OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

1st Order Differential Equations by Substitution

Worked solutions

12 questions · 118 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

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1. The variables x and y, where $x > 0$, are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 3xy + y^2}{x^2} \qquad (*)$$

(a) Use the substitution $y = ux$, where u is a function of x, to show that

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = (u+1)^2$$ **[3]**

(b) Hence find the particular solution of equation $(*)$ for which $y = 0$ when $x = 1$, giving your answer in the form $y = f(x)$. **[5]**

Solution

(a) Differentiating $y = ux$ with respect to x using the product rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$$

Dividing each term of the right-hand side of equation $(*)$ by x^2 and substituting $y = ux$:

$$\frac{x^2 + 3xy + y^2}{x^2} = 1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2 = 1 + 3u + u^2$$

So equation $(*)$ becomes

$$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + 3u + u^2$$

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 + 2u + 1$$

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = (u+1)^2$$

as required.

(b) Using part (a), separate the variables:

$$\int \frac{1}{(u+1)^2}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x$$

Since $x > 0$, integrating both sides gives

$$-\frac{1}{u+1} = \ln x + c$$

When $x = 1$, $y = 0$, so $u = \frac{y}{x} = 0$. Substituting:

$$-\frac{1}{0+1} = \ln 1 + c \quad \Longrightarrow \quad c = -1$$

Hence

$$-\frac{1}{u+1} = \ln x - 1$$

$$u + 1 = \frac{1}{1 - \ln x}$$

$$u = \frac{1}{1 - \ln x} - 1 = \frac{\ln x}{1 - \ln x}$$

Since $y = ux$, the particular solution is

$$y = \frac{x \ln x}{1 - \ln x}$$

2. **(a)** Show that the substitution $y^3 = \frac{1}{t}$ transforms the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2y^4, \qquad x > 0, \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{3t}{x} = -3x \qquad (**)$$

[3]

(b) Solve equation $(**)$, and hence find y in terms of x, given that $y = \frac{1}{2}$ when $x = 1$. **[6]**

Solution

(a) Since $y^3 = \frac{1}{t}$, $y \neq 0$ and $t = y^{-3}$. Differentiating with respect to x:

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -3y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} \quad \Longrightarrow \quad y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}\frac{\mathrm{d}t}{\mathrm{d}x}$$

Dividing equation $(*)$ by y^4 gives

$$xy^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} + y^{-3} = x^2$$

Substituting $y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}\frac{\mathrm{d}t}{\mathrm{d}x}$ and $y^{-3} = t$:

$$-\frac{x}{3}\frac{\mathrm{d}t}{\mathrm{d}x} + t = x^2$$

Multiplying through by $-\frac{3}{x}$ gives

$$\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{3t}{x} = -3x$$

as required.

(b) Since $x > 0$, the logarithm needs no modulus. The integrating factor is

$$\mu(x) = e^{\int -\frac{3}{x}\,\mathrm{d}x} = e^{-3\ln x} = x^{-3}$$

Multiplying equation $(**)$ by x^{-3}:

$$x^{-3}\frac{\mathrm{d}t}{\mathrm{d}x} - 3x^{-4}t = -3x^{-2}$$

The left-hand side is the derivative of $\mu(x)t = x^{-3}t$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-3}t\right) = -3x^{-2}$$

Integrating both sides:

$$x^{-3}t = 3x^{-1} + c$$

$$t = 3x^2 + cx^3$$

Since $y^3 = \frac{1}{t}$, when $x = 1$ and $y = \frac{1}{2}$, $y^3 = \frac{1}{8}$, so $t = 8$. Substituting:

$$8 = 3 + c \quad \Longrightarrow \quad c = 5$$

So $t = 3x^2 + 5x^3$, and hence

$$y^3 = \frac{1}{3x^2 + 5x^3} \quad \Longrightarrow \quad y = \frac{1}{\sqrt[3]{3x^2 + 5x^3}}$$

3. In this question, $0 < x < \pi$.

(a) Show that the substitution $z = y^{\frac{1}{2}}$ transforms the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cot x = 4y^{\frac{1}{2}}\cos x \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} + z\cot x = 2\cos x \qquad (**)$$

