OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

2nd Order Differential Equations by Substitution

Worked solutions

12 questions · 180 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

Question 1	*12 marks*	2
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1. It is given that, for $x \neq 0$, y satisfies the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x(1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(2x^2 - 4x + 1\right)y = 4x^2 \qquad (*)$$

(a) Show that the substitution $u = x^2y$ transforms this differential equation into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 4\frac{\mathrm{d}u}{\mathrm{d}x} + 4u = 4x^2 \qquad (**)$$ **[5]**

(b) Hence find the general solution of differential equation $(*)$, giving your answer in the form $y = f(x)$. **[7]**

Solution

(a) Differentiate $u = x^2y$ with respect to x, using the product rule:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y$$

Expanding the brackets in the differential equation gives

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y - 4x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^2y - 8xy = 4x^2$$

The first three terms are $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$, and $4x^2y = 4u$. Rearranging the expression for $\frac{\mathrm{d}u}{\mathrm{d}x}$:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 2xy$$

so

$$-4x^2\frac{\mathrm{d}y}{\mathrm{d}x} = -4\frac{\mathrm{d}u}{\mathrm{d}x} + 8xy$$

Substituting these into the expanded equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 4\frac{\mathrm{d}u}{\mathrm{d}x} + 8xy + 4u - 8xy = 4x^2$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 4\frac{\mathrm{d}u}{\mathrm{d}x} + 4u = 4x^2$$

as required.

(b) The auxiliary equation is

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Longrightarrow \quad (\lambda - 2)^2 = 0 \quad \Longrightarrow \quad \lambda = 2 \text{ (repeated)}$$

so the complementary function is

$$u_c = (A + Bx)e^{2x}$$

For the particular integral, try $u_p = ax^2 + bx + c$, so $\frac{\mathrm{d}u_p}{\mathrm{d}x} = 2ax + b$ and $\frac{\mathrm{d}^2u_p}{\mathrm{d}x^2} = 2a$. Substituting:

$$2a - 4(2ax + b) + 4\left(ax^2 + bx + c\right) = 4x^2$$

$$4ax^2 + (4b - 8a)x + (2a - 4b + 4c) = 4x^2$$

Equating coefficients:

$$\begin{aligned} x^2: &\quad 4a = 4 \quad \Longrightarrow \quad a = 1 \\ x: &\quad 4b - 8a = 0 \quad \Longrightarrow \quad b = 2 \\ \text{constant}: &\quad 2a - 4b + 4c = 0 \quad \Longrightarrow \quad 4c = 6 \quad \Longrightarrow \quad c = \tfrac{3}{2} \end{aligned}$$

So $u_p = x^2 + 2x + \frac{3}{2}$, and the general solution for u is

$$u = (A + Bx)e^{2x} + x^2 + 2x + \frac{3}{2}$$

Since $u = x^2 y$ and $x \neq 0$, dividing by x^2 gives

$$y = \frac{(A + Bx)e^{2x}}{x^2} + 1 + \frac{2}{x} + \frac{3}{2x^2}$$

2. A particle P moves along a straight line.
At time t minutes, where $t > 0$, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (4t+2)\frac{\mathrm{d}x}{\mathrm{d}t} + (13t+4)x = 40\cos t \qquad (*)$$

(a) Show that the substitution $x = \frac{y}{t}$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = 40\cos t \qquad (**)$$ **[5]**

(b) Hence find a general solution for the displacement of P from O at time t minutes. **[8]**

Solution

(a) With $x = \frac{y}{t} = yt^{-1}$, differentiating using the product rule gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{y}{t^2}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{1}{t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{1}{t^2}\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{t^2}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2y}{t^3}$$

$$= \frac{1}{t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{2}{t^2}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2y}{t^3}$$

Now find each term on the left-hand side of $(*)$:

$$t\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{2}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2y}{t^2}$$

$$(4t+2)\frac{\mathrm{d}x}{\mathrm{d}t} = 4\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2}{t}\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{4y}{t} - \frac{2y}{t^2}$$

$$(13t+4)x = 13y + \frac{4y}{t}$$

Adding these three lines, the terms in $\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t}$, $\frac{y}{t}$ and $\frac{y}{t^2}$ cancel, so $(*)$ becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = 40\cos t$$

as required.

(b) The auxiliary equation is

$$\lambda^2 + 4\lambda + 13 = 0 \quad \Longrightarrow \quad \lambda = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$$

so the complementary function is

$$y_c = e^{-2t}(A\cos 3t + B\sin 3t)$$

For the particular integral, try $y_p = a\cos t + b\sin t$. Then

$$\frac{\mathrm{d}y_p}{\mathrm{d}t} = -a\sin t + b\cos t$$

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}t^2} = -a\cos t - b\sin t$$

Substituting into $(**)$:

$$(-a\cos t - b\sin t) + 4(-a\sin t + b\cos t) + 13(a\cos t + b\sin t) = 40\cos t$$

Equating coefficients of $\cos t$ and $\sin t$:

$$12a + 4b = 40$$

$$-4a + 12b = 0$$

The second equation gives $a = 3b$, so $36b + 4b = 40$, giving $b = 1$ and $a = 3$. Hence $y_p = 3\cos t + \sin t$, and the general solution for y is

$$y = e^{-2t}(A\cos 3t + B\sin 3t) + 3\cos t + \sin t$$

Since $x = \frac{y}{t}$, the general solution for the displacement is

$$x = \frac{e^{-2t}(A\cos 3t + B\sin 3t) + 3\cos t + \sin t}{t}$$

3. **(a)** Given that $x = t^{\frac{1}{3}}$, determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t and derivatives of y with respect to t. **[4]**

