OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

1st Order Differential Equations by Substitution

Questions by topic 12 questions · 118 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

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1. The variables x and y, where $x > 0$, are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 3xy + y^2}{x^2} \qquad (*)$$

(a) Use the substitution $y = ux$, where u is a function of x, to show that

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = (u+1)^2$$ **[3]**

(b) Hence find the particular solution of equation $(*)$ for which $y = 0$ when $x = 1$, giving your answer in the form $y = f(x)$. **[5]**

2. **(a)** Show that the substitution $y^3 = \frac{1}{t}$ transforms the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2y^4, \qquad x > 0, \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{3t}{x} = -3x \qquad (**)$$

[3]

(b) Solve equation $(**)$, and hence find y in terms of x, given that $y = \frac{1}{2}$ when $x = 1$. **[6]**

3. In this question, $0 < x < \pi$.

(a) Show that the substitution $z = y^{\frac{1}{2}}$ transforms the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y \cot x = 4y^{\frac{1}{2}} \cos x \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} + z \cot x = 2 \cos x \qquad (**)$$

[3]

(b) Solve equation $(**)$ to find z as a function of x. **[4]**

(c) Hence find the particular solution of equation $(*)$ for which $y = 4$ when $x = \frac{\pi}{6}$, giving y in terms of x. **[2]**

4. For $x > 0$, a curve satisfies the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x\cos^2\left(\frac{y}{x}\right)$$

(a) Use the substitution $y = xz$ to find the general solution of the differential equation, giving y explicitly in terms of x. **[6]**

(b) Find the particular solution for which $y = \frac{\pi}{2}$ when $x = 2$. Hence find the exact value of y when $x = 2e^{-2}$. **[3]**

5. Use the substitution $u = y^2$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \frac{\cos^3 x}{y}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

given that $y = 3$ when $x = 0$. Give your answer in the form $y = f(x)$. **[9]**

6. The concentration, $P\,\text{mg}\,\text{m}^{-3}$, of a pollutant in a reservoir, t days after the pollutant entered the reservoir, is modelled by the differential equation

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{t+1} - 2(t+1)P^2 \qquad (*)$$

(a) Show that the substitution $x = \frac{1}{P^2}$ transforms equation $(*)$ into the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{t+1} = 4(t+1) \qquad (**)$$ **[3]**

(b) Given that $P = 0.5$ when $t = 0$, solve equation $(**)$ to show that, according to the model,

$$P^2 = \frac{(t+1)^2}{(t+1)^4 + k}$$

where k is a constant to be determined. **[6]**

(c) Measurements taken 60 days after the pollutant entered the reservoir showed a concentration of $0.2\,\text{mg}\,\text{m}^{-3}$.
Comment on the reliability of the model in the long term, giving a reason for your answer. **[2]**

7. Use the substitution $y = vx$ to find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{4x^2 - y^2}, \qquad x > 0$$

for which $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$. **[9]**

8. The substitution $y = u^k$, where k is an integer and $u > 0$, is to be used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 2x^3\sqrt{y} \qquad (*)$$

for $x > 0$ and $y > 0$, by changing it into an equation in the variables u and x.

(a) Show that this equation in u and x may be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{kx}u = \frac{2}{k}x^2u^{1-\frac{k}{2}} \qquad (**)$$ **[3]**

(b) Write down the value of k for which the integrating factor method may be used to solve equation $(**)$. **[1]**

(c) Using this value of k, solve equation $(**)$ and hence find the general solution of equation $(*)$, giving your answer in the form $y = f(x)$. **[4]**

9. The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\tan x - y^2\cos^2 x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

is to be solved subject to the condition $y = 1$ when $x = 0$.

(a) Show that the substitution $y = \frac{1}{u}$ transforms the differential equation into

$$\frac{\mathrm{d}u}{\mathrm{d}x} + u\tan x = \cos^2 x$$ **[3]**

(b) Find y in terms of x. **[6]**

10. **(a)** Show that the substitution $v = y - \frac{1}{x}$, $x \neq 0$, transforms the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + xy(xy - 2) = -2 \qquad (*)$$

into the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -v^2 \qquad (**)$$

[3]

(b) Solve equation $(**)$ to determine v as a function of x. **[2]**

(c) Hence obtain the general solution of equation $(*)$, giving y in terms of x. **[1]**

(d) Sketch the solution curve that passes through the point $\left(1, \frac{1}{2}\right)$.
On your sketch show clearly the equations of any asymptotes and the coordinates of any points where the curve meets the coordinate axes. **[6]**

11. Use the substitution $z = x + y$ to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+y)^2 - 2}{(x+y)^2}$$

for which $y = 2$ when $x = 1$. Give your answer in the form $\frac{x+y-1}{x+y+1} = Ae^{k(x-y)}$, where A and k are constants to be determined. **[8]**

12. A doctor monitors the concentration of a drug in a patient's bloodstream after an injection.
The concentration, x milligrams per litre, of the drug t hours after the injection is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3 + 2\cosh t}{x\cosh^2 t} - x\tanh t \qquad (*)$$

Immediately after the injection, the concentration of the drug is 2 milligrams per litre.

(a) The concentration can be estimated step by step, at the times $t_0 = 0$ (the injection), $t_1 = t_0 + h$, $t_2 = t_1 + h$, and so on, where the step length h is the time in hours from one step to the next. At each step, x_n is the estimate of the concentration at time t_n, and $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n$ is the value of $\frac{\mathrm{d}x}{\mathrm{d}t}$ given by equation $(*)$ when $t = t_n$ and $x = x_n$. Each estimate is found from the one before it using the approximation

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_n \approx \frac{x_{n+1} - x_n}{h}$$

Starting from $x_0 = 2$, use this approximation twice, with $h = 0.1$, to estimate the concentration of the drug 12 minutes after the injection. Give your answer to 3 decimal places. **[5]**

(b) Show that the substitution $u = x^2$ transforms equation $(*)$ into the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + 2u\tanh t = \frac{4}{\cosh t} + \frac{6}{\cosh^2 t} \qquad (**)$$ **[3]**

(c) Determine the general solution of equation $(**)$. **[4]**

(d) Hence find an expression for x in terms of t. **[2]**

(e) Find the percentage error of the estimate found in part (a) relative to the value given by the model, stating whether it is an overestimate or an underestimate. **[3]**