[3]

(b) Solve equation $(**)$ to find z as a function of x. **[4]**

(c) Hence find the particular solution of equation $(*)$ for which $y = 4$ when $x = \frac{\pi}{6}$, giving y in terms of x. **[2]**

Solution

(a) Since $z = y^{\frac{1}{2}}$, we have $y = z^2$. Differentiating with respect to x, using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2z\frac{\mathrm{d}z}{\mathrm{d}x}$$

Substituting $y = z^2$, $y^{\frac{1}{2}} = z$ and this expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ into equation $(*)$:

$$2z\frac{\mathrm{d}z}{\mathrm{d}x} + 2z^2\cot x = 4z\cos x$$

For $z \neq 0$, divide through by $2z$:

$$\frac{\mathrm{d}z}{\mathrm{d}x} + z\cot x = 2\cos x$$

This is equation $(**)$, as required.

(b) Equation $(**)$ is linear in z. Since $\sin x > 0$ for $0 < x < \pi$, the logarithm needs no modulus. The integrating factor is

$$\mu(x) = e^{\int \cot x\,\mathrm{d}x} = e^{\ln(\sin x)} = \sin x$$

Multiplying equation $(**)$ by $\sin x$:

$$\sin x\frac{\mathrm{d}z}{\mathrm{d}x} + z\cos x = 2\sin x\cos x$$

The left-hand side is the derivative of $\mu(x)z = z\sin x$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(z\sin x) = 2\sin x\cos x$$

By the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^2 x\right) = 2\sin x\cos x$. Integrating both sides therefore gives

$$z\sin x = \int 2\sin x\cos x\,\mathrm{d}x$$

$$= \sin^2 x + c$$

Dividing by $\sin x$:

$$z = \sin x + \frac{c}{\sin x}$$

(c) When $x = \frac{\pi}{6}$, $y = 4$, so $z = 4^{\frac{1}{2}} = 2$. Also $\sin \frac{\pi}{6} = \frac{1}{2}$. Substituting these into the result of part (b):

$$2 = \frac{1}{2} + \frac{c}{\frac{1}{2}}$$

$$2 = \frac{1}{2} + 2c$$

$$c = \frac{3}{4}$$

So $z = \sin x + \frac{3}{4 \sin x}$. Since $y = z^2$, the particular solution of equation $(*)$ is

$$y = \left(\sin x + \frac{3}{4 \sin x} \right)^2 = \frac{\left(4 \sin^2 x + 3\right)^2}{16 \sin^2 x}$$

4. For $x > 0$, a curve satisfies the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x\cos^2\left(\frac{y}{x}\right)$$

(a) Use the substitution $y = xz$ to find the general solution of the differential equation, giving y explicitly in terms of x. **[6]**

(b) Find the particular solution for which $y = \frac{\pi}{2}$ when $x = 2$. Hence find the exact value of y when $x = 2e^{-2}$. **[3]**

Solution

(a) Differentiating $y = xz$ with respect to x using the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}$$

Substituting into the differential equation, with $\frac{y}{x} = z$:

$$x\left(z + x\frac{\mathrm{d}z}{\mathrm{d}x}\right) - xz = x\cos^2 z$$

$$x^2\frac{\mathrm{d}z}{\mathrm{d}x} = x\cos^2 z$$

$$x\frac{\mathrm{d}z}{\mathrm{d}x} = \cos^2 z$$

This is separable. Separating the variables:

$$\int \sec^2 z\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

Since $x > 0$, the logarithm needs no modulus, so integrating gives

$$\tan z = \ln x + c$$

Replacing z by $\frac{y}{x}$:

$$\tan\left(\frac{y}{x}\right) = \ln x + c$$

$$\frac{y}{x} = \arctan(\ln x + c)$$

$$y = x\arctan(\ln x + c)$$

(b) When $x = 2$, $y = \frac{\pi}{2}$, so $z = \frac{y}{x} = \frac{\pi}{4}$. Substituting into $\tan z = \ln x + c$:

$$\tan\frac{\pi}{4} = \ln 2 + c \quad\Longrightarrow\quad 1 = \ln 2 + c \quad\Longrightarrow\quad c = 1 - \ln 2$$