(b) Hence show that the substitution $x = t^{\frac{1}{3}}$, where $t > 0$, transforms the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(4x^3 - 2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^5y = 50x^5e^{x^3} \qquad (*)$$

into the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 12\frac{\mathrm{d}y}{\mathrm{d}t} + 4y = 50e^t \qquad (**)$$ **[2]**

(c) Solve differential equation $(**)$ to find a general solution for y in terms of t. **[6]**

(d) Hence find the general solution of differential equation $(*)$, giving y in terms of x. **[1]**

Solution

(a) Since $x = t^{\frac{1}{3}}$, $t = x^3$, so

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 3x^2 = 3t^{\frac{2}{3}}$$

Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = 3x^2\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^{\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Differentiating $3x^2\frac{\mathrm{d}y}{\mathrm{d}t}$ with respect to x using the product rule, and the chain rule on $\frac{\mathrm{d}y}{\mathrm{d}t}$:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6x\frac{\mathrm{d}y}{\mathrm{d}t} + 3x^2 \times \frac{\mathrm{d}^2y}{\mathrm{d}t^2} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= 6x\frac{\mathrm{d}y}{\mathrm{d}t} + 9x^4\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \\ &= 9t^{\frac{4}{3}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6t^{\frac{1}{3}}\frac{\mathrm{d}y}{\mathrm{d}t}\end{aligned}$$

(b) Using part (a), with $x = t^{\frac{1}{3}}$, $x^3 = t$ and $x^5 = t^{\frac{5}{3}}$, each term of $(*)$ becomes

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = t^{\frac{1}{3}}\left(9t^{\frac{4}{3}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6t^{\frac{1}{3}}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = 9t^{\frac{5}{3}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6t^{\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\left(4x^3 - 2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = (4t - 2) \times 3t^{\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}t} = 12t^{\frac{5}{3}}\frac{\mathrm{d}y}{\mathrm{d}t} - 6t^{\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$4x^5y = 4t^{\frac{5}{3}}y$$

$$50x^5e^{x^3} = 50t^{\frac{5}{3}}e^t$$

Adding the left-hand terms, the $6t^{\frac{2}{3}}\frac{\mathrm{d}y}{\mathrm{d}t}$ terms cancel, so $(*)$ becomes

$$9t^{\frac{5}{3}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 12t^{\frac{5}{3}}\frac{\mathrm{d}y}{\mathrm{d}t} + 4t^{\frac{5}{3}}y = 50t^{\frac{5}{3}}e^t$$

Since $t > 0$, $t^{\frac{5}{3}} \neq 0$, so dividing through by $t^{\frac{5}{3}}$ gives

$$9\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 12\frac{\mathrm{d}y}{\mathrm{d}t} + 4y = 50e^t$$

as required.

(c) The auxiliary equation is

$$9\lambda^2 + 12\lambda + 4 = 0 \quad \Longrightarrow \quad (3\lambda + 2)^2 = 0 \quad \Longrightarrow \quad \lambda = -\frac{2}{3} \text{ (repeated)}$$

so the complementary function is

$$y_c = (A + Bt)e^{-\frac{2}{3}t}$$

For the particular integral, try $y_p = ae^t$, so $\frac{\mathrm{d}y_p}{\mathrm{d}t} = ae^t$ and $\frac{\mathrm{d}^2y_p}{\mathrm{d}t^2} = ae^t$. Substituting into $(**)$:

$$9ae^t + 12ae^t + 4ae^t = 50e^t \quad \Longrightarrow \quad 25a = 50 \quad \Longrightarrow \quad a = 2$$

The general solution is $y = y_c + y_p$:

$$y = (A + Bt)e^{-\frac{2}{3}t} + 2e^t$$

(d) Substituting $t = x^3$ into the answer to part (c):

$$y = \left(A + Bx^3\right)e^{-\frac{2}{3}x^3} + 2e^{x^3}$$

4. It is given that $v = y^2$ and

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 5y^2 = 5x^2 + 2x + 6$$

(a) Show that

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 10x^2 + 4x + 12 \qquad \textbf{[5]}$$

(b) Find y in terms of x, given that, when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}$. **[10]**

Solution

(a) Differentiate $v = y^2$ with respect to x, using the chain rule and then the product rule:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

Hence

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{1}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}, \qquad 2y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}, \qquad 5y^2 = 5v$$

Substituting these into the given equation gives

$$\frac{1}{2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \frac{\mathrm{d}v}{\mathrm{d}x} + 5v = 5x^2 + 2x + 6$$

and multiplying by 2:

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 10x^2 + 4x + 12$$

as required.

