OPTIONAL PURE / CONIC SECTIONS

Parabolas and Rectangular Hyperbolas

Worked solutions 11 questions · 121 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*9 marks*	2
Question 2	*13 marks*	3
Question 3	*11 marks*	5
Question 4	*10 marks*	7
Question 5	*11 marks*	9
Question 6	*8 marks*	11
Question 7	*13 marks*	13
Question 8	*11 marks*	15
Question 9	*12 marks*	17
Question 10	*9 marks*	19
Question 11	*14 marks*	21

1. A rectangular hyperbola H has equation $xy = 16$.
The point $P\left(4t, \frac{4}{t}\right)$, $t \neq 0$, is a general point on H.

(a) Show that an equation of the tangent to H at P is $x + t^2y = 8t$. **[4]**

The distinct points Q and R lie on H. The tangent to H at Q and the tangent to H at R meet at the point $T(k, 3)$, where k is a constant. The point Q has coordinates $(8, 2)$.

(b) Find the value of k. **[2]**

(c) Find the coordinates of R. **[3]**

Solution

(a) Write the equation of H as $y = 16x^{-1}$. Differentiating gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -16x^{-2} = -\frac{16}{x^2}$$

At P, $x = 4t$, so the gradient of the tangent is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{16}{16t^2} = -\frac{1}{t^2}$$

The tangent passes through $P\left(4t, \frac{4}{t}\right)$, so its equation is

$$y - \frac{4}{t} = -\frac{1}{t^2}(x - 4t)$$

$$t^2y - 4t = -x + 4t$$

$$x + t^2y = 8t$$

as required.

(b) At $Q(8, 2)$, $4t = 8$, so $t = 2$. Using part (a) with $t = 2$, the tangent at Q is

$$x + 4y = 16$$

The point $T(k, 3)$ lies on this tangent, so

$$k + 12 = 16 \quad \Longrightarrow \quad k = 4$$

(c) Let R have parameter t. By part (a), the tangent at R is $x + t^2y = 8t$, and it passes through $T(4, 3)$, so

$$4 + 3t^2 = 8t$$

$$3t^2 - 8t + 4 = 0$$

$$(3t - 2)(t - 2) = 0$$

so $t = \frac{2}{3}$ or $t = 2$. The value $t = 2$ gives Q, and R is distinct from Q, so $t = \frac{2}{3}$. Then

$$x = 4 \times \frac{2}{3} = \frac{8}{3}, \qquad y = \frac{4}{\frac{2}{3}} = 6$$

so R has coordinates $\left(\frac{8}{3}, 6\right)$.

2. The parabola C has equation $y^2 = 8x$, and the point S is the focus of C.
The point P lies on C and has y-coordinate -8. The point T lies on the directrix of C, and the line segment PT is parallel to the x-axis.

(a) Write down the coordinates of S. **[1]**

(b) Find the area of triangle PST. **[2]**

(c) Use calculus to show that the normal to C at the point P has equation

$$y = 2x - 24$$ **[4]**

The rectangular hyperbola H has equation $xy = 16$. The normal to C at P meets H at the points L and M.

(d) Find the exact coordinates of the points L and M, giving your answers in their simplest form. **[6]**

Solution

(a) The parabola $y^2 = 4ax$ has focus $(a, 0)$. Here $4a = 8$, so $a = 2$ and

$$S = (2, 0)$$

(b) When $y = -8$:

$$64 = 8x \quad \Longrightarrow \quad x = 8$$

so $P = (8, -8)$. The parabola $y^2 = 4ax$ has directrix $x = -a$, so the directrix of C is $x = -2$. Since PT is parallel to the x-axis, T also has y-coordinate -8, so $T = (-2, -8)$.

Taking PT as the base, its length is $8 - (-2) = 10$. The perpendicular height is the distance from $S(2, 0)$ to the line $y = -8$, which is 8. So

$$\text{Area} = \frac{1}{2} \times 10 \times 8 = 40$$

(c) Differentiating $y^2 = 8x$ implicitly with respect to x:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 8 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{y}$$

At $P(8, -8)$, the gradient of the tangent is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{-8} = -\frac{1}{2}$$

The normal is perpendicular to the tangent, so its gradient is 2. The equation of the normal at P is

$$y - (-8) = 2(x - 8)$$

$$y + 8 = 2x - 16$$

$$y = 2x - 24$$

as required.