The particular solution is therefore

$$y = x\arctan(1 + \ln x - \ln 2) = x\arctan\left(1 + \ln\left(\frac{x}{2}\right)\right)$$

When $x = 2e^{-2}$, $\ln\left(\frac{x}{2}\right) = \ln\left(e^{-2}\right) = -2$, so

$$\begin{aligned} y &= 2e^{-2}\arctan(1-2) \\ &= 2e^{-2}\arctan(-1) \\ &= 2e^{-2} \times \left(-\frac{\pi}{4}\right) \\ &= -\frac{\pi}{2e^2} \end{aligned}$$

5. Use the substitution $u = y^2$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \frac{\cos^3 x}{y}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

given that $y = 3$ when $x = 0$. Give your answer in the form $y = f(x)$. **[9]**

Solution

With $u = y^2$, differentiating with respect to x gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

Multiplying the differential equation through by $2y$:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2\tan x = 2\cos^3 x$$

Replacing $2y\frac{\mathrm{d}y}{\mathrm{d}x}$ by $\frac{\mathrm{d}u}{\mathrm{d}x}$ and y^2 by u gives the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2u\tan x = 2\cos^3 x$$

On $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\cos x > 0$, so $\sec x > 0$ and $\int \tan x\,\mathrm{d}x = \ln(\sec x)$. The integrating factor is

$$\mu(x) = e^{\int 2\tan x\,\mathrm{d}x} = e^{2\ln(\sec x)} = \sec^2 x$$

Multiplying the linear equation by $\sec^2 x$:

$$\sec^2 x\frac{\mathrm{d}u}{\mathrm{d}x} + 2u\sec^2 x\tan x = 2\cos^3 x\sec^2 x = 2\cos x$$

The left-hand side is the derivative of $\mu(x)u = u\sec^2 x$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u\sec^2 x\right) = 2\cos x$$

Integrating both sides:

$$u\sec^2 x = 2\sin x + c$$

$$u = \cos^2 x\,(2\sin x + c)$$

Since $u = y^2$, this is

$$y^2 = \cos^2 x\,(2\sin x + c)$$

When $x = 0$, $y = 3$:

$$9 = 1 \times (0 + c) \quad \Longrightarrow \quad c = 9$$

so

$$y^2 = \cos^2 x\,(2\sin x + 9)$$

Since $y = 3 > 0$ when $x = 0$ and $\cos x > 0$ on the interval, taking the positive square root gives

$$y = \cos x\sqrt{2\sin x + 9}$$

6. The concentration, $P\,\text{mg m}^{-3}$, of a pollutant in a reservoir, t days after the pollutant entered the reservoir, is modelled by the differential equation

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{t+1} - 2(t+1)P^2 \qquad (*)$$

(a) Show that the substitution $x = \frac{1}{P^2}$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{t+1} = 4(t+1) \qquad (**)$$ **[3]**

(b) Given that $P = 0.5$ when $t = 0$, solve equation $(**)$ to show that, according to the model,

$$P^2 = \frac{(t+1)^2}{(t+1)^4 + k}$$

where k is a constant to be determined. **[6]**

(c) Measurements taken 60 days after the pollutant entered the reservoir showed a concentration of $0.2\,\text{mg m}^{-3}$.
Comment on the reliability of the model in the long term, giving a reason for your answer. **[2]**

Solution

(a) Since $x = P^{-2}$, the chain rule gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2P^{-3}\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{2}{P^2} \times \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}$$

Substituting from equation $(*)$, and using $\frac{1}{P^2} = x$:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -2x\left(\frac{1}{t+1} - 2(t+1)P^2\right)\\ &= -\frac{2x}{t+1} + 4(t+1)xP^2\end{aligned}$$

Since $xP^2 = 1$, this becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2x}{t+1} + 4(t+1)$$

so

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{t+1} = 4(t+1)$$

as required.