(b) The auxiliary equation is

$$\lambda^2 + 2\lambda + 10 = 0 \implies (\lambda + 1)^2 = -9 \implies \lambda = -1 \pm 3i$$

so the complementary function is

$$v_c = e^{-x}(A\cos 3x + B\sin 3x)$$

For the particular integral, try $v_p = ax^2 + bx + c$, so that $\frac{\mathrm{d}v_p}{\mathrm{d}x} = 2ax + b$ and $\frac{\mathrm{d}^2v_p}{\mathrm{d}x^2} = 2a$. Substituting into the equation from part (a):

$$2a + 2(2ax + b) + 10(ax^2 + bx + c) = 10x^2 + 4x + 12$$

$$10ax^2 + (4a + 10b)x + (2a + 2b + 10c) = 10x^2 + 4x + 12$$

Equating coefficients:

$$10a = 10, \qquad 4a + 10b = 4, \qquad 2a + 2b + 10c = 12$$

so $a = 1$, $b = 0$ and $c = 1$, giving $v_p = x^2 + 1$. The general solution is

$$v = e^{-x}(A\cos 3x + B\sin 3x) + x^2 + 1$$

When $x = 0$, $y = 1$, so $v = 1^2 = 1$:

$$A + 1 = 1 \quad \Longrightarrow \quad A = 0$$

Also $\frac{\mathrm{d}v}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$, so when $x = 0$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 2 \times 1 \times \frac{3}{2} = 3$$

Differentiating the general solution with $A = 0$:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -Be^{-x}\sin 3x + 3Be^{-x}\cos 3x + 2x$$

so at $x = 0$:

$$3B = 3 \quad \Longrightarrow \quad B = 1$$

Hence

$$y^2 = v = x^2 + 1 + e^{-x}\sin 3x$$

Since $y = 1 > 0$ when $x = 0$, the positive square root is taken:

$$y = \sqrt{x^2 + 1 + e^{-x}\sin 3x}$$

5. **(a)** Show that the substitution $x = t^2$, where $t \geq 0$, transforms the differential equation

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(1 - 6\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 12e^{\sqrt{x}} \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 8y = 12e^{t} \qquad (**)$$

[5]

(b) Solve differential equation $(**)$ to determine y in terms of t. **[6]**

(c) Hence find the particular solution of differential equation $(*)$ for which $y = 32$ when $x = (\ln 2)^2$ and $y = 111$ when $x = (\ln 3)^2$. **[2]**

Solution

(a) Since $x = t^2$ with $t \geq 0$, $t = \sqrt{x}$ and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \quad \Longrightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2t}$$

By the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2t}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Differentiating again with respect to x, using the chain rule and the product rule:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \left(\frac{1}{2t}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{1}{2t^2}\frac{\mathrm{d}y}{\mathrm{d}t}\right) \times \frac{1}{2t} \\ &= \frac{1}{4t^2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{1}{4t^3}\frac{\mathrm{d}y}{\mathrm{d}t}\end{aligned}$$

Now substitute into each term of $(*)$, using $4x = 4t^2$ and $\sqrt{x} = t$:

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4t^2\left(\frac{1}{4t^2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{1}{4t^3}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$2\left(1 - 6\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1 - 6t) \times \frac{1}{2t}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t} - 6\frac{\mathrm{d}y}{\mathrm{d}t}$$

Adding these, the $\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}t}$ terms cancel, and $12e^{\sqrt{x}} = 12e^{t}$, so $(*)$ becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 8y = 12e^{t}$$

as required.

(b) The auxiliary equation is

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Longrightarrow \quad (\lambda - 2)(\lambda - 4) = 0 \quad \Longrightarrow \quad \lambda = 2,\ \lambda = 4$$

so the complementary function is

$$y_c = Ae^{2t} + Be^{4t}$$

For the particular integral, try $y_p = ae^t$, so $\frac{\mathrm{d}y_p}{\mathrm{d}t} = ae^t$ and $\frac{\mathrm{d}^2y_p}{\mathrm{d}t^2} = ae^t$. Substituting into $(**)$:

$$ae^t - 6ae^t + 8ae^t = 12e^t \quad \Longrightarrow \quad 3a = 12 \quad \Longrightarrow \quad a = 4$$

The general solution is $y = y_c + y_p$:

$$y = Ae^{2t} + Be^{4t} + 4e^t$$

(c) Using part (b) with $t = \sqrt{x}$, since $t \geq 0$, the general solution of $(*)$ is

$$y = Ae^{2\sqrt{x}} + Be^{4\sqrt{x}} + 4e^{\sqrt{x}}$$

When $x = (\ln 2)^2$, $t = \ln 2$, so $e^t = 2$, $e^{2t} = 4$, $e^{4t} = 16$, and $y = 32$:

$$4A + 16B + 8 = 32 \quad \Longrightarrow \quad A + 4B = 6$$

When $x = (\ln 3)^2$, $t = \ln 3$, so $e^t = 3$, $e^{2t} = 9$, $e^{4t} = 81$, and $y = 111$:

$$9A + 81B + 12 = 111 \quad \Longrightarrow \quad A + 9B = 11$$

Solving these simultaneously gives $B = 1$ and $A = 2$, so the particular solution is

$$y = 2e^{2\sqrt{x}} + e^{4\sqrt{x}} + 4e^{\sqrt{x}}$$

6. A differential equation is given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(1+\tan x\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(2+\tan x+\sec^2 x\right)y = 25x\cos x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2} \qquad (*)$$

(a) Show that the substitution $y = u\cos x$, where u is a function of x, transforms this differential equation into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + 5u = 25x \qquad (**)$$

[5]

(b) Hence find the general solution of differential equation $(*)$, giving your answer in the form $y = f(x)$. **[7]**

Solution

(a) Differentiating $y = u\cos x$ using the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}\cos x - u\sin x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\cos x - 2\frac{\mathrm{d}u}{\mathrm{d}x}\sin x - u\cos x$$

Substituting into the differential equation and collecting the terms in $\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and u:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\cos x + \frac{\mathrm{d}u}{\mathrm{d}x}\Big[-2\sin x + 2(1+\tan x)\cos x\Big]$$
$$+ u\Big[-\cos x - 2(1+\tan x)\sin x + 2\big(2+\tan x+\sec^2 x\big)\cos x\Big] = 25x\cos x$$

