OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

Series Solutions of Differential Equations

Questions by topic 12 questions · 105 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*6 marks*	2
Question 2	*8 marks*	3
Question 3	*7 marks*	4
Question 4	*8 marks*	5
Question 5	*6 marks*	6
Question 6	*10 marks*	7
Question 7	*10 marks*	8
Question 8	*9 marks*	9
Question 9	*10 marks*	10
Question 10	*10 marks*	11
Question 11	*11 marks*	12
Question 12	*10 marks*	13

1. The variable y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos 2x$$

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 0$, find a series expansion for y in terms of x, up to and including the term in x^4. **[6]**

2. The variable y satisfies the differential equation

$$(2+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y^2 - 3x\frac{\mathrm{d}y}{\mathrm{d}x}$$

(a) Show that

$$(2+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (3x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2y-3)\frac{\mathrm{d}y}{\mathrm{d}x}$$ **[3]**

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 0$,

(b) find a series solution for y in ascending powers of x, up to and including the term in x^3. **[3]**

(c) Hence estimate the value of y when $x = 0.2$, giving your answer to 3 decimal places. **[2]**

3. A function y satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = e^{2x-2}$$

It is given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 1$.

(a) Find a Taylor series for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$. **[5]**

(b) Hence find an estimate of the value of y when $x = 1.1$, giving your answer to 3 significant figures. **[2]**

4. A function y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

(a) Show that

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \textbf{[3]}$$

When $x = 1$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$.

(b) Find the Taylor series solution for y, in ascending powers of $(x-1)$ up to and including the term in $(x-1)^4$, giving each coefficient in simplest form. **[5]**

5. The displacement x metres of a particle at time t seconds satisfies the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x\frac{\mathrm{d}x}{\mathrm{d}t} + 2\cos x = 0$$

At time $t = 0$, the particle is at $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 3$.
Find a Taylor series solution for x in ascending powers of t, up to and including the term in t^4. **[6]**

6. The variable y satisfies the differential equation

$$(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2xy=4$$

At $x=0$, $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$.

(a) Show that, at $x=0$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{7}{2}$. **[6]**

(b) Find a series solution for y in ascending powers of x, up to and including the term in x^3, and use it to estimate the value of y when $x=0.2$, giving your answer to 3 decimal places. **[4]**

7. The function y satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x^2y = 0$$

It is given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 1$.

(a) Determine the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 1$. **[6]**

(b) Hence find the series expansion for y about $x = 1$, in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$, giving each coefficient in simplest form, and use your expansion to estimate the value of y at $x = 1.1$, giving your answer to 3 decimal places. **[4]**

8. The variable y satisfies the differential equation

$$(1+x)\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2 \qquad (*)$$

(a) Show that

$$(1+x)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = (ay+b)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + c\,\frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

where a, b and c are integers to be determined. **[4]**

(b) Given that $y = 1$ when $x = 0$, find a series solution of $(*)$ in ascending powers of x, up to and including the term in x^4. Hence find an approximate value of y when $x = 0.2$, giving your answer to 3 decimal places. **[5]**

9. The function y satisfies the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 2$$

(a) Show that

$$x\frac{\mathrm{d}^4y}{\mathrm{d}x^4} + a\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

where a and b are constants to be found. **[4]**

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 1$,

(b) find a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^4$. Write each term in its simplest form. **[3]**

(c) Use your answer to part (b) to find an approximate value for $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0.9$, giving your answer to 3 decimal places. **[3]**

10. Given that

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 6x$$

(a) show that

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{6}{y} + \frac{10}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 - \frac{30x}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} \quad \textbf{[5]}$$

It is also given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 1$.

(b) Find a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$, simplifying the coefficients where possible. **[3]**

(c) Use your series to estimate the value of y when $x = 1.2$, giving your answer to 3 decimal places. **[2]**

11. The variable y satisfies the differential equation

$$(1+x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}-6y=0$$

(a) Show that

$$(1+x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+ax\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+b\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

where a and b are constants to be found. **[4]**

Given that $y=2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ at $x=0$,

(b) find a series solution for y in ascending powers of x up to and including the term in x^5, **[5]**

(c) use your series to estimate the value of y at $x=0.1$, giving your answer to four decimal places. **[2]**

12. The Taylor series expansion of $f(x)$ about $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \cdots$$

The curve with equation $y = f(x)$ satisfies the differential equation

$$\sin x \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \cos x$$

The point $\left(\frac{\pi}{6}, 1\right)$ is a stationary point of the curve.

(a) Determine the nature of this stationary point, giving a reason for your answer. **[2]**

(b) Find the exact value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at this stationary point. **[4]**

(c) Hence find a series solution for y, in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$, giving each coefficient in exact form. Use your series to estimate the value of y when $x = 0.6$, giving your answer to 4 decimal places. **[4]**