(b) Since $t + 1 > 0$ for $t \geq 0$, the logarithm needs no modulus. The integrating factor is

$$\mu(t) = e^{\int \frac{2}{t+1}\,\mathrm{d}t} = e^{2\ln(t+1)} = (t+1)^2$$

Multiplying equation $(**)$ by $(t+1)^2$:

$$(t+1)^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2(t+1)x = 4(t+1)^3$$

The left-hand side is the derivative of $\mu(t)x = (t+1)^2 x$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(t+1)^2 x\right] = 4(t+1)^3$$

Integrating both sides:

$$(t+1)^2 x = (t+1)^4 + c$$

Since $x = \frac{1}{P^2}$, this gives

$$\frac{(t+1)^2}{P^2} = (t+1)^4 + c$$

$$P^2 = \frac{(t+1)^2}{(t+1)^4 + c}$$

which is the required form with $k = c$. When $t = 0$, $P = 0.5$, so $P^2 = 0.25$:

$$0.25 = \frac{1^2}{1^4 + k}$$

$$1 + k = 4$$

$$k = 3$$

Therefore the model gives

$$P^2 = \frac{(t+1)^2}{(t+1)^4 + 3}$$

(c) When $t = 60$:

$$P^2 = \frac{61^2}{61^4 + 3} = \frac{3721}{13\,845\,844} \quad \Longrightarrow \quad P \approx 0.0164$$

So the model predicts a concentration of about $0.0164\,\mathrm{mg\,m^{-3}}$ after 60 days, far less than the measured $0.2\,\mathrm{mg\,m^{-3}}$. Moreover, for large t, $P^2 \approx \frac{1}{(t+1)^2}$, so the model predicts that $P \to 0$ as $t \to \infty$, whereas the measurement shows the pollutant persisting at a much higher level. The model greatly underestimates the concentration, so it is not reliable in the long term.

7. Use the substitution $y = vx$ to find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{4x^2 - y^2}, \qquad x > 0$$

for which $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$. **[9]**

Solution

With $y = vx$, the product rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Substituting into the differential equation:

$$x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = vx + \sqrt{4x^2 - v^2x^2}$$

$$vx + x^2\frac{\mathrm{d}v}{\mathrm{d}x} = vx + \sqrt{x^2(4 - v^2)}$$

Since $x > 0$, $\sqrt{x^2} = x$, so

$$x^2\frac{\mathrm{d}v}{\mathrm{d}x} = x\sqrt{4 - v^2}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{4 - v^2}$$

Separating the variables:

$$\int \frac{1}{\sqrt{4 - v^2}}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$$

The standard result, with v as its variable, is

$$\int \frac{1}{\sqrt{a^2 - v^2}}\,\mathrm{d}v = \arcsin\left(\frac{v}{a}\right) + c \qquad (|v| < a)$$

Using this with $a = 2$, and noting that $x > 0$ so the logarithm needs no modulus:

$$\arcsin\left(\frac{v}{2}\right) = \ln x + c$$

When $x = 1$, $y = 1$, so $v = \frac{y}{x} = 1$:

$$\arcsin\left(\frac{1}{2}\right) = \ln 1 + c \quad \Longrightarrow \quad c = \frac{\pi}{6}$$

Replacing v by $\frac{y}{x}$:

$$\arcsin\left(\frac{y}{2x}\right) = \ln x + \frac{\pi}{6}$$

$$\frac{y}{2x} = \sin\left(\ln x + \frac{\pi}{6}\right)$$

$$y = 2x\sin\left(\ln x + \frac{\pi}{6}\right)$$

8. The substitution $y = u^k$, where k is an integer and $u > 0$, is to be used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 2x^3\sqrt{y} \qquad (*)$$

for $x > 0$ and $y > 0$, by changing it into an equation in the variables u and x.

