OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

2nd Order Differential Equations by Substitution

Questions by topic

12 questions · 180 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

Question 1	*12 marks*	2
Question 2	*13 marks*	3
Question 3	*13 marks*	4
Question 4	*15 marks*	5
Question 5	*13 marks*	6
Question 6	*12 marks*	7
Question 7	*19 marks*	8
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Question 10	*16 marks*	11
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Question 12	*17 marks*	13

1. It is given that, for $x \neq 0$, y satisfies the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x(1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(2x^2 - 4x + 1\right)y = 4x^2 \qquad (*)$$

(a) Show that the substitution $u = x^2y$ transforms this differential equation into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 4\frac{\mathrm{d}u}{\mathrm{d}x} + 4u = 4x^2 \qquad (**)$$ **[5]**

(b) Hence find the general solution of differential equation $(*)$, giving your answer in the form $y = f(x)$. **[7]**

2. A particle P moves along a straight line.
At time t minutes, where $t > 0$, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (4t+2)\frac{\mathrm{d}x}{\mathrm{d}t} + (13t+4)x = 40\cos t \qquad (*)$$

(a) Show that the substitution $x = \frac{y}{t}$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = 40\cos t \qquad (**)$$

[5]

(b) Hence find a general solution for the displacement of P from O at time t minutes. **[8]**

3. **(a)** Given that $x = t^{\frac{1}{3}}$, determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t and derivatives of y with respect to t. **[4]**

(b) Hence show that the substitution $x = t^{\frac{1}{3}}$, where $t > 0$, transforms the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(4x^3 - 2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^5y = 50x^5e^{x^3} \qquad (*)$$

into the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 12\frac{\mathrm{d}y}{\mathrm{d}t} + 4y = 50e^t \qquad (**)$$ **[2]**

(c) Solve differential equation $(**)$ to find a general solution for y in terms of t. **[6]**

(d) Hence find the general solution of differential equation $(*)$, giving y in terms of x. **[1]**

4. It is given that $v = y^2$ and

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 5y^2 = 5x^2 + 2x + 6$$

(a) Show that

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x} + 10v = 10x^2 + 4x + 12$$ **[5]**

(b) Find y in terms of x, given that, when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}$. **[10]**

5. **(a)** Show that the substitution $x = t^2$, where $t \geq 0$, transforms the differential equation

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(1 - 6\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 12e^{\sqrt{x}} \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 8y = 12e^{t} \qquad (**)$$

[5]

(b) Solve differential equation $(**)$ to determine y in terms of t. **[6]**

(c) Hence find the particular solution of differential equation $(*)$ for which $y = 32$ when $x = (\ln 2)^2$ and $y = 111$ when $x = (\ln 3)^2$. **[2]**

6. A differential equation is given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(1+\tan x\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 2\left(2+\tan x+\sec^2 x\right)y = 25x\cos x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2} \qquad (*)$$

(a) Show that the substitution $y = u\cos x$, where u is a function of x, transforms this differential equation into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + 5u = 25x \qquad (**)$$

[5]

(b) Hence find the general solution of differential equation $(*)$, giving your answer in the form $y = f(x)$. **[7]**

7. The vertical height, h m, above horizontal ground, of a passenger on a fairground ride, t seconds after the ride starts, where $0 < t \leq 5$, is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} - 4t\frac{\mathrm{d}h}{\mathrm{d}t} + 6h = 6t + 12 \qquad (*)$$

(a) Given that $t = e^x$, show that

$$t\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}x}$$

and

$$t^2\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = \frac{\mathrm{d}^2h}{\mathrm{d}x^2} - \frac{\mathrm{d}h}{\mathrm{d}x}$$

[4]

(b) Hence show that the substitution $t = e^x$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}^2h}{\mathrm{d}x^2} - 5\frac{\mathrm{d}h}{\mathrm{d}x} + 6h = 6e^x + 12 \qquad (**)$$

[2]

(c) Hence show that

$$h = At^2 + Bt^3 + 3t + 2$$

where A and B are constants. **[7]**

Given that when $t = 1$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 4$, and when $t = 2$, $h = \frac{28}{3}$,

(d) find the maximum height of the passenger above the ground during the ride, justifying that it is a maximum. **[6]**

8. A biologist models the number of bacteria in a culture, t days after observations begin, where $1 \leq t \leq 10$, by the differential equation

$$t^2\frac{\mathrm{d}^2N}{\mathrm{d}t^2} + 2t\frac{\mathrm{d}N}{\mathrm{d}t} - 2N = 4t^2 \qquad (*)$$

where N is the number of bacteria, in thousands.

(a) Show that the substitution $t = e^x$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}^2N}{\mathrm{d}x^2} + \frac{\mathrm{d}N}{\mathrm{d}x} - 2N = 4e^{2x} \qquad (**)$$ **[5]**

(b) Hence find the general solution for N in terms of t. **[7]**

Given that when $t = 1$, $N = 37$ and $\frac{\mathrm{d}N}{\mathrm{d}t} = -58$,

(c) find the minimum number of bacteria in the culture, justifying that the value you find is a minimum. **[6]**

9. The motion of a particle P along the x-axis is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2t(2t+1)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(3t^2+4t+2\right)x = -6t^3e^{2t} \qquad (*)$$

where P has displacement x metres from the origin O at time t minutes, $t > 0$.

(a) Show that the substitution $x = tu$ transforms the differential equation $(*)$ into the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - 4\frac{\mathrm{d}u}{\mathrm{d}t} + 3u = -6e^{2t} \qquad (**)$$

[5]

(b) Hence find the general solution of the differential equation $(*)$, giving x in terms of t. **[7]**

Given that P is at O when $t = \ln 2$ and when $t = \ln 4$,

(c) find the particular solution of the differential equation $(*)$, and find the interval of time during which P is on the positive x-axis. **[4]**

10. A curve satisfies the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 8x^2, \qquad -1 < x < 1 \qquad (*)$$

The substitution $x = \sin t$, where $-\frac{\pi}{2} < t < \frac{\pi}{2}$, is used to transform $(*)$.

(a) Show that

$$\sqrt{1-x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \text{and} \qquad (1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$ **[4]**

(b) Hence show that the substitution $x = \sin t$ transforms $(*)$ into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y = 4 - 4\cos 2t \qquad (**)$$ **[2]**

(c) Find the general solution of $(**)$, using a particular integral of the form $y = p + qt\sin 2t$, where p and q are constants to be found. **[5]**

(d) Hence find the solution of $(*)$ for which $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$, giving your answer in the form $y = f(x)$, with no trigonometric function of $\arcsin x$. **[5]**

11. It is given that $y = \sinh u$ and

$$\sqrt{1+\sinh^2 u}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+3\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\sinh u\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+2\sinh u = 10\sin x$$

(a) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y = 10\sin x$$ **[5]**

(b) Find u in terms of x, given that, when $x = 0$, $u = 0$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$. **[11]**

12. **(a)** Given that $x = \sinh t$, show that

$$\sqrt{1+x^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \text{and} \qquad \left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} \qquad \textbf{[4]}$$

(b) Hence show that the substitution $x = \sinh t$ transforms the differential equation

$$\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 6x \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 4y = 6\sinh t \qquad (**) \qquad \textbf{[2]}$$

(c) Solve differential equation $(**)$ to find y in terms of t. **[6]**

(d) Hence find the solution of differential equation $(*)$ for which $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$, giving y in terms of x only, with no hyperbolic or exponential functions. **[5]**