The coefficient of $\frac{\mathrm{d}u}{\mathrm{d}x}$ is

$$-2\sin x + 2\cos x + 2\sin x = 2\cos x$$

The coefficient of u is

$$-\cos x - 2\sin x - \frac{2\sin^2 x}{\cos x} + 4\cos x + 2\sin x + \frac{2}{\cos x}$$
$$= 3\cos x + \frac{2\big(1-\sin^2 x\big)}{\cos x}$$
$$= 3\cos x + \frac{2\cos^2 x}{\cos x}$$
$$= 5\cos x$$

So the equation becomes

$$\cos x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\cos x\frac{\mathrm{d}u}{\mathrm{d}x} + 5u\cos x = 25x\cos x$$

Since $\cos x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, dividing through by $\cos x$ gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + 5u = 25x$$

as required.

(b) The auxiliary equation is

$$\lambda^2 + 2\lambda + 5 = 0 \quad \Longrightarrow \quad \lambda = -1 \pm 2i$$

so the complementary function is

$$u_c = e^{-x}(A\cos 2x + B\sin 2x)$$

For the particular integral, try $u_p = ax + b$, so $\frac{\mathrm{d}u_p}{\mathrm{d}x} = a$ and $\frac{\mathrm{d}^2u_p}{\mathrm{d}x^2} = 0$. Substituting:

$$2a + 5(ax + b) = 25x$$

Equating coefficients of x gives $5a = 25$, so $a = 5$. Equating constant terms gives $2a + 5b = 0$, so $b = -2$. Hence $u_p = 5x - 2$, and

$$u = e^{-x}(A\cos 2x + B\sin 2x) + 5x - 2$$

Since $y = u\cos x$, the general solution of $(*)$ is

$$y = \cos x\big[e^{-x}(A\cos 2x + B\sin 2x) + 5x - 2\big]$$

7. The vertical height, h m, above horizontal ground, of a passenger on a fairground ride, t seconds after the ride starts, where $0 < t \leq 5$, is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} - 4t\frac{\mathrm{d}h}{\mathrm{d}t} + 6h = 6t + 12 \qquad (*)$$

(a) Given that $t = e^x$, show that

$$t\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}x}$$

and

$$t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = \frac{\mathrm{d}^2h}{\mathrm{d}x^2} - \frac{\mathrm{d}h}{\mathrm{d}x}$$

[4]

(b) Hence show that the substitution $t = e^x$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}^2h}{\mathrm{d}x^2} - 5\frac{\mathrm{d}h}{\mathrm{d}x} + 6h = 6e^x + 12 \qquad (**)$$

[2]

(c) Hence show that

$$h = At^2 + Bt^3 + 3t + 2$$

where A and B are constants. **[7]**

Given that when $t = 1$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 4$, and when $t = 2$, $h = \frac{28}{3}$,

(d) find the maximum height of the passenger above the ground during the ride, justifying that it is a maximum. **[6]**

Solution

(a) Differentiating $t = e^x$ with respect to x:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = e^x = t$$

Using the chain rule:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = t\frac{\mathrm{d}h}{\mathrm{d}t}$$

so $t\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}x}$, as required.

Differentiating $\frac{\mathrm{d}h}{\mathrm{d}x} = t\frac{\mathrm{d}h}{\mathrm{d}t}$ with respect to x, using the product rule and the chain rule:

$$\begin{aligned}\frac{\mathrm{d}^2h}{\mathrm{d}x^2} &= \frac{\mathrm{d}t}{\mathrm{d}x} \times \frac{\mathrm{d}h}{\mathrm{d}t} + t \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right) \\ &= t\frac{\mathrm{d}h}{\mathrm{d}t} + t \times \frac{\mathrm{d}^2h}{\mathrm{d}t^2} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= t\frac{\mathrm{d}h}{\mathrm{d}t} + t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} \\ &= \frac{\mathrm{d}h}{\mathrm{d}x} + t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\end{aligned}$$

Rearranging:

$$t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = \frac{\mathrm{d}^2h}{\mathrm{d}x^2} - \frac{\mathrm{d}h}{\mathrm{d}x}$$

as required.

(b) Substituting both results from part (a) and $t = e^x$ into $(*)$:

$$\left(\frac{\mathrm{d}^2h}{\mathrm{d}x^2} - \frac{\mathrm{d}h}{\mathrm{d}x}\right) - 4\frac{\mathrm{d}h}{\mathrm{d}x} + 6h = 6e^x + 12$$

so

$$\frac{\mathrm{d}^2h}{\mathrm{d}x^2} - 5\frac{\mathrm{d}h}{\mathrm{d}x} + 6h = 6e^x + 12$$

as required.

(c) The auxiliary equation is

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Longrightarrow \quad (\lambda - 2)(\lambda - 3) = 0 \quad \Longrightarrow \quad \lambda = 2,\ \lambda = 3$$

so the complementary function is

$$h_c = Ae^{2x} + Be^{3x}$$

For the particular integral, try $h_p = ae^x + b$, so that

$$\frac{\mathrm{d}h_p}{\mathrm{d}x} = ae^x, \qquad \frac{\mathrm{d}^2h_p}{\mathrm{d}x^2} = ae^x$$

Substituting into $(**)$:

$$ae^x - 5ae^x + 6(ae^x + b) = 6e^x + 12$$
$$2ae^x + 6b = 6e^x + 12$$

Equating coefficients, $2a = 6$ and $6b = 12$, so $a = 3$ and $b = 2$. The general solution is

$$h = Ae^{2x} + Be^{3x} + 3e^x + 2$$

Since $t = e^x$, $e^{2x} = t^2$ and $e^{3x} = t^3$, so

$$h = At^2 + Bt^3 + 3t + 2$$

as required.