(a) Show that this equation in u and x may be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{kx}u = \frac{2}{k}x^2u^{1-\frac{k}{2}} \qquad (**)$$ **[3]**

(b) Write down the value of k for which the integrating factor method may be used to solve equation $(**)$. **[1]**

(c) Using this value of k, solve equation $(**)$ and hence find the general solution of equation $(*)$, giving your answer in the form $y = f(x)$. **[4]**

Solution

(a) Differentiating $y = u^k$ with respect to x gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ku^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Since $u > 0$, $\sqrt{y} = \sqrt{u^k} = u^{\frac{k}{2}}$. Substituting into equation $(*)$:

$$kxu^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x} - 4u^k = 2x^3u^{\frac{k}{2}}$$

Dividing every term by kxu^{k-1}, which is non-zero since $x > 0$ and $u > 0$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4u^k}{kxu^{k-1}} = \frac{2x^3u^{\frac{k}{2}}}{kxu^{k-1}}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{kx}u = \frac{2}{k}x^2u^{\frac{k}{2}-(k-1)}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{kx}u = \frac{2}{k}x^2u^{1-\frac{k}{2}}$$

as required.

(b) The integrating factor method needs equation $(**)$ to be linear in u, so the right-hand side must not involve u. This needs $1 - \frac{k}{2} = 0$, so $k = 2$.

(c) With $k = 2$, equation $(**)$ becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2}{x}u = x^2$$

Since $x > 0$, the logarithm needs no modulus. The integrating factor is

$$\mu(x) = e^{\int -\frac{2}{x}\,\mathrm{d}x} = e^{-2\ln x} = x^{-2}$$

Multiplying equation $(**)$ by x^{-2}:

$$x^{-2}\frac{\mathrm{d}u}{\mathrm{d}x} - 2x^{-3}u = 1$$

The left-hand side is the derivative of $\mu(x)u = x^{-2}u$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-2}u\right) = 1$$

$$x^{-2}u = x + c$$

$$u = x^3 + cx^2$$

Since $y = u^2$, the general solution of equation $(*)$ is

$$y = \left(x^3 + cx^2\right)^2 = x^4(x + c)^2$$

9. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\tan x - y^2\cos^2 x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

is to be solved subject to the condition $y = 1$ when $x = 0$.

(a) Show that the substitution $y = \frac{1}{u}$ transforms the differential equation into

$$\frac{\mathrm{d}u}{\mathrm{d}x} + u\tan x = \cos^2 x \qquad \textbf{[3]}$$

(b) Find y in terms of x. **[6]**

Solution

(a) Differentiating $y = \frac{1}{u}$ with respect to x gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Substituting into the differential equation:

$$-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\tan x}{u} - \frac{\cos^2 x}{u^2}$$

Multiplying through by $-u^2$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -u\tan x + \cos^2 x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} + u\tan x = \cos^2 x$$

as required.

(b) Since $\sec x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the logarithm needs no modulus. The integrating factor is

$$\mu(x) = e^{\int \tan x\,\mathrm{d}x} = e^{\ln(\sec x)} = \sec x$$

Multiplying the equation from part (a) by $\sec x$:

$$\sec x\frac{\mathrm{d}u}{\mathrm{d}x} + u\sec x\tan x = \cos^2 x\sec x = \cos x$$

The left-hand side is the derivative of $\mu(x)u = u\sec x$, so

$$\frac{\mathrm{d}}{\mathrm{d}x}(u\sec x) = \cos x$$

$$u\sec x = \sin x + c$$

$$u = \sin x\cos x + c\cos x$$

When $x = 0$, $y = 1$, so $u = \frac{1}{y} = 1$. Substituting:

$$1 = 0 + c \quad \Longrightarrow \quad c = 1$$

Hence

$$u = \sin x\cos x + \cos x = \cos x(1 + \sin x)$$

Since $y = \frac{1}{u}$, the particular solution is

$$y = \frac{1}{\cos x(1 + \sin x)}$$

10. **(a)** Show that the substitution $v = y - \frac{1}{x}$, $x \neq 0$, transforms the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy(xy-2) = -2 \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -v^2 \qquad (**)$$

[3]

(b) Solve equation $(**)$ to determine v as a function of x. **[2]**

(c) Hence obtain the general solution of equation $(*)$, giving y in terms of x. **[1]**

(d) Sketch the solution curve that passes through the point $\left(1, \frac{1}{2}\right)$.
On your sketch show clearly the equations of any asymptotes and the coordinates of any points where the curve meets the coordinate axes. **[6]**

