OPTIONAL PURE / DIFFERENTIAL EQUATIONS & NUMERICAL METHODS

1st Order Differential Equations by Substitution

Mark scheme

12 questions · 118 marks

EXAM BOARDS Edexcel FP1 9FM0/3A · CIE 9231

Questions

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Question 6	(a) 3 (b) 6 (c) 2	11	9
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Question 8	(a) 3 (b) 1 (c) 4	8	11
Question 9	(a) 3 (b) 6	9	12
Question 10	(a) 3 (b) 2 (c) 1 (d) 6	12	13
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Question 12	(a) 5 (b) 3 (c) 4 (d) 2 (e) 3	17	16
	Total	**118**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$	**B1**	
	$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + 3u + u^2$	**M1**	Substitutes $y = ux$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into equation $(*)$, with the right-hand side in u only. Condone slips.
	$x\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 + 2u + 1 = (u+1)^2$	**A1 AG**	Correct substituted equation leading without error to the printed result.
		[3]	
1(b)	$\int \frac{1}{(u+1)^2}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x$	**M1**	Separates the variables and integrates both sides, at least one side integrated correctly.
	$-\frac{1}{u+1} = \ln x + c$	**A1**	oe, including $\ln\lvert x\rvert$; the constant may be seen here or on the next line.
	$x = 1,\ u = 0 \Rightarrow -1 = \ln 1 + c \Rightarrow c = \ldots\ [= -1]$	**M1**	Substitutes $x = 1$ with $u = 0$, or $y = 0$ after replacing u by $\frac{y}{x}$, into their integrated equation to find its constant.
	$u = \frac{1}{\text{“1”} - \ln x} - 1 \Rightarrow y = ux = \ldots$	**dM1**	Rearranges for u and multiplies by x, or substitutes $u = \frac{y}{x}$ and makes y the subject. Condone slips.
	$y = \frac{x\ln x}{1-\ln x}$	**A1**	oe, such as $y = \frac{x}{1-\ln x} - x$, with y the subject.
		[5]	
	Total for Question 1	**8**	

Question	Scheme	Marks	Notes
2(a)	$t = y^{-3} \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = -3y^{-4}\frac{\mathrm{d}y}{\mathrm{d}x}$	**B1**	Correct relation in any form, such as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3}t^{-\frac{4}{3}}\frac{\mathrm{d}t}{\mathrm{d}x}$.
	$x\left(-\frac{1}{3}y^4\frac{\mathrm{d}t}{\mathrm{d}x}\right) + y = x^2y^4$ $\Rightarrow -\frac{x}{3}\frac{\mathrm{d}t}{\mathrm{d}x} + t = x^2$	**M1**	Substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into equation $(*)$ and eliminates y, reaching an equation in t, $\frac{\mathrm{d}t}{\mathrm{d}x}$ and x only. Condone sign and coefficient slips.
	$\frac{\mathrm{d}t}{\mathrm{d}x} - \frac{3t}{x} = -3x$	**A1 AG**	Correct substituted equation leading without error to equation $(**)$; combined steps accepted.
		[3]	
2(b)	$\mu = e^{\int -\frac{3}{x}\,\mathrm{d}x} = \ldots$	**M1**	Forms $e^{\int P\,\mathrm{d}x}$ with $P = \pm\frac{3}{x}$ and integrates to a multiple of $\ln x$.
	$\mu = x^{-3}$	**A1**	oe, such as $\frac{1}{x^3}$.
	$x^{-3}t = \int -3x^{-2}\,\mathrm{d}x = \ldots$	**dM1**	Multiplies equation $(**)$ by their integrating factor and integrates, with μt on the left and at least one power increased by 1 on the right.
	$x^{-3}t = 3x^{-1} + c \Rightarrow t = 3x^2 + cx^3$	**A1**	Correct general solution in any equivalent form, including the constant.
	$x = 1,\ t = 8 \Rightarrow 8 = \text{“3”} + c \Rightarrow c = \ldots$	**M1**	Uses the condition, with $t = y^{-3}$, in their general solution to find the constant.
	$y = \frac{1}{\sqrt[3]{3x^2+5x^3}}$	**A1**	oe with y the subject, such as $y = (3x^2 + 5x^3)^{-\frac{1}{3}}$.
		[6]	
Total for Question 2		**9**	

