OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

Numerical Solution of Differential Equations

Questions by topic 13 questions · 94 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · AQA 7367

Instructions

AQA (7367): numerical solution of second-order differential equations is not on the specification. Questions 11 to 13 can be left out.

AQA (7367): the questions give each step as an approximation to the derivative. Rearranged, these are the formulae booklet's Euler's method and midpoint formula, where $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h} \quad \text{is Euler's method,} \quad y_{r+1} = y_r + hf(x_r, y_r)$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h} \quad \text{is the midpoint formula,} \quad y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

Questions

Question 1	*6 marks*	2
Question 2	*9 marks*	3
Question 3	*6 marks*	4
Question 4	*10 marks*	5
Question 5	*9 marks*	6
Question 6	*6 marks*	7
Question 7	*10 marks*	8
Question 8	*4 marks*	9
Question 9	*7 marks*	10
Question 10	*6 marks*	11
Question 11	*6 marks*	12
Question 12	*8 marks*	13
Question 13	*7 marks*	14

1. It is given that $y(1) = 3$ and that y satisfies the differential equation

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 2$$

(a) Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. Hence use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

with $h = 0.1$ to obtain an estimate for $y(1.1)$. **[3]**

(b) Use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

with your answer to part **(a)** to obtain an estimate for $y(1.2)$, giving your answer to 3 decimal places. **[3]**

2. A parachutist is falling vertically. At time $t = 0$ her parachute opens, when her speed is $20\,\mathrm{m\,s^{-1}}$. Her speed $v\,\mathrm{m\,s^{-1}}$, t seconds after the parachute opens, is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - 0.05v^2 \qquad t \geq 0$$

(a) Use three iterations of the approximation formula

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_n \approx \frac{v_{n+1} - v_n}{h}$$

with a step length of 0.2 to estimate the speed of the parachutist 0.6 seconds after her parachute opens. Give your answer to 3 significant figures. **[7]**

(b) The constant 9.8 in the model represents the acceleration due to gravity, which is known accurately. Observations show that the parachutist's speed settles at a steady value of $12\,\mathrm{m\,s^{-1}}$, which is less than the steady speed predicted by the current model.
Refine the model by changing the value of one constant, giving its new value to 3 significant figures. **[2]**

3. Water flows into a tank through a pipe and, at the same time, leaks out through a hole in the base of the tank. The depth of water, H metres, in the tank is modelled by the differential equation

$$2(H+3)\frac{\mathrm{d}H}{\mathrm{d}t} = 12 - 5t\sqrt{H} \qquad H > 0$$

where t is the time in hours after the pipe was opened.
The depth of water in the tank 15 minutes after the pipe was opened was 1 m.

(a) Use two applications of the approximation formula

$$\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right)_n \approx \frac{H_{n+1} - H_n}{h}$$

to estimate the depth of water in the tank 1 hour 45 minutes after the pipe was opened. Give your answer to 3 significant figures. **[5]**

(b) Explain how the same approximation formula could be used to obtain a more accurate estimate of this depth. **[1]**

4. Priya stocks a new pond with 50 carp, which she plans to breed and sell.
After t months the number of carp, P, is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.2P - 0.0008P^2 \qquad t \geq 0$$

Priya will begin selling carp only once the pond holds at least 75 of them.

(a) Use three iterations of the approximation formula

$$\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)_n \approx \frac{P_{n+1} - P_n}{h}$$

with a step length of 1 month to determine whether, according to the model, Priya will be able to begin selling carp after 3 months. **[8]**

(b) By considering how $\frac{\mathrm{d}P}{\mathrm{d}t}$ changes as P increases from 50 to 80, explain whether your estimate in part (a) is likely to be an underestimate or an overestimate of the number of carp predicted by the model, and state what this means for your conclusion in part (a). **[2]**

5. A particle of mass 2 kg moves in a straight line on a smooth horizontal surface.
At time t seconds, the velocity of the particle is $v\,\mathrm{m\,s^{-1}}$.
The following horizontal forces act on the particle:

- a driving force of magnitude $(8 + 2t)$ newtons, acting in the direction of motion
- a resistive force of magnitude $0.3v^2$ newtons

When $t = 0$, $v = 1$.