Solution

(a) Rearranging the substitution gives $y = v + \frac{1}{x}$, so differentiating with respect to x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{x^2} \quad \Longrightarrow \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\frac{\mathrm{d}v}{\mathrm{d}x} - 1$$

Also $xy = xv + 1$, so $xy - 2 = xv - 1$ and

$$xy(xy-2) = (xv+1)(xv-1) = x^2v^2 - 1$$

Substituting both into equation $(*)$:

$$x^2\frac{\mathrm{d}v}{\mathrm{d}x} - 1 + x^2v^2 - 1 = -2$$

$$x^2\frac{\mathrm{d}v}{\mathrm{d}x} + x^2v^2 = 0$$

$$x^2\left(\frac{\mathrm{d}v}{\mathrm{d}x} + v^2\right) = 0$$

Since $x \neq 0$, dividing by x^2 gives

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -v^2$$

as required.

(b) Separating the variables:

$$\int -\frac{1}{v^2}\,\mathrm{d}v = \int 1\,\mathrm{d}x$$

Integrating both sides gives

$$\frac{1}{v} = x + c$$

where c is a constant, so

$$v = \frac{1}{x+c}$$

(c) Using part (b) with $y = v + \frac{1}{x}$, the general solution of equation $(*)$ is

$$y = \frac{1}{x} + \frac{1}{x+c}$$

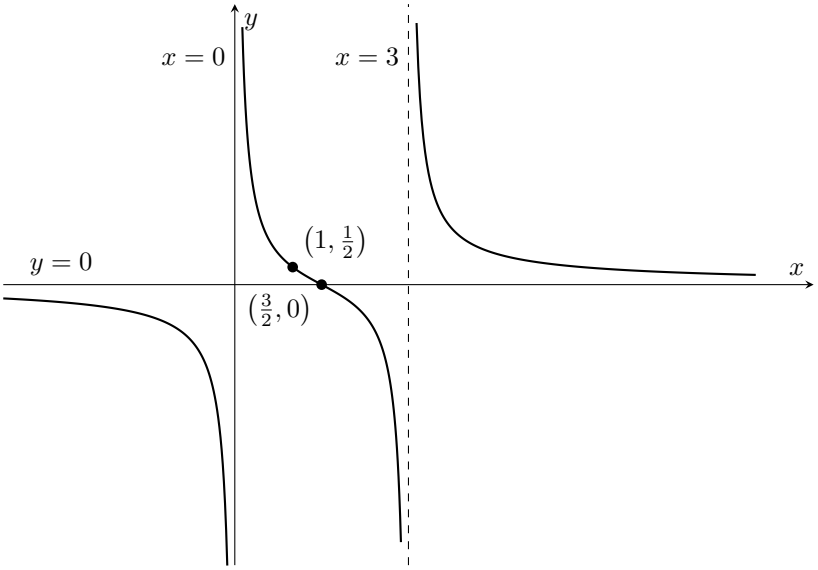
(d) Substituting $x = 1$, $y = \frac{1}{2}$ into the general solution:

$$\begin{aligned} \frac{1}{2} &= 1 + \frac{1}{1+c} \\ \frac{1}{1+c} &= -\frac{1}{2} \\ 1 + c &= -2 \\ c &= -3 \end{aligned}$$

So the solution curve is

$$y = \frac{1}{x} + \frac{1}{x-3} = \frac{2x-3}{x(x-3)}$$

The sketch should look like:



The denominator $x(x-3)$ is zero at $x = 0$ and $x = 3$, where the numerator $2x - 3$ is non-zero, so the curve has vertical asymptotes $x = 0$ and $x = 3$. As $x \to \pm\infty$, $\frac{1}{x} \to 0$ and $\frac{1}{x-3} \to 0$, so $y \to 0$. Hence the curve has horizontal asymptote $y = 0$.

Setting $y = 0$ gives $2x - 3 = 0$, so $x = \frac{3}{2}$ and the curve meets the x-axis at $\left(\frac{3}{2}, 0\right)$. The curve does not meet the y-axis, since $x = 0$ is an asymptote.

