= y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \\ &= 3 + 0.2 \times 6 \\ &= 4.2 \end{aligned}$$

So $y \approx 4.2$ when $x_1 = 1.2$.

Rearranging the second approximation gives

$$y_{n+1} \approx y_{n-1} + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$.

Taking $n = 1$:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 = 1.2\ln 1.2 + \frac{2 \times 4.2}{1.2} \approx 7.218786$$

$$\begin{aligned} y_2 &= y_0 + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 \\ &\approx 3 + 0.4 \times 7.218786 \\ &\approx 5.887514 \end{aligned}$$

So $y \approx 5.887514$ when $x_2 = 1.4$.

Taking $n = 2$:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_2 \approx 1.4\ln 1.4 + \frac{2 \times 5.887514}{1.4} \approx 8.881796$$

$$\begin{aligned} y_3 &= y_1 + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_2 \\ &\approx 4.2 + 0.4 \times 8.881796 \\ &\approx 7.752718 \end{aligned}$$

Since $x_3 = 1.6$, the estimate when $x = 1.6$ is

$$y \approx 7.7527 \qquad (5 \text{ s.f.})$$

(b) Substituting $x = 1.6$ into the equation of the curve:

$$\begin{aligned} y &= \frac{1}{2} \times 1.6^2\big((\ln 1.6)^2 + 6\big) \\ &= 1.28\big((\ln 1.6)^2 + 6\big) \\ &\approx 7.962756 \end{aligned}$$

The percentage error is

$$\begin{aligned} \frac{\text{estimate} - \text{exact value}}{\text{exact value}} \times 100\% &\approx \frac{7.7527 - 7.962756}{7.962756} \times 100\% \\ &\approx -2.64\% \end{aligned}$$

The estimate from part (a) is an underestimate, with a percentage error of 2.64% to 3 significant figures.

8. A curve passes through the point $(a, 2)$, where $a > 0$.
The curve satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 2y}{xy}$$

Using the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

once, with starting point $(a, 2)$ and a step length of 0.25, gives an estimate of $y = 2.6$ when $x = a + 0.25$.
Find the two possible values of a, giving your answers in exact form. **[4]**

Solution

Rearranging the approximation gives

$$y_{n+1} \approx y_n + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n$$

This formula is also sometimes written as $y_{r+1} = y_r + hf(x_r, y_r)$.

At the starting point $(x_0, y_0) = (a, 2)$ the gradient is

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = \frac{a^2 + 2 \times 2}{a \times 2} = \frac{a^2 + 4}{2a}$$

With $h = 0.25$, one step gives the estimate $y = 2.6$ at $x = a + 0.25$, so

$$2 + 0.25 \times \frac{a^2 + 4}{2a} = 2.6$$

$$\frac{a^2 + 4}{8a} = 0.6$$

$$a^2 + 4 = 4.8a$$

$$5a^2 - 24a + 20 = 0$$

The discriminant is $(-24)^2 - 4 \times 5 \times 20 = 176$, and $\sqrt{176} = 4\sqrt{11}$. Using the quadratic formula:

$$a = \frac{24 \pm 4\sqrt{11}}{10} = \frac{12 \pm 2\sqrt{11}}{5}$$

Since $2\sqrt{11} = \sqrt{44} < 12$, both values satisfy $a > 0$, so the two possible values are

$$a = \frac{12 - 2\sqrt{11}}{5} \quad \text{or} \quad a = \frac{12 + 2\sqrt{11}}{5}$$

9. The temperature, $\theta\,^\circ$C, of soup in a bowl, t minutes after the bowl is placed on a table in a room, is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20)^2$$

where k is a positive constant.
The soup has an initial temperature of $80\,^\circ$C, and $k = 0.002$.