Question	Scheme	Marks	Notes
3(a)	$y = z^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2z\frac{\mathrm{d}z}{\mathrm{d}x}$	**B1**	oe, such as $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$2z\frac{\mathrm{d}z}{\mathrm{d}x} + 2z^2\cot x = 4z\cos x$	**M1**	Substitutes into equation $(*)$ using their derivative relation to get an equation in z and x only, then divides through by their factor. Condone coefficient slips.
	$\frac{\mathrm{d}z}{\mathrm{d}x} + z\cot x = 2\cos x$	**A1 AG**	Correct substituted equation leading without error to equation $(**)$.
		[3]	
3(b)	$\mu = e^{\int \cot x\,\mathrm{d}x} = e^{\ln(\sin x)}$	**M1**	Uses $e^{\int \cot x\,\mathrm{d}x}$ and integrates to a multiple of $\ln(\sin x)$. Condone a modulus.
	$\mu = \sin x$	**A1**	
	$z\sin x = \int 2\sin x\cos x\,\mathrm{d}x = \ldots$	**dM1**	Multiplies equation $(**)$ by their integrating factor to form "μ" $z = \int 2\cos x$ "μ" $\mathrm{d}x$ and integrates the right, raising the power of $\sin x$ by 1, or reaching $k\cos^2 x$ or $k\cos 2x$.
	$z\sin x = \sin^2 x + c$ $\Rightarrow z = \sin x + \frac{c}{\sin x}$	**A1**	Correct, with constant and z as subject, oe such as $z = \frac{c-\cos^2 x}{\sin x}$ or $z = \frac{c-\frac{1}{2}\cos 2x}{\sin x}$.
		[4]	
3(c)	$2 = \frac{1}{2} + \frac{c}{\frac{1}{2}} \Rightarrow c = \ldots\ [= \frac{3}{4}]$	**M1**	Substitutes $x = \frac{\pi}{6}$ with $z = 2$, or $y = 4$ in a solution for y, into their general solution with a constant and solves for the constant.
	$y = \left(\sin x + \frac{3}{4\sin x}\right)^2$	**A1**	oe, with y as subject; z alone scores A0.
		[2]	
	Total for Question 3	**9**	

Question	Scheme	Marks	Notes
4(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}$	**B1**	
	$x\left(z + x\frac{\mathrm{d}z}{\mathrm{d}x}\right) - xz = x\cos^2 z$	**M1**	Substitutes $y = xz$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$, with $\frac{y}{x}$ replaced by z, reaching an equation in x and z only. Condone slips.
	$x\frac{\mathrm{d}z}{\mathrm{d}x} = \cos^2 z$	**A1**	oe, such as $x^2\frac{\mathrm{d}z}{\mathrm{d}x} = x\cos^2 z$.
	$\int \sec^2 z\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x \Rightarrow \ldots$	**M1**	Separates their equation into $\int f(z)\,\mathrm{d}z = \int g(x)\,\mathrm{d}x$ and integrates at least one side correctly.
	$\tan z = \ln x + c$	**A1**	Needs the constant. Condone $\ln\lvert x\rvert$.
	$y = x\arctan(\ln x + c)$	**A1**	oe with y explicit and a constant, such as $y = x\arctan(\ln kx)$.
		[6]	
4(b)	$\tan\frac{\pi}{4} = \ln 2 + c \Rightarrow c = \ldots$	**M1**	Substitutes $x = 2$, $y = \frac{\pi}{2}$ into their general solution, in any form with a constant, and solves for the constant.
	$y = x\arctan\left(1 + \ln\left(\frac{x}{2}\right)\right)$	**A1**	oe, such as $y = x\arctan(1 - \ln 2 + \ln x)$.
	$y = 2e^{-2}\arctan(-1) = -\frac{\pi}{2e^2}$	**A1**	Exact, oe such as $-\frac{\pi}{2}e^{-2}$.
		[3]	
	Total for Question 4	**9**	