(d) Differentiating:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 2At + 3Bt^2 + 3$$

When $t = 1$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 4$, and when $t = 2$, $h = \frac{28}{3}$:

$$2A + 3B + 3 = 4 \quad \Longrightarrow \quad 2A + 3B = 1$$

$$4A + 8B + 6 + 2 = \frac{28}{3} \quad \Longrightarrow \quad A + 2B = \frac{1}{3}$$

Solving these simultaneously gives $A = 1$ and $B = -\frac{1}{3}$, so

$$h = t^2 - \frac{1}{3}t^3 + 3t + 2$$

At a stationary point $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$:

$$2t - t^2 + 3 = 0 \quad \Longrightarrow \quad t^2 - 2t - 3 = 0 \quad \Longrightarrow \quad (t-3)(t+1) = 0$$

Since $0 < t \leq 5$, $t = 3$. Differentiating again:

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = 2 - 2t$$

When $t = 3$, $\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -4 < 0$, so this is a maximum. It is the only stationary point for $0 < t \leq 5$, so h increases up to $t = 3$ and decreases after it, and this is the greatest height during the ride:

$$h = 3^2 - \frac{1}{3} \times 3^3 + 3 \times 3 + 2 = 11$$

The maximum height of the passenger is 11 m, when $t = 3$.

8. A biologist models the number of bacteria in a culture, t days after observations begin, where $1 \leq t \leq 10$, by the differential equation

$$t^2\frac{\mathrm{d}^2N}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}N}{\mathrm{d}t} - 2N = 4t^2 \qquad (*)$$

where N is the number of bacteria, in thousands.

(a) Show that the substitution $t = e^x$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}^2N}{\mathrm{d}x^2} + \frac{\mathrm{d}N}{\mathrm{d}x} - 2N = 4e^{2x} \qquad (**)$$ **[5]**

(b) Hence find the general solution for N in terms of t. **[7]**

Given that when $t = 1$, $N = 37$ and $\frac{\mathrm{d}N}{\mathrm{d}t} = -58$,

(c) find the minimum number of bacteria in the culture, justifying that the value you find is a minimum. **[6]**

Solution

(a) Differentiating $t = e^x$ with respect to x:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = e^x = t$$

Using the chain rule:

$$\frac{\mathrm{d}N}{\mathrm{d}x} = \frac{\mathrm{d}N}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = t\frac{\mathrm{d}N}{\mathrm{d}t} \quad \Longrightarrow \quad t\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}N}{\mathrm{d}x}$$

Differentiating again with respect to x, using the product rule and the chain rule:

$$\begin{aligned}\frac{\mathrm{d}^2N}{\mathrm{d}x^2} &= \frac{\mathrm{d}t}{\mathrm{d}x} \times \frac{\mathrm{d}N}{\mathrm{d}t} + t \times \frac{\mathrm{d}^2N}{\mathrm{d}t^2} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= t\frac{\mathrm{d}N}{\mathrm{d}t} + t^2\frac{\mathrm{d}^2N}{\mathrm{d}t^2} \\ &= \frac{\mathrm{d}N}{\mathrm{d}x} + t^2\frac{\mathrm{d}^2N}{\mathrm{d}t^2}\end{aligned}$$

so

$$t^2\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \frac{\mathrm{d}^2N}{\mathrm{d}x^2} - \frac{\mathrm{d}N}{\mathrm{d}x}$$

Substituting into $(*)$, with $4t^2 = 4e^{2x}$:

$$\left(\frac{\mathrm{d}^2N}{\mathrm{d}x^2} - \frac{\mathrm{d}N}{\mathrm{d}x}\right) + 2\frac{\mathrm{d}N}{\mathrm{d}x} - 2N = 4e^{2x}$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}x^2} + \frac{\mathrm{d}N}{\mathrm{d}x} - 2N = 4e^{2x}$$

as required.

(b) The auxiliary equation of $(**)$ is

$$\lambda^2 + \lambda - 2 = 0 \quad \Longrightarrow \quad (\lambda + 2)(\lambda - 1) = 0 \quad \Longrightarrow \quad \lambda = 1,\ \lambda = -2$$

so the complementary function is

$$N_c = Ae^{x} + Be^{-2x}$$

For the particular integral, try $N_p = ae^{2x}$, so that $\frac{\mathrm{d}N_p}{\mathrm{d}x} = 2ae^{2x}$ and $\frac{\mathrm{d}^2N_p}{\mathrm{d}x^2} = 4ae^{2x}$. Substituting into $(**)$:

$$4ae^{2x} + 2ae^{2x} - 2ae^{2x} = 4e^{2x} \quad \Longrightarrow \quad 4a = 4 \quad \Longrightarrow \quad a = 1$$