(a) Use two iterations of the approximation formula

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_n \approx \frac{\theta_{n+1} - \theta_n}{h}$$

to estimate the temperature of the soup 4 minutes after the bowl was placed on the table, giving your answer to 1 decimal place. **[5]**

(b) By finding an expression for $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$ in terms of θ and k, explain whether your answer to part (a) is likely to be an overestimate or an underestimate of the temperature predicted by the model. **[2]**

Solution

(a) Two iterations are needed to reach $t = 4$, so take the step length $h = 2$. Rearranging the approximation formula gives

$$\theta_{n+1} \approx \theta_n + h\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_n, \qquad \text{where } \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_n = -0.002(\theta_n - 20)^2$$

First iteration, with $t_0 = 0$ and $\theta_0 = 80$:

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_0 = -0.002 \times (80 - 20)^2 = -0.002 \times 3600 = -7.2$$

$$\theta_1 \approx 80 + 2 \times (-7.2) = 65.6$$

Second iteration, with $t_1 = 2$ and $\theta_1 = 65.6$:

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_1 = -0.002 \times (65.6 - 20)^2 = -0.002 \times 2079.36 = -4.15872$$

$$\theta_2 \approx 65.6 + 2 \times (-4.15872) = 57.28256$$

$$\theta_2 \approx 57.3 \qquad (1\text{ d.p.})$$

So the estimated temperature of the soup after 4 minutes is $57.3\,^\circ$C.

(b) Differentiate $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20)^2$ with respect to t, using the chain rule:

$$\begin{aligned}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &= -2k(\theta - 20)\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= -2k(\theta - 20) \times \left(-k(\theta - 20)^2\right) \\ &= 2k^2(\theta - 20)^3\end{aligned}$$

The soup cools from $80\,^\circ$C towards the room temperature of $20\,^\circ$C, so $\theta > 20$ throughout, and hence $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = 2k^2(\theta - 20)^3 > 0$. The graph of θ against t is therefore convex (its gradient is increasing), so each tangent lies below the curve. Each step of the approximation follows the tangent at the start of the step, so it lands below the curve, and the answer to part (a) is likely to be an **underestimate** of the temperature predicted by the model.

10. A survey of a lake records the proportion, x, of the lake's surface that is covered by weed. This proportion is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x(0.7-x)(t+2)}{3t+x} \qquad t > 0$$

where t is the number of years since the start of the survey.
Exactly 2 years after the start of the survey, 40% of the lake's surface was covered by weed.

(a) Use two applications of the approximation formula

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n \approx \frac{x_{n+1}-x_n}{h}$$

with a step length of 0.25 years to estimate the proportion of the lake's surface covered by weed six months later. Give your answer to 3 decimal places. **[5]**

(b) Deduce from the differential equation the proportion of the lake's surface that will be covered by weed in the long term. **[1]**

Solution

(a) Rearranging the approximation formula gives

$$x_{n+1} \approx x_n + h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n$$

where

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n = \frac{x_n(0.7-x_n)(t_n+2)}{3t_n+x_n}$$

The starting values are $t_0 = 2$ and $x_0 = 0.4$, with $h = 0.25$. Six months later is $t = 2.5$, which is reached after two steps.

First step:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_0 = \frac{0.4\times(0.7-0.4)\times(2+2)}{3\times 2+0.4} = \frac{0.48}{6.4} = 0.075$$

$$x_1 \approx 0.4 + 0.25\times 0.075 = 0.41875$$

Second step: now $t_1 = 2.25$ and $x_1 = 0.41875$, so

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_1 = \frac{0.41875\times(0.7-0.41875)\times(2.25+2)}{3\times 2.25+0.41875}$$

$$= \frac{0.41875\times 0.28125\times 4.25}{7.16875}$$

$$\approx 0.0698221$$

$$x_2 \approx 0.41875 + 0.25\times 0.0698221$$

$$\approx 0.4362055$$

The estimated proportion of the lake's surface covered by weed six months later is

$$x \approx 0.436 \qquad (3 \text{ d.p.})$$

(b) For $t > 0$ and $0 < x < 0.7$, every factor in $\frac{x(0.7-x)(t+2)}{3t+x}$ is positive, so $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$ and x keeps increasing, while $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $x = 0.7$. Starting from $x = 0.4$, the proportion therefore increases towards 0.7, so in the long term 70% of the lake's surface will be covered by weed.