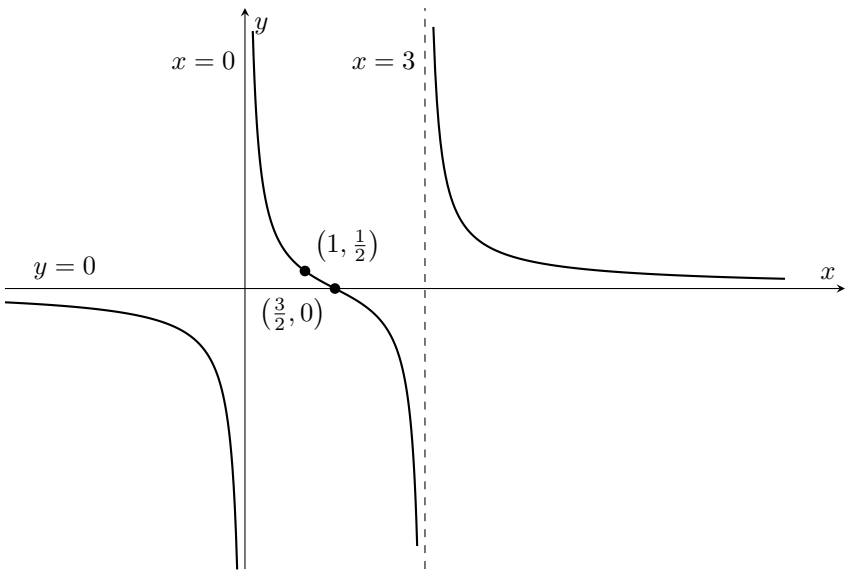
Question	Scheme	Marks	Notes
5	$\frac{\mathrm{d}u}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$	**B1**	oe, such as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{u}}\frac{\mathrm{d}u}{\mathrm{d}x}$.
	$2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2\tan x = 2\cos^3 x$ $\frac{\mathrm{d}u}{\mathrm{d}x} + 2u\tan x = 2\cos^3 x$	**M1 A1**	**M1** Substitutes their derivative relation and $u = y^2$ to obtain an equation in u and x only. Condone coefficient slips. **A1** Correct linear equation, oe.
	$\mu = e^{\int 2\tan x\,\mathrm{d}x} = e^{2\ln(\sec x)} = \sec^2 x$	**M1 A1**	**M1** Forms $e^{\int P\,\mathrm{d}x}$ with P their coefficient of u in a linear equation, and integrates. Condone sign slips. **A1** $\sec^2 x$ or $\frac{1}{\cos^2 x}$.
	$u\sec^2 x = \int 2\cos^3 x\sec^2 x\,\mathrm{d}x$	**dM1**	Multiplies their linear equation by their integrating factor, reaching $u\times$ factor $= \int$ factor $\times$ right-hand side, and integrates at least one term.
	$u\sec^2 x = 2\sin x + c$	**A1**	Including the constant, oe such as $y^2\sec^2 x = 2\sin x + c$.
	$9 = 1\times(0 + c) \Rightarrow c = \ldots$	**M1**	Substitutes $x = 0$ and $y = 3$, or $u = 9$, into their solution containing a constant to find the constant.
	$y = \cos x\sqrt{2\sin x + 9}$	**A1**	oe in the form $y = f(x)$, such as $y = \sqrt{\cos^2 x\,(2\sin x + 9)}$. A0 for $y^2 = \ldots$ or a $\pm$ root.
	Total for Question 5	**9**	