So the general solution of $(**)$ is

$$N = Ae^{x} + Be^{-2x} + e^{2x}$$

Since $e^{x} = t$, $e^{-2x} = \frac{1}{t^2}$ and $e^{2x} = t^2$, the general solution in terms of t is

$$N = At + \frac{B}{t^2} + t^2$$

(c) Differentiating the general solution from part (b):

$$\frac{\mathrm{d}N}{\mathrm{d}t} = A - \frac{2B}{t^3} + 2t$$

When $t = 1$, $N = 37$ and $\frac{\mathrm{d}N}{\mathrm{d}t} = -58$, so

$$A + B + 1 = 37 \quad \Longrightarrow \quad A + B = 36$$

$$A - 2B + 2 = -58 \quad \Longrightarrow \quad A - 2B = -60$$

Solving gives $A = 4$ and $B = 32$, so

$$N = 4t + \frac{32}{t^2} + t^2, \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = 4 - \frac{64}{t^3} + 2t$$

At a stationary point $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$. Multiplying by $\frac{t^3}{2}$:

$$t^4 + 2t^3 - 32 = 0$$

$$(t-2)\left(t^3 + 4t^2 + 8t + 16\right) = 0$$

The cubic factor is positive for $t > 0$, so the only stationary point in $1 \le t \le 10$ is at $t = 2$. Then

$$N = 4 \times 2 + \frac{32}{4} + 4 = 20$$

To classify it:

$$\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \frac{192}{t^4} + 2, \qquad \text{so at } t = 2, \ \frac{\mathrm{d}^2N}{\mathrm{d}t^2} = 12 + 2 = 14 > 0$$

so this is a minimum, and as it is the only stationary point in $1 \le t \le 10$, it is the least value of N there. Since N is in thousands, the minimum number of bacteria is 20 000, when $t = 2$.

9. The motion of a particle P along the x-axis is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2t(2t+1)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(3t^2+4t+2\right)x = -6t^3e^{2t} \qquad (*)$$

where P has displacement x metres from the origin O at time t minutes, $t > 0$.

(a) Show that the substitution $x = tu$ transforms the differential equation $(*)$ into the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - 4\frac{\mathrm{d}u}{\mathrm{d}t} + 3u = -6e^{2t} \qquad (**)$$

[5]

(b) Hence find the general solution of the differential equation $(*)$, giving x in terms of t. **[7]**

Given that P is at O when $t = \ln 2$ and when $t = \ln 4$,

(c) find the particular solution of the differential equation $(*)$, and find the interval of time during which P is on the positive x-axis. **[4]**

Solution

(a) Differentiate $x = tu$ with respect to t, using the product rule:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + t\frac{\mathrm{d}u}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}t} + t\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = 2\frac{\mathrm{d}u}{\mathrm{d}t} + t\frac{\mathrm{d}^2u}{\mathrm{d}t^2}$$

Substituting into $(*)$:

$$t^2\left(2\frac{\mathrm{d}u}{\mathrm{d}t} + t\frac{\mathrm{d}^2u}{\mathrm{d}t^2}\right) - 2t(2t+1)\left(u + t\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \left(3t^2+4t+2\right)tu = -6t^3e^{2t}$$

Expanding and collecting terms:

$$t^3\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \left(2t^2 - 4t^3 - 2t^2\right)\frac{\mathrm{d}u}{\mathrm{d}t} + \left(-4t^2 - 2t + 3t^3 + 4t^2 + 2t\right)u = -6t^3e^{2t}$$

$$t^3\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - 4t^3\frac{\mathrm{d}u}{\mathrm{d}t} + 3t^3u = -6t^3e^{2t}$$

Since $t > 0$, dividing by t^3 gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - 4\frac{\mathrm{d}u}{\mathrm{d}t} + 3u = -6e^{2t}$$

as required.

(b) The auxiliary equation is

$$\lambda^2 - 4\lambda + 3 = 0 \implies (\lambda - 1)(\lambda - 3) = 0 \implies \lambda = 1,\ \lambda = 3$$

so the complementary function is

$$u_c = Ae^t + Be^{3t}$$

For the particular integral, try $u_p = ae^{2t}$, so $\frac{\mathrm{d}u_p}{\mathrm{d}t} = 2ae^{2t}$ and $\frac{\mathrm{d}^2u_p}{\mathrm{d}t^2} = 4ae^{2t}$. Substituting:

$$4ae^{2t} - 8ae^{2t} + 3ae^{2t} = -6e^{2t} \quad \Longrightarrow \quad -a = -6 \quad \Longrightarrow \quad a = 6$$

So

$$u = Ae^t + Be^{3t} + 6e^{2t}$$

and, since $x = tu$, the general solution of $(*)$ is

$$x = t\big(Ae^t + Be^{3t} + 6e^{2t}\big)$$

(c) When $t = \ln 2$, $e^t = 2$ and $x = 0$. Since $\ln 2 \neq 0$:

$$2A + 8B + 24 = 0 \quad \Longrightarrow \quad A + 4B = -12$$

When $t = \ln 4$, $e^t = 4$ and $x = 0$. Since $\ln 4 \neq 0$:

$$4A + 64B + 96 = 0 \quad \Longrightarrow \quad A + 16B = -24$$

Subtracting, $12B = -12$, so $B = -1$ and $A = -8$. The particular solution is

$$x = t\big(6e^{2t} - 8e^t - e^{3t}\big)$$

Factorising:

$$x = -te^t\big(e^{2t} - 6e^t + 8\big) = -te^t\big(e^t - 2\big)\big(e^t - 4\big)$$

Since $t > 0$ and $e^t > 0$, $x > 0$ exactly when $(e^t - 2)(e^t - 4) < 0$, that is when $2 < e^t < 4$. So P is on the positive x-axis for

$$\ln 2 < t < \ln 4$$

10. A curve satisfies the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x^2, \qquad -1 < x < 1 \qquad (*)$$

The substitution $x = \sin t$, where $-\frac{\pi}{2} < t < \frac{\pi}{2}$, is used to transform $(*)$.