11. The angle, θ radians, that a pendulum makes with the downward vertical, t seconds after it is first observed, is modelled by the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 0.2\frac{\mathrm{d}\theta}{\mathrm{d}t} + 9\sin\theta = 0$$

Two measurements are taken: $\theta = 0.5$ when $t = 0$, and $\theta = 0.48$ when $t = 0.1$.
The approximations

$$\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right)_n \approx \frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_n \approx \frac{\theta_{n+1} - \theta_{n-1}}{2h}$$

are used with $h = 0.1$, where θ_n is the estimate of θ when $t = 0.1n$.

(a) Use the two measurements to estimate the value of θ when $t = 0.2$, giving your answer to 4 decimal places. **[4]**

(b) Hence estimate the value of θ when $t = 0.3$, giving your answer to 3 decimal places. **[2]**

Solution

(a) Substituting the two approximations into the differential equation at $t = 0.1n$ gives

$$\frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{h^2} + 0.2 \times \frac{\theta_{n+1} - \theta_{n-1}}{2h} + 9\sin\theta_n = 0$$

With $h = 0.1$, $\frac{1}{h^2} = 100$ and $\frac{0.2}{2h} = 1$, so

$$100(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + (\theta_{n+1} - \theta_{n-1}) + 9\sin\theta_n = 0$$

$$101\theta_{n+1} - 200\theta_n + 99\theta_{n-1} + 9\sin\theta_n = 0$$

$$\theta_{n+1} = \frac{200\theta_n - 99\theta_{n-1} - 9\sin\theta_n}{101}$$

The measurements give $\theta_0 = 0.5$ and $\theta_1 = 0.48$, and $t = 0.2$ corresponds to θ_2. Taking $n = 1$, with the calculator in radians:

$$\theta_2 = \frac{200 \times 0.48 - 99 \times 0.5 - 9\sin 0.48}{101}$$

$$\approx \frac{96 - 49.5 - 4.156013}{101}$$

$$\approx 0.4192474$$

So the estimate for θ when $t = 0.2$ is

$$\theta \approx 0.4192 \qquad (4 \text{ d.p.})$$

(b) The value $t = 0.3$ corresponds to θ_3. Taking $n = 2$ in the formula from part (a), with $\theta_1 = 0.48$ and the unrounded value $\theta_2 \approx 0.4192474$:

$$\begin{aligned}
\theta_3 &= \frac{200\theta_2 - 99\theta_1 - 9\sin\theta_2}{101} \\
&\approx \frac{200 \times 0.4192474 - 99 \times 0.48 - 9\sin 0.4192474}{101} \\
&\approx \frac{83.849480 - 47.52 - 3.663658}{101} \\
&\approx 0.323424
\end{aligned}$$

So the estimate for θ when $t = 0.3$ is

$$\theta \approx 0.323 \qquad (3 \text{ d.p.})$$

12. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 3y - x^2$$

where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 1$.
The approximations

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

are used with $h = 0.1$, where $x_n = 1 + 0.1n$ and y_n is the approximation to y at $x = x_n$.

(a) Show that

$$(20 + x_n)\, y_{n+1} = 40.6\, y_n - (20 - x_n)\, y_{n-1} - 0.2x_n^2 \qquad \textbf{[3]}$$

(b) Hence find an estimate for the value of y at $x = 1.2$, giving your answer to 3 decimal places. **[5]**

Solution

(a) With $h = 0.1$, $h^2 = 0.01$ and $2h = 0.2$. Substituting both approximations into the differential equation at $x = x_n$:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{0.01} + x_n \frac{y_{n+1} - y_{n-1}}{0.2} = 3y_n - x_n^2$$

Multiplying through by 0.2:

$$20(y_{n+1} - 2y_n + y_{n-1}) + x_n(y_{n+1} - y_{n-1}) = 0.6y_n - 0.2x_n^2$$

Expanding and collecting the y_{n+1} terms on the left:

$$20y_{n+1} + x_n y_{n+1} = 40y_n + 0.6y_n - 20y_{n-1} + x_n y_{n-1} - 0.2x_n^2$$

$$(20 + x_n)\, y_{n+1} = 40.6\, y_n - (20 - x_n)\, y_{n-1} - 0.2x_n^2$$

as required.