Question	Scheme	Marks	Notes
6(a)	$\frac{\mathrm{d}x}{\mathrm{d}t} = -2P^{-3}\frac{\mathrm{d}P}{\mathrm{d}t}$	**B1**	Correct, oe such as $\frac{\mathrm{d}P}{\mathrm{d}t} = -\frac{P^3}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$.
	$\frac{\mathrm{d}x}{\mathrm{d}t} = -2x\left(\frac{1}{t+1} - 2(t+1)P^2\right)$	**M1**	Substitutes equation $(*)$, or $\frac{\mathrm{d}P}{\mathrm{d}t}$ from it, into their chain-rule relation and uses $x = \frac{1}{P^2}$ to eliminate P. Condone sign slips.
	$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2x}{t+1} + 4(t+1)$ $\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2x}{t+1} = 4(t+1)$	**A1 AG**	Correct substituted equation leading without error to equation $(**)$; combined steps accepted.
		[3]	
6(b)	$\mu = e^{\int \frac{2}{t+1}\,\mathrm{d}t} = e^{2\ln(t+1)}$	**M1**	Uses $e^{\int \frac{2}{t+1}\,\mathrm{d}t}$, or reaches the form $e^{a\ln(t+1)}$. Condone a coefficient slip.
	$\mu = (t+1)^2$	**A1**	
	$(t+1)^2 x = \int 4(t+1)^3\,\mathrm{d}t = \ldots$	**dM1**	Multiplies equation $(**)$ by their integrating factor and integrates both sides, the power of $(t+1)$ on the right increased by 1.
	$(t+1)^2 x = (t+1)^4 + c$	**A1**	Correct, with the constant.
	$\frac{(t+1)^2}{P^2} = (t+1)^4 + c$ $t = 0,\ P = 0.5 \Rightarrow 4 = 1 + c \Rightarrow c = \ldots$	**M1**	Uses $x = \frac{1}{P^2}$ and $P = 0.5$ at $t = 0$, in either order, in their solution with a constant to find it.
	$P^2 = \frac{(t+1)^2}{(t+1)^4+3}$	**A1**	cso. The printed form with $k = 3$, or $k = 3$ stated after reaching that form.
		[6]	
6(c)	$t = 60:\ P^2 = \frac{61^2}{61^4+3} \Rightarrow P = \ldots$	**M1**	Substitutes $t = 60$ into their model from (b) to find P or P^2.
	$P \approx 0.0164$, much less than 0.2 Model underestimates: not reliable in the long term	**A1ft**	ft their P at $t = 60$. Compares it with 0.2, or P^2 with 0.04, and concludes not reliable where their P is not within 10% of 0.2, reliable otherwise.
		[2]	
	Total for Question 6	**11**	

Question	Scheme	Marks	Notes
7	$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$	**B1**	
	$x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = vx + \sqrt{4x^2 - v^2x^2}$	**M1**	Substitutes $y = vx$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into the equation and simplifies, taking a factor of x from the root. Condone sign slips.
	$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{4 - v^2}$	**A1**	oe, such as $x^2\frac{\mathrm{d}v}{\mathrm{d}x} = x\sqrt{4 - v^2}$.
	$\int \frac{1}{\sqrt{4-v^2}}\,\mathrm{d}v = \int \frac{1}{x}\,\mathrm{d}x$ $\Rightarrow \arcsin(\ldots) = \ldots$	**M1**	Separates their variables and integrates to the form $\arcsin(kv) = \alpha\ln x$, or the inverse function their root gives. Condone a missing constant here.
	$\arcsin\left(\frac{v}{2}\right) = \ln x + c$	**A1**	Needs the constant; $\ln\lvert x\rvert$ accepted.
	$\arcsin\left(\frac{1}{2}\right) = \ln 1 + c \Rightarrow c = \ldots$	**M1**	Substitutes $v = 1$, or $x = 1$ and $y = 1$, into their integrated equation containing a constant, to find it.
	$c = \frac{\pi}{6}$	**A1**	oe, or awrt 0.524.
	$\arcsin\left(\frac{y}{2x}\right) = \ln x + \text{“}\frac{\pi}{6}\text{”}$ $\Rightarrow y = \ldots$	**dM1**	Replaces v by $\frac{y}{x}$ in their integrated equation and makes y the subject, taking sine, or the inverse of their function, of the whole right-hand side.
	$y = 2x\sin\left(\ln x + \frac{\pi}{6}\right)$	**A1**	cso, in the form $y = f(x)$; an unrearranged arcsin form scores nothing here.
	Total for Question 7	**9**	