(a) Show that

$$\sqrt{1-x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \text{and} \qquad (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$ **[4]**

(b) Hence show that the substitution $x = \sin t$ transforms $(*)$ into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y = 4 - 4\cos 2t \qquad (**)$$ **[2]**

(c) Find the general solution of $(**)$, using a particular integral of the form $y = p + qt\sin 2t$, where p and q are constants to be found. **[5]**

(d) Hence find the solution of $(*)$ for which $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$, giving your answer in the form $y = f(x)$, with no trigonometric function of $\arcsin x$. **[5]**

Solution

(a) Differentiating $x = \sin t$ with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t$$

Since $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $\cos t > 0$, so $\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$. By the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = \cos t\,\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1-x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}$$

as required.

Differentiate $\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t\,\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to t, using the product rule and the chain rule:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\sin t\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= -\sin t\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos t\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \times \frac{\mathrm{d}x}{\mathrm{d}t}\\ &= -\sin t\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cos^2 t\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\end{aligned}$$

Since $\sin t = x$ and $\cos^2 t = 1 - x^2$, this gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x}$$

as required.

(b) Using part (a), the left-hand side of $(*)$ is $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y$, and $8x^2 = 8\sin^2 t$. Using $\cos 2t = 1 - 2\sin^2 t$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y = 8\sin^2 t = 4(1 - \cos 2t) = 4 - 4\cos 2t$$

as required.

(c) The auxiliary equation is

$$\lambda^2 + 4 = 0 \quad \Longrightarrow \quad \lambda = \pm 2i$$

so the complementary function is

$$y_c = A\cos 2t + B\sin 2t$$

For the particular integral $y_p = p + qt\sin 2t$:

$$\frac{\mathrm{d}y_p}{\mathrm{d}t} = q\sin 2t + 2qt\cos 2t$$

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}t^2} = 2q\cos 2t + 2q\cos 2t - 4qt\sin 2t = 4q\cos 2t - 4qt\sin 2t$$

Substituting into $(**)$:

$$4q\cos 2t - 4qt\sin 2t + 4p + 4qt\sin 2t = 4 - 4\cos 2t$$

$$4p + 4q\cos 2t = 4 - 4\cos 2t$$

Equating coefficients gives $4p = 4$ and $4q = -4$, so $p = 1$ and $q = -1$. The general solution is

$$y = A\cos 2t + B\sin 2t + 1 - t\sin 2t$$

(d) When $x = 0$, $t = 0$. Since $y = 3$ when $t = 0$:

$$A + 1 = 3 \quad \Longrightarrow \quad A = 2$$

By part (a), when $x = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{1 - 0^2} \times 4 = 4$$

Differentiating the general solution:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2A\sin 2t + 2B\cos 2t - \sin 2t - 2t\cos 2t$$

so at $t = 0$, $2B = 4$, giving $B = 2$. Hence

$$y = 2\cos 2t + 2\sin 2t + 1 - t\sin 2t$$

Now return to x, using $t = \arcsin x$, $\cos 2t = 1 - 2\sin^2 t = 1 - 2x^2$ and $\sin 2t = 2\sin t\cos t = 2x\sqrt{1 - x^2}$:

$$y = 2(1 - 2x^2) + 4x\sqrt{1 - x^2} + 1 - 2x\sqrt{1 - x^2}\arcsin x$$

$$= 3 - 4x^2 + 4x\sqrt{1 - x^2} - 2x\sqrt{1 - x^2}\arcsin x$$

11. It is given that $y = \sinh u$ and

$$\sqrt{1+\sinh^2 u}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+3\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\sinh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+2\sinh u = 10\sin x$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = 10\sin x \qquad \textbf{[5]}$$

(b) Find u in terms of x, given that, when $x = 0$, $u = 0$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$. **[11]**

Solution

(a) Differentiating $y = \sinh u$ with respect to x, using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cosh u\,\frac{\mathrm{d}u}{\mathrm{d}x}$$

Differentiating again, using the product rule and the chain rule:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \cosh u\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+\sinh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2$$

Since $\cosh^2 u - \sinh^2 u = 1$ and $\cosh u > 0$, $\sqrt{1+\sinh^2 u} = \cosh u$. The given equation therefore becomes

$$\cosh u\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+3\cosh u\,\frac{\mathrm{d}u}{\mathrm{d}x}+\sinh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+2\sinh u = 10\sin x$$

$$\left(\cosh u\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+\sinh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right)+3\left(\cosh u\,\frac{\mathrm{d}u}{\mathrm{d}x}\right)+2\sinh u = 10\sin x$$

Replacing each bracket by the derivatives above, and $\sinh u$ by y:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = 10\sin x$$

as required.