(d) Substituting $y = 2x - 24$ into $xy = 16$:

$$x(2x - 24) = 16$$
$$2x^2 - 24x - 16 = 0$$
$$x^2 - 12x - 8 = 0$$

Completing the square:

$$(x - 6)^2 - 36 - 8 = 0$$
$$(x - 6)^2 = 44$$
$$x - 6 = \pm\sqrt{44} = \pm 2\sqrt{11}$$
$$x = 6 \pm 2\sqrt{11}$$

Using $y = 2x - 24$:

$$y = 2\left(6 \pm 2\sqrt{11}\right) - 24 = -12 \pm 4\sqrt{11}$$

So the points L and M are

$$\left(6 + 2\sqrt{11},\ -12 + 4\sqrt{11}\right) \quad \text{and} \quad \left(6 - 2\sqrt{11},\ -12 - 4\sqrt{11}\right)$$

3. The rectangular hyperbola H has equation $xy = 9$.

(a) Use calculus to show that an equation of the tangent to H at the point $P\left(3t, \frac{3}{t}\right)$, where $t \neq 0$, is

$$x + t^2 y = 6t$$ **[4]**

The tangent to H at P meets the x-axis at the point A. The point Q lies on H and has the same x-coordinate as A.

(b) Find the coordinates of Q in terms of t, and find an equation of the tangent to H at Q. **[3]**

The tangent at P and the tangent at Q meet at the point R.

(c) Show that, as t varies, the locus of R is a rectangular hyperbola, and state its equation. **[4]**

Solution

(a) Write the equation of H as $y = \frac{9}{x}$ and differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{9}{x^2}$$

At P, $x = 3t$, so the gradient of the tangent is

$$-\frac{9}{(3t)^2} = -\frac{9}{9t^2} = -\frac{1}{t^2}$$

The tangent passes through $P\left(3t, \frac{3}{t}\right)$, so its equation is

$$y - \frac{3}{t} = -\frac{1}{t^2}(x - 3t)$$

$$t^2 y - 3t = -x + 3t$$

$$x + t^2 y = 6t$$

as required.

(b) On the x-axis $y = 0$, so the tangent at P gives $x = 6t$, and A is $(6t, 0)$.

Q lies on H with $x = 6t$, so

$$y = \frac{9}{6t} = \frac{3}{2t}$$

Hence Q is $\left(6t, \frac{3}{2t}\right)$.

This is $\left(3(2t), \frac{3}{2t}\right)$, the point of H given by the parameter $2t$ in place of t. Replacing t by $2t$ in the result of part (a), the tangent to H at Q is

$$x + (2t)^2 y = 6(2t)$$

$$x + 4t^2 y = 12t$$

(c) At R both tangent equations hold:

$$x + 4t^2 y = 12t$$

$$x + t^2 y = 6t$$

Subtracting the second equation from the first:

$$3t^2 y = 6t \quad \Longrightarrow \quad y = \frac{2}{t}$$

since $t \neq 0$. Substituting into $x + t^2 y = 6t$:

$$x = 6t - t^2 \times \frac{2}{t} = 6t - 2t = 4t$$

So R is $\left(4t, \frac{2}{t}\right)$. Eliminating t by multiplying the coordinates:

$$xy = 4t \times \frac{2}{t} = 8$$

As t varies over all non-zero values, R lies on the curve $xy = 8$, which is of the form $xy = c^2$ with $c = 2\sqrt{2}$. Hence the locus of R is a rectangular hyperbola, with equation $xy = 8$.

4. The parabola C has equation $y^2 = 12x$ and focus S.
The point $P(3t^2, 6t)$, where $t \neq 0$, lies on C.

(a) Show that an equation of the normal to C at P is

$$y + tx = 6t + 3t^3 \qquad \textbf{[4]}$$

The line ℓ passes through S and is perpendicular to the normal to C at P.
The line ℓ and the normal to C at P intersect at the point Q.

(b) Determine, in simplest form in terms of t, the coordinates of Q. **[3]**

(c) Hence show that, as t varies, the point Q lies on a parabola, and find the coordinates of the focus of this parabola. **[3]**

Solution

(a) With $x = 3t^2$ and $y = 6t$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 6$$

so the gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{6}{6t} = \frac{1}{t}$$

The normal is perpendicular to the tangent, so its gradient is $-t$. The normal passes through $P(3t^2, 6t)$, so its equation is

$$y - 6t = -t\left(x - 3t^2\right)$$

$$y - 6t = -tx + 3t^3$$

$$y + tx = 6t + 3t^3$$

as required.