Question	Scheme	Marks	Notes
8(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = ku^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x}$	**B1**	oe, such as $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{k}u^{1-k}\frac{\mathrm{d}y}{\mathrm{d}x}$.
	$kxu^{k-1}\frac{\mathrm{d}u}{\mathrm{d}x} - 4u^k = 2x^3u^{\frac{k}{2}}$ $\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4u^k}{kxu^{k-1}} = \frac{2x^3u^{\frac{k}{2}}}{kxu^{k-1}}$	**M1**	Substitutes their derivative and $\sqrt{y} = u^{\frac{k}{2}}$ into equation $(*)$ and divides every term by their coefficient of $\frac{\mathrm{d}u}{\mathrm{d}x}$. Condone index slips.
	$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{kx}u = \frac{2}{k}x^2u^{1-\frac{k}{2}}$	**A1 AG**	Correct substituted equation leading without error to the printed result; combined index steps accepted.
		[3]	
8(b)	$k = 2$	**B1**	
		[1]	
8(c)	$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2}{x}u = x^2$ $\mu = e^{\int -\frac{2}{x}\,\mathrm{d}x} = e^{-2\ln x}$	**M1**	Uses $e^{\int P\,\mathrm{d}x}$ with $P = \pm\frac{2}{x}$ and integrates to a multiple of $\ln x$.
	$\mu = x^{-2}$	**A1**	oe, such as $\frac{1}{x^2}$.
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-2}u\right) = 1$ $\Rightarrow x^{-2}u = x + c$	**dM1**	Multiplies equation $(**)$ by their integrating factor and integrates both sides, the LHS to their integrating factor times u. Condone the constant omitted.
	$u = x^3 + cx^2$ $\Rightarrow y = \left(x^3 + cx^2\right)^2$	**A1**	oe, such as $y = x^4(x+c)^2$, with the constant. Not for $u = \ldots$ alone.
		[4]	
	Total for Question 8	**8**	

Question	Scheme	Marks	Notes
9(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x}$	**B1**	oe, including $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = -y^2\frac{\mathrm{d}u}{\mathrm{d}x}$.
	$-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\tan x}{u} - \frac{\cos^2 x}{u^2}$	**M1**	Substitutes their derivative and $y = \frac{1}{u}$ into the equation and multiplies through by a power of u to clear fractions. Condone sign slips.
	$\frac{\mathrm{d}u}{\mathrm{d}x} + u\tan x = \cos^2 x$	**A1 AG**	Correct substituted equation leading without error to the printed result.
		[3]	
9(b)	$\mu = e^{\int \tan x\,\mathrm{d}x} = e^{\cdots}$	**M1**	Uses $e^{\int \tan x\,\mathrm{d}x}$ and integrates $\tan x$ to a form $\pm\ln(\sec x)$ or $\pm\ln(\cos x)$.
	$\mu = \sec x$	**A1**	oe, such as $\frac{1}{\cos x}$. Condone a modulus.
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(u\text{“}\sec x\text{”}\right) = \cos^2 x\text{“}\sec x\text{”}$ $\Rightarrow u\sec x = \ldots$	**dM1**	Multiplies through by their integrating factor and integrates both sides, with $u\times$ their integrating factor on the left. Condone a missing constant.
	$u\sec x = \sin x + c$	**A1**	oe, such as $u = \sin x\cos x + c\cos x$. Needs the constant.
	$x = 0,\ u = 1:\ 1 = 0 + c \Rightarrow c = \ldots\ [= 1]$	**M1**	Substitutes $x = 0$ with $u = 1$, or $y = 1$ in a solution for y, into a general solution containing a constant to find it.
	$y = \frac{1}{\cos x(1+\sin x)}$	**A1**	oe, such as $y = \frac{\sec x}{1+\sin x}$ or $\frac{1}{\sin x\cos x+\cos x}$. Must be y in terms of x; u alone scores nothing.
		[6]	
	Total for Question 9	**9**	