(a) Show that

$$20\frac{\mathrm{d}v}{\mathrm{d}t} = 80 + 20t - 3v^2$$ **[2]**

(b) Use the approximation formula

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_n \approx \frac{v_{n+1} - v_n}{h}$$

with a step length of 0.2 to estimate the velocity of the particle when $t = 0.6$, giving your answer to 4 significant figures. **[7]**

6. A curve passes through the point $(0.5, -1.2)$ and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 2y}{3x - y}$$

(a) Use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

once, and then the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

once, each with a step length of 0.05, to estimate the value of y when $x = 0.6$. Give your answer to 4 decimal places. **[5]**

(b) Explain why the second approximation in part (a) cannot be used for the first step from $x = 0.5$. **[1]**

7. A curve passes through the point $(1, 3)$ and satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \ln x + \frac{2y}{x} \qquad x > 0$$

(a) Use the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

once, and then the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

twice, each with a step length of 0.2, to estimate the value of y when $x = 1.6$, giving your answer to 5 significant figures. **[7]**

(b) It is given that the curve has equation

$$y = \frac{1}{2}x^2 \left((\ln x)^2 + 6\right)$$

Find the percentage error in your estimate from part (a). **[3]**

8. A curve passes through the point $(a, 2)$, where $a > 0$.
The curve satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 2y}{xy}$$

Using the approximation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h}$$

once, with starting point $(a, 2)$ and a step length of 0.25, gives an estimate of $y = 2.6$ when $x = a + 0.25$.
Find the two possible values of a, giving your answers in exact form. **[4]**

9. The temperature, $\theta\,^{\circ}\mathrm{C}$, of soup in a bowl, t minutes after the bowl is placed on a table in a room, is modelled by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20)^2$$

where k is a positive constant.
The soup has an initial temperature of $80\,^{\circ}\mathrm{C}$, and $k = 0.002$.

(a) Use two iterations of the approximation formula

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_n \approx \frac{\theta_{n+1} - \theta_n}{h}$$

to estimate the temperature of the soup 4 minutes after the bowl was placed on the table, giving your answer to 1 decimal place. **[5]**

(b) By finding an expression for $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$ in terms of θ and k, explain whether your answer to part (a) is likely to be an overestimate or an underestimate of the temperature predicted by the model. **[2]**

10. A survey of a lake records the proportion, x, of the lake's surface that is covered by weed. This proportion is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x(0.7 - x)(t + 2)}{3t + x} \qquad t > 0$$

where t is the number of years since the start of the survey.
Exactly 2 years after the start of the survey, 40% of the lake's surface was covered by weed.

(a) Use two applications of the approximation formula

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n \approx \frac{x_{n+1} - x_n}{h}$$

with a step length of 0.25 years to estimate the proportion of the lake's surface covered by weed six months later. Give your answer to 3 decimal places. **[5]**

(b) Deduce from the differential equation the proportion of the lake's surface that will be covered by weed in the long term. **[1]**

11. The angle, θ radians, that a pendulum makes with the downward vertical, t seconds after it is first observed, is modelled by the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 0.2\frac{\mathrm{d}\theta}{\mathrm{d}t} + 9\sin\theta = 0$$

Two measurements are taken: $\theta = 0.5$ when $t = 0$, and $\theta = 0.48$ when $t = 0.1$.
The approximations

$$\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right)_n \approx \frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)_n \approx \frac{\theta_{n+1} - \theta_{n-1}}{2h}$$

are used with $h = 0.1$, where θ_n is the estimate of θ when $t = 0.1n$.

(a) Use the two measurements to estimate the value of θ when $t = 0.2$, giving your answer to 4 decimal places. **[4]**

(b) Hence estimate the value of θ when $t = 0.3$, giving your answer to 3 decimal places. **[2]**

12. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 3y - x^2$$

where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ at $x = 1$.
The approximations

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

are used with $h = 0.1$, where $x_n = 1 + 0.1n$ and y_n is the approximation to y at $x = x_n$.

(a) Show that

$$(20 + x_n)\, y_{n+1} = 40.6\, y_n - (20 - x_n)\, y_{n-1} - 0.2x_n^2$$ **[3]**

(b) Hence find an estimate for the value of y at $x = 1.2$, giving your answer to 3 decimal places. **[5]**

13. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 3x$$

where $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5$ when $x = 0$.
Use the approximations

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \quad \text{and} \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

with $h = 0.2$, where y_n denotes the estimate of y at $x = nh$, to find an estimate for the value of y when $x = 0.4$. Give your answer to 3 significant figures. **[7]**