11. Use the substitution $z = x + y$ to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+y)^2 - 2}{(x+y)^2}$$

for which $y = 2$ when $x = 1$. Give your answer in the form $\frac{x+y-1}{x+y+1} = Ae^{k(x-y)}$, where A and k are constants to be determined. **[8]**

Solution

With $z = x + y$, differentiating with respect to x gives

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x} \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} - 1$$

Substituting into the differential equation:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} - 1 &= \frac{z^2 - 2}{z^2} \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{z^2 - 2 + z^2}{z^2} \\ &= \frac{2\left(z^2 - 1\right)}{z^2} \end{aligned}$$

Separating the variables:

$$\int \frac{z^2}{z^2 - 1}\,\mathrm{d}z = \int 2\,\mathrm{d}x$$

The integrand on the left can be written as

$$\frac{z^2}{z^2 - 1} = \frac{\left(z^2 - 1\right) + 1}{z^2 - 1} = 1 + \frac{1}{(z-1)(z+1)}$$

Using partial fractions:

$$\frac{1}{(z-1)(z+1)} = \frac{P}{z-1} + \frac{Q}{z+1} \quad \Longrightarrow \quad 1 = P(z+1) + Q(z-1)$$

Setting $z = 1$ gives $P = \frac{1}{2}$, and setting $z = -1$ gives $Q = -\frac{1}{2}$. So

$$\int \left(1 + \frac{1}{2(z-1)} - \frac{1}{2(z+1)}\right)\mathrm{d}z = \int 2\,\mathrm{d}x$$

Integrating both sides:

$$\begin{aligned} z + \tfrac{1}{2}\ln|z-1| - \tfrac{1}{2}\ln|z+1| &= 2x + c \\ \tfrac{1}{2}\ln\left|\frac{z-1}{z+1}\right| &= 2x - z + c \end{aligned}$$

Now replace z by $x+y$, so that $2x-z=2x-(x+y)=x-y$:

$$\begin{aligned}\tfrac{1}{2}\ln\left|\frac{x+y-1}{x+y+1}\right| &= x-y+c\\ \ln\left|\frac{x+y-1}{x+y+1}\right| &= 2(x-y)+2c\\ \left|\frac{x+y-1}{x+y+1}\right| &= e^{2c}e^{2(x-y)}\end{aligned}$$

so

$$\frac{x+y-1}{x+y+1}=Ae^{2(x-y)}$$

where $A=\pm e^{2c}$ is a constant. Hence $k=2$.

Using $y=2$ when $x=1$:

$$\begin{aligned}\frac{1+2-1}{1+2+1} &= Ae^{2(1-2)}\\ \frac{1}{2} &= Ae^{-2}\\ A &= \frac{e^2}{2}\end{aligned}$$

Therefore the solution is

$$\frac{x+y-1}{x+y+1}=\frac{e^2}{2}\,e^{2(x-y)}$$

which is in the required form with $A=\frac{e^2}{2}\approx 3.69$ and $k=2$.

12. A doctor monitors the concentration of a drug in a patient's bloodstream after an injection.
The concentration, x milligrams per litre, of the drug t hours after the injection is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3 + 2\cosh t}{x\cosh^2 t} - x\tanh t \qquad (*)$$

Immediately after the injection, the concentration of the drug is 2 milligrams per litre.

(a) The concentration can be estimated step by step, at the times $t_0 = 0$ (the injection), $t_1 = t_0 + h$, $t_2 = t_1 + h$, and so on, where the step length h is the time in hours from one step to the next. At each step, x_n is the estimate of the concentration at time t_n, and $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n$ is the value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ given by equation $(*)$ when $t = t_n$ and $x = x_n$. Each estimate is found from the one before it using the approximation

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n \approx \frac{x_{n+1} - x_n}{h}$$

Starting from $x_0 = 2$, use this approximation twice, with $h = 0.1$, to estimate the concentration of the drug 12 minutes after the injection. Give your answer to 3 decimal places. **[5]**

(b) Show that the substitution $u = x^2$ transforms equation $(*)$ into the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + 2u\tanh t = \frac{4}{\cosh t} + \frac{6}{\cosh^2 t} \qquad (**)$$ **[3]**

(c) Determine the general solution of equation $(**)$. **[4]**

(d) Hence find an expression for x in terms of t. **[2]**

(e) Find the percentage error of the estimate found in part (a) relative to the value given by the model, stating whether it is an overestimate or an underestimate. **[3]**

Solution

(a) Rearranging the approximation gives

$$x_{n+1} \approx x_n + h\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n$$

Here $h = 0.1$, $t_0 = 0$ and $x_0 = 2$. Twelve minutes is 0.2 hours, so two steps are needed.