(b) The parabola $y^2 = 4ax$ has focus $(a, 0)$. Here $4a = 12$, so $a = 3$ and S is $(3, 0)$.

The normal has gradient $-t$, so ℓ, which is perpendicular to it, has gradient $\frac{1}{t}$. Since ℓ passes through $S(3, 0)$, its equation is

$$y = \frac{1}{t}(x - 3) \quad \Longrightarrow \quad x = 3 + ty$$

Substituting $x = 3 + ty$ into the equation of the normal:

$$y + t(3 + ty) = 6t + 3t^3$$

$$y + t^2 y = 3t + 3t^3$$

$$y\left(1 + t^2\right) = 3t\left(1 + t^2\right)$$

$$y = 3t$$

since $1 + t^2 \neq 0$. Then $x = 3 + t \times 3t = 3 + 3t^2$, so Q is the point $\left(3 + 3t^2,\ 3t\right)$.

(c) From part (b), Q has coordinates $x = 3 + 3t^2$, $y = 3t$. Eliminating t, with $t = \frac{y}{3}$:

$$x = 3 + 3\left(\frac{y}{3}\right)^2$$

$$x - 3 = \frac{y^2}{3}$$

$$y^2 = 3(x - 3)$$

This is the parabola $y^2 = 3x$ translated by 3 units in the positive x-direction, so Q lies on a parabola.

The parabola $y^2 = 3x$ is of the form $y^2 = 4bx$ with $4b = 3$, so $b = \frac{3}{4}$ and its focus is $\left(\frac{3}{4}, 0\right)$. Translating by 3 units in the positive x-direction, the focus of the parabola on which Q lies is

$$\left(3 + \frac{3}{4}, 0\right) = \left(\frac{15}{4}, 0\right)$$

5. The parabola C has equation $y^2 = 24x$.
The point $P\left(6t^2, 12t\right)$ lies on C, where t is a positive constant.

(a) Use calculus to show that an equation of the normal to C at P is

$$y + tx = 12t + 6t^3$$ **[4]**

(b) Write down the coordinates of the focus of C. **[1]**

The line ℓ is the reflection of the normal to C at P in the line through P parallel to the x-axis.

(c) Given that ℓ passes through the focus of C, determine the exact coordinates of P. **[6]**

Solution

(a) At P, $x = 6t^2$ and $y = 12t$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 12t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 12$$

The gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{12}{12t} = \frac{1}{t}$$

The normal is perpendicular to the tangent, so its gradient is $-t$. The normal passes through $P\left(6t^2, 12t\right)$, so its equation is

$$y - 12t = -t\left(x - 6t^2\right)$$

$$y - 12t = -tx + 6t^3$$

$$y + tx = 12t + 6t^3$$

as required.

(b) The parabola $y^2 = 4ax$ has focus $(a, 0)$. Here $4a = 24$, so $a = 6$ and the focus of C is

$$(6, 0)$$

(c) The line through P parallel to the x-axis is $y = 12t$. Reflection in this line keeps the x-coordinate of each point and sends a point at height y to a point at the same distance on the other side of $y = 12t$, so it maps (x, y) to $(x, 24t - y)$.

A point (x, y) lies on ℓ exactly when its reflection $(x, 24t - y)$ lies on the normal. Substituting $24t - y$ for y in the equation of the normal from part (a):

$$(24t - y) + tx = 12t + 6t^3$$

$$y = tx + 12t - 6t^3$$

The focus $(6, 0)$ lies on ℓ, so

$$0 = 6t + 12t - 6t^3$$

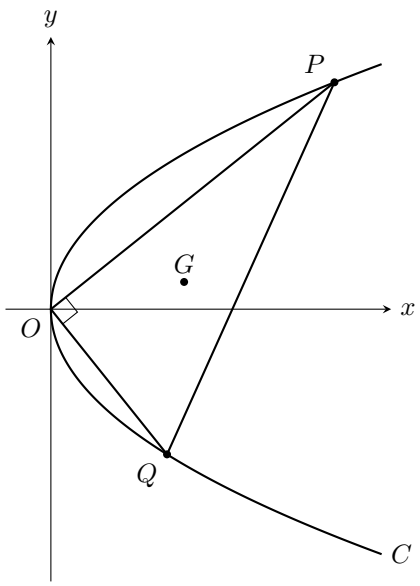
$$0 = 18t - 6t^3$$

$$0 = 6t\left(3 - t^2\right)$$

Since $t > 0$, $t \neq 0$, so $t^2 = 3$ and $t = \sqrt{3}$. Then

$$x = 6t^2 = 6 \times 3 = 18, \qquad y = 12t = 12\sqrt{3}$$

so P has coordinates $\left(18, 12\sqrt{3}\right)$.