Question	Scheme	Marks	Notes
10(a)	$y = v + \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{x^2}$	**B1**	oe, such as $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x^2}$.
	$x^2\left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{x^2}\right) + (xv+1)(xv-1) = -2$	**M1**	Substitutes for both y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ in equation $(*)$ and expands or simplifies. Condone sign slips.
	$x^2\frac{\mathrm{d}v}{\mathrm{d}x} + x^2v^2 = 0 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -v^2$	**A1 AG**	Correct substituted equation leading without error to the printed result; collected steps accepted. Division by x^2 need not be justified.
		[3]	
10(b)	$\int -\frac{1}{v^2}\,\mathrm{d}v = \int 1\,\mathrm{d}x \Rightarrow \frac{1}{v} = x + c$	**M1**	Separates the variables and integrates both sides, at least one to the form kv^{-1} or kx. Condone sign slips.
	$v = \frac{1}{x+c}$	**A1**	v in terms of x with an arbitrary constant, oe such as $v = \frac{A}{Ax+1}$.
		[2]	
10(c)	$y = \frac{1}{x} + \frac{1}{x+c}$	**B1ft**	ft their v from (b), which must contain an arbitrary constant: $y = \frac{1}{x}+$ their v.
		[1]	
10(d)	$\frac{1}{2} = 1 + \frac{1}{1+c} \Rightarrow c = \ldots$	**M1**	Substitutes $x = 1$, $y = \frac{1}{2}$ into their general solution from (c), containing a constant, and solves for the constant.
	$c = -3 \Rightarrow y = \frac{1}{x} + \frac{1}{x-3}$	**A1**	Or the equivalent constant for their form of solution, or the correct particular solution.
			

Question 10 continued

Question	Scheme	Marks	Notes
	Shape	**B1**	Three branches: below the x-axis for $x < 0$, falling from $+\infty$ to $-\infty$ through $\left(1, \frac{1}{2}\right)$ for $0 < x < 3$, above it for $x > 3$; the point need not be labelled.
	Asymptotes	**B1**	Vertical asymptotes $x = 0$ and $x = 3$, both equations stated, with branches approaching them.
	Asymptotes	**B1**	$y = 0$ stated, with the outer branches approaching the x-axis.
	Intercepts	**B1**	$\left(\frac{3}{2}, 0\right)$ marked, oe $x = \frac{3}{2}$, and no other point where the curve meets an axis.
		[6]	
	Total for Question 10	**12**	

Question	Scheme	Marks	Notes
11	$\frac{\mathrm{d}z}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$	**B1**	
	$\frac{\mathrm{d}z}{\mathrm{d}x} - 1 = \frac{z^2-2}{z^2}$ $\Rightarrow \int \frac{z^2}{z^2-1}\,\mathrm{d}z = \int 2\,\mathrm{d}x$	**M1 A1**	**M1** Substitutes for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $x+y$ to get an equation in z and x only, then separates the variables. Condone sign slips. **A1** Correct separated form, oe.
	$\frac{z^2}{z^2-1} = 1 + \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$	**M1**	Writes the integrand as 1 plus partial fractions in $z-1$ and $z+1$, or uses a standard result, and integrates to the form $z + p\ln\lvert z-1\rvert + q\ln\lvert z+1\rvert$. Condone coefficient slips.
	$z + \frac{1}{2}\ln\lvert z-1\rvert - \frac{1}{2}\ln\lvert z+1\rvert = 2x + c$	**A1**	Correct integration with a constant, oe.
	$\frac{1}{2}\ln\left\lvert\frac{x+y-1}{x+y+1}\right\rvert = x - y + c$ $\Rightarrow \frac{x+y-1}{x+y+1} = Ae^{2(x-y)}$	**dM1**	Replaces z by $x+y$, combines the logarithms and removes them to reach the form $\frac{x+y-1}{x+y+1} = Ae^{k(x-y)}$. Condone a missing modulus.
	$\frac{1+2-1}{1+2+1} = Ae^{\text{“2”}(1-2)} \Rightarrow A = \ldots$	**M1**	Substitutes $x=1$, $y=2$ into their equation containing a constant, at any stage, and solves for that constant.
	$A = \frac{e^2}{2}, \quad k = 2$	**A1**	Both values, or the solution in the required form with them; A oe or awrt 3.69.
	Total for Question 11	**8**	