(b) The auxiliary equation is

$$\lambda^2+3\lambda+2 = 0 \quad\Longrightarrow\quad (\lambda+1)(\lambda+2) = 0 \quad\Longrightarrow\quad \lambda = -1,\ \lambda = -2$$

so the complementary function is

$$y_c = Ae^{-x}+Be^{-2x}$$

For the particular integral, try $y_p = a\sin x + b\cos x$. Then

$$\frac{\mathrm{d}y_p}{\mathrm{d}x} = a\cos x - b\sin x$$

$$\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} = -a\sin x - b\cos x$$

Substituting into the differential equation:

$$(-a\sin x - b\cos x) + 3(a\cos x - b\sin x) + 2(a\sin x + b\cos x) = 10\sin x$$
$$(a - 3b)\sin x + (3a + b)\cos x = 10\sin x$$

Equating coefficients of $\sin x$ and $\cos x$:

$$a - 3b = 10, \qquad 3a + b = 0$$

so $a = 1$ and $b = -3$. The general solution is

$$y = Ae^{-x} + Be^{-2x} + \sin x - 3\cos x$$

Now use the initial conditions. When $x = 0$, $u = 0$, so $y = \sinh 0 = 0$, and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cosh u \frac{\mathrm{d}u}{\mathrm{d}x} = \cosh 0 \times 2 = 2$$

Setting $y = 0$ when $x = 0$:

$$A + B - 3 = 0 \quad \Longrightarrow \quad A + B = 3$$

Differentiating the general solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -Ae^{-x} - 2Be^{-2x} + \cos x + 3\sin x$$

Setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$:

$$-A - 2B + 1 = 2 \quad \Longrightarrow \quad A + 2B = -1$$

Solving these simultaneously gives $B = -4$ and $A = 7$, so

$$y = 7e^{-x} - 4e^{-2x} + \sin x - 3\cos x$$

Since $y = \sinh u$, $u = \operatorname{arsinh} y$, so

$$u = \operatorname{arsinh}\left(7e^{-x} - 4e^{-2x} + \sin x - 3\cos x\right)$$

12. **(a)** Given that $x = \sinh t$, show that

$$\sqrt{1+x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \text{and} \qquad \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} \qquad \textbf{[4]}$$

(b) Hence show that the substitution $x = \sinh t$ transforms the differential equation

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 6x \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = 6\sinh t \qquad (**) \qquad \textbf{[2]}$$

(c) Solve differential equation $(**)$ to find y in terms of t. **[6]**

(d) Hence find the solution of differential equation $(*)$ for which $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$, giving y in terms of x only, with no hyperbolic or exponential functions. **[5]**

Solution

(a) Differentiating $x = \sinh t$ with respect to t, and using $\cosh^2 t - \sinh^2 t = 1$ with $\cosh t > 0$, gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cosh t = \sqrt{1+\sinh^2 t} = \sqrt{1+x^2}$$

By the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{1+x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}$$

as required.

Now write $\frac{\mathrm{d}y}{\mathrm{d}t} = \cosh t\,\frac{\mathrm{d}y}{\mathrm{d}x}$ and differentiate with respect to t, using the product rule and the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \sinh t\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cosh t \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= \sinh t\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cosh t \times \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\
&= \sinh t\,\frac{\mathrm{d}y}{\mathrm{d}x} + \cosh^2 t\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
&= x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}
\end{aligned}$$

So $\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$, as required.

(b) By part (a), the first two terms of $(*)$ together equal $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$, and $6x = 6\sinh t$, so $(*)$ becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = 6\sinh t$$

as required.

(c) The auxiliary equation is

$$\lambda^2 - 4 = 0 \quad \Longrightarrow \quad \lambda = \pm 2$$

so the complementary function is

$$y_c = Ae^{2t} + Be^{-2t}$$

For the particular integral, try $y_p = a\sinh t + b\cosh t$. Then

$$\frac{\mathrm{d}^2 y_p}{\mathrm{d}t^2} = a\sinh t + b\cosh t$$

Substituting into $(**)$:

$$a\sinh t + b\cosh t - 4(a\sinh t + b\cosh t) = 6\sinh t$$

$$-3a\sinh t - 3b\cosh t = 6\sinh t$$

Equating coefficients gives $a = -2$ and $b = 0$, so $y_p = -2\sinh t$. The general solution is

$$y = Ae^{2t} + Be^{-2t} - 2\sinh t$$

(d) When $x = 0$, $\sinh t = 0$, so $t = 0$. Since $y = 3$ when $t = 0$:

$$A + B = 3$$

By part (a), $\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{1+x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x}$, so when $t = 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 \times 0 = 0$. Differentiating the solution from part (c):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2Ae^{2t} - 2Be^{-2t} - 2\cosh t$$

so

$$2A - 2B - 2 = 0 \quad \Longrightarrow \quad A - B = 1$$

Solving gives $A = 2$ and $B = 1$, so

$$y = 2e^{2t} + e^{-2t} - 2\sinh t$$

Now $e^t = \cosh t + \sinh t$ and $e^{-t} = \cosh t - \sinh t$, with $\cosh t = \sqrt{1+x^2}$ from part (a), so

$$e^t = x + \sqrt{1+x^2}, \qquad e^{-t} = \sqrt{1+x^2} - x$$

Squaring:

$$e^{2t} = x^2 + 2x\sqrt{1+x^2} + 1 + x^2 = 2x^2 + 1 + 2x\sqrt{1+x^2}$$

$$e^{-2t} = x^2 - 2x\sqrt{1+x^2} + 1 + x^2 = 2x^2 + 1 - 2x\sqrt{1+x^2}$$

Substituting these and $\sinh t = x$:

$$y = 2\left(2x^2 + 1 + 2x\sqrt{1+x^2}\right) + \left(2x^2 + 1 - 2x\sqrt{1+x^2}\right) - 2x$$

$$= 6x^2 - 2x + 3 + 2x\sqrt{1+x^2}$$