(b) Here $x_0 = 1$ and $y_0 = 2$, and y_{-1} is the approximation to y at $x = 0.9$. Using the first derivative approximation with $n = 0$ and the given condition $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 1$:

$$\frac{y_1 - y_{-1}}{0.2} = -1 \quad \Longrightarrow \quad y_{-1} = y_1 + 0.2$$

Putting $n = 0$ into the result of part (a), with $x_0 = 1$:

$$21y_1 = 40.6 \times 2 - 19(y_1 + 0.2) - 0.2 \times 1^2$$

$$21y_1 = 81.2 - 19y_1 - 3.8 - 0.2$$

$$40y_1 = 77.2$$

$$y_1 = 1.93$$

Putting $n = 1$ into the result of part (a), with $x_1 = 1.1$, $y_1 = 1.93$ and $y_0 = 2$:

$$21.1y_2 = 40.6 \times 1.93 - 18.9 \times 2 - 0.2 \times 1.1^2$$

$$21.1y_2 = 78.358 - 37.8 - 0.242$$

$$21.1y_2 = 40.316$$

$$y_2 = \frac{40.316}{21.1}$$

Since $x_2 = 1.2$, the estimate for y at $x = 1.2$ is

$$y_2 \approx 1.911 \qquad (3 \text{ d.p.})$$

13. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 3x$$

where $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5$ when $x = 0$.
Use the approximations

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

with $h = 0.2$, where y_n denotes the estimate of y at $x = nh$, to find an estimate for the value of y when $x = 0.4$. Give your answer to 3 significant figures. **[7]**

Solution

With $h = 0.2$, the values of x are $x_0 = 0$, $x_1 = 0.2$ and $x_2 = 0.4$, and $y_0 = 1$. The estimate needed is y_2.

Substituting the two approximations into the differential equation at $x = x_n$ gives

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + 2x_n\left(\frac{y_{n+1} - y_{n-1}}{2h}\right) - y_n^2 = 3x_n$$

Here $h^2 = 0.04$ and $2h = 0.4$.

Finding y_1: when $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5$, so the approximation for the first derivative with $n = 0$ gives

$$\frac{y_1 - y_{-1}}{0.4} = 0.5 \quad \Longrightarrow \quad y_{-1} = y_1 - 0.2$$

Putting $n = 0$ in the equation, with $x_0 = 0$ and $y_0 = 1$:

$$\frac{y_1 - 2 + y_{-1}}{0.04} + 0 - 1^2 = 0$$

$$y_1 - 2 + y_{-1} = 0.04$$

$$y_1 + y_{-1} = 2.04$$

Substituting $y_{-1} = y_1 - 0.2$:

$$2y_1 - 0.2 = 2.04$$

$$y_1 = 1.12$$

Finding y_2: putting $n = 1$ in the equation, with $x_1 = 0.2$, $y_0 = 1$ and $y_1 = 1.12$:

$$\frac{y_2 - 2 \times 1.12 + 1}{0.04} + 2 \times 0.2 \times \frac{y_2 - 1}{0.4} - 1.12^2 = 3 \times 0.2$$

$$25(y_2 - 1.24) + (y_2 - 1) - 1.2544 = 0.6$$

$$26y_2 - 33.2544 = 0.6$$

$$26y_2 = 33.8544$$

The estimate for the value of y when $x = 0.4$ is therefore

$$y_2 = \frac{33.8544}{26} \approx 1.30 \qquad (3 \text{ s.f.})$$