Question	Scheme	Marks	Notes
12(a)	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_0 = \frac{3+2}{2} - 0 = 2.5$	**B1**	
	$x_1 = 2 + 0.1 \times \text{“2.5”} = 2.25$	**M1 A1**	**M1** Uses $x_1 = 2 + 0.1\times$ their gradient at $t = 0$.
	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_1 = \frac{3+2\cosh 0.1}{\text{“2.25”}\cosh^2 0.1} - \text{“2.25”}\tanh 0.1$ $x_2 = \text{“2.25”} + 0.1 \times \ldots$	**dM1**	Substitutes $t = 0.1$ and their x_1 into equation $(*)$, then adds $0.1\times$ that gradient to their x_1.
	$x_2 \approx 2.448$	**A1**	awrt 2.448.
		[5]	
12(b)	$\frac{\mathrm{d}u}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}$	**B1**	oe, such as $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2x}\frac{\mathrm{d}u}{\mathrm{d}t}$ or $\frac{1}{2}u^{-\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}t}$.
	$\frac{\mathrm{d}u}{\mathrm{d}t} = 2x\left(\frac{3+2\cosh t}{x\cosh^2 t} - x\tanh t\right)$ $= \frac{6+4\cosh t}{\cosh^2 t} - 2u\tanh t$	**M1**	Substitutes equation $(*)$ into their chain-rule relation and replaces x^2 by u, eliminating x. Condone sign slips.
	$\frac{\mathrm{d}u}{\mathrm{d}t} + 2u\tanh t = \frac{4}{\cosh t} + \frac{6}{\cosh^2 t}$	**A1 AG**	Correct substituted equation leading without error to equation $(**)$; combined steps accepted.
		[3]	
12(c)	$\mu = e^{\int 2\tanh t\,\mathrm{d}t} = e^{2\ln(\cosh t)}$	**M1**	Forms $e^{\int 2\tanh t\,\mathrm{d}t}$ and integrates to the form $e^{k\ln(\cosh t)}$.
	$\mu = \cosh^2 t$	**A1**	
	$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\cosh^2 t\right) = 4\cosh t + 6$ $\Rightarrow u\cosh^2 t = \ldots$	**dM1**	Multiplies equation $(**)$ by their integrating factor and integrates both sides, the left as their integrating factor times u. Condone a missing constant.
	$u = \frac{4\sinh t + 6t + c}{\cosh^2 t}$	**A1**	oe, including $u\cosh^2 t = 4\sinh t + 6t + c$. Needs the constant.
		[4]	
12(d)	$4\cosh^2 0 = 4\sinh 0 + 0 + c \Rightarrow c = \ldots\ [= 4]$	**M1**	Uses $t = 0$ with $u = 4$, or $x = 2$ in an expression for x, in their general solution to find their constant.
	$x = \frac{\sqrt{4\sinh t + 6t + 4}}{\cosh t}$	**A1**	oe, as $x = \ldots$ with the positive root only.
		[2]	
12(e)	$x = \frac{\sqrt{4\sinh 0.2 + 1.2 + 4}}{\cosh 0.2}\ [\approx 2.4024]$	**M1**	Substitutes $t = 0.2$ into their expression for x in terms of t.

Question 12 continued

Question	Scheme	Marks	Notes
	$\frac{\text{“2.4480”} - \text{“2.4024”}}{\text{“2.4024”}} \times 100$	**dM1**	Divides the difference of their estimate and their model value by their model value, times 100. Condone the difference either way round.
	$\approx 1.90\%$, overestimate	**A1**	awrt 1.9%, with overestimate stated.
		[3]	
	Total for Question 12	**17**	