At $t_0 = 0$, $\cosh 0 = 1$ and $\tanh 0 = 0$, so

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_0 = \frac{3+2}{2\times 1} - 0 = 2.5$$

$$x_1 \approx 2 + 0.1 \times 2.5 = 2.25$$

At $t_1 = 0.1$:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_1 \approx \frac{3 + 2\cosh 0.1}{2.25\cosh^2 0.1} - 2.25\tanh 0.1$$

$$\approx 2.204551 - 0.224253$$

$$= 1.980298$$

$$x_2 \approx 2.25 + 0.1 \times 1.980298$$

$$= 2.448030$$

So the estimated concentration 12 minutes after the injection, in milligrams per litre, is

$$x_2 \approx 2.448 \qquad (3 \text{ d.p.})$$

(b) Differentiating $u = x^2$ with respect to t using the chain rule:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}$$

Substituting equation $(*)$:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= 2x\left(\frac{3+2\cosh t}{x\cosh^2 t} - x\tanh t\right)\\ &= \frac{2(3+2\cosh t)}{\cosh^2 t} - 2x^2\tanh t\\ &= \frac{6}{\cosh^2 t} + \frac{4\cosh t}{\cosh^2 t} - 2u\tanh t\\ &= \frac{6}{\cosh^2 t} + \frac{4}{\cosh t} - 2u\tanh t\end{aligned}$$

Rearranging gives

$$\frac{\mathrm{d}u}{\mathrm{d}t} + 2u\tanh t = \frac{4}{\cosh t} + \frac{6}{\cosh^2 t}$$

which is equation $(**)$, as required.

(c) Equation $(**)$ is a linear first-order equation. Since $\cosh t > 0$ for all t, the logarithm needs no modulus. The integrating factor is

$$\mu(t) = e^{\int 2\tanh t\,\mathrm{d}t} = e^{2\ln(\cosh t)} = \cosh^2 t$$

Multiplying equation $(**)$ by $\cosh^2 t$, and using $\tanh t\cosh^2 t = \sinh t\cosh t$:

$$\cosh^2 t\,\frac{\mathrm{d}u}{\mathrm{d}t} + 2u\sinh t\cosh t = 4\cosh t + 6$$

The left-hand side is the derivative of $\mu(t)u = u\cosh^2 t$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\cosh^2 t\right) = 4\cosh t + 6$$

Integrating both sides:

$$u\cosh^2 t = 4\sinh t + 6t + c$$

so the general solution is

$$u = \frac{4\sinh t + 6t + c}{\cosh^2 t}$$

(d) When $t = 0$, $x = 2$, so $u = x^2 = 4$. Substituting into the result of part (c):

$$4\cosh^2 0 = 4\sinh 0 + 0 + c \quad\Longrightarrow\quad c = 4$$

So

$$x^2 = \frac{4\sinh t + 6t + 4}{\cosh^2 t}$$

The concentration is positive, so taking the positive square root:

$$x = \frac{\sqrt{4\sinh t + 6t + 4}}{\cosh t}$$

(e) Twelve minutes is $t = 0.2$, where the model gives

$$x = \frac{\sqrt{4\sinh 0.2 + 1.2 + 4}}{\cosh 0.2} \approx \frac{\sqrt{6.005344}}{1.020067} \approx 2.402373$$

The percentage error of the estimate relative to the model's value is

$$\frac{2.448030 - 2.402373}{2.402373} \times 100\% \approx 1.90\%$$

The estimate is greater than the value given by the model, so it is an overestimate, by about 1.90%.