6. The diagram above shows the parabola C with equation $y^2 = 4ax$, where a is a positive constant. The point $P(ap^2, 2ap)$, where $p > 0$, lies on C, and O is the origin.

(a) Write down the gradient of OP in terms of p. **[1]**

A second point $Q(aq^2, 2aq)$ on C is such that $\angle POQ = 90°$.

(b) Show that $pq = -4$. **[2]**

(c) The point G is the centroid of triangle OPQ, the point whose coordinates are the means of the coordinates of O, P and Q.
Prove that, as P varies, the locus of G has equation

$$9y^2 = 4a(3x - 8a)$$ **[5]**

Solution

(a) The line OP joins $(0, 0)$ to $(ap^2, 2ap)$, so its gradient is

$$m_{OP} = \frac{2ap - 0}{ap^2 - 0} = \frac{2}{p}$$

(b) In the same way, with q in place of p, the gradient of OQ is

$$m_{OQ} = \frac{2aq}{aq^2} = \frac{2}{q}$$

Since $\angle POQ = 90°$, the lines OP and OQ are perpendicular, so the product of their gradients is -1:

$$\frac{2}{p} \times \frac{2}{q} = -1$$

$$\frac{4}{pq} = -1$$

$$pq = -4$$

as required.

(c) Let G have coordinates (x, y). Taking the means of the coordinates of $O(0,0)$, $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$:

$$x = \frac{ap^2 + aq^2}{3}, \qquad y = \frac{2ap + 2aq}{3}$$

From part (b), $q = -\frac{4}{p}$. Substituting this into both coordinates:

$$\frac{3x}{a} = p^2 + \frac{16}{p^2}$$

$$\frac{3y}{2a} = p - \frac{4}{p}$$

Squaring the second equation gives

$$\frac{9y^2}{4a^2} = p^2 - 8 + \frac{16}{p^2}$$

and substituting $p^2 + \frac{16}{p^2} = \frac{3x}{a}$ from the first equation:

$$\frac{9y^2}{4a^2} = \frac{3x}{a} - 8$$

$$9y^2 = 12ax - 32a^2$$

$$9y^2 = 4a(3x - 8a)$$

This holds for every position of P, so the locus of G has equation $9y^2 = 4a(3x - 8a)$, as required.

7. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant.
The point $P\left(ct, \frac{c}{t}\right)$, where $t > 0$, lies on H.

(a) Use calculus to show that an equation of the normal to H at P is

$$t^3x - ty = c\left(t^4 - 1\right)$$ **[4]**

The parabola C has equation $y^2 = 16x$.
The point $(4, 9)$ lies on H. The normal to H at $(4, 9)$ meets C at the points A and B, where the x-coordinate of A is less than the x-coordinate of B.

(b) Determine the exact coordinates of A and of B. **[4]**

The point R is the focus of C. The line through A and R meets C again at the point T.

(c) Determine the exact length of AT. **[5]**

Solution

(a) Write the equation of H as $y = c^2x^{-1}$ and differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2}$$

At P, $x = ct$, so the gradient of the tangent is

$$-\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

The normal is perpendicular to the tangent, so its gradient is t^2. The normal at P is therefore

$$y - \frac{c}{t} = t^2(x - ct)$$

$$ty - c = t^3x - ct^4$$

$$t^3x - ty = ct^4 - c$$

so

$$t^3x - ty = c\left(t^4 - 1\right)$$

as required.

(b) Since $(4, 9)$ lies on H, $c^2 = 4 \times 9 = 36$, so $c = 6$ as $c > 0$. Then $ct = 4$ gives $t = \frac{2}{3}$.
Substituting $c = 6$ and $t = \frac{2}{3}$ into the equation from part (a):

$$\frac{8}{27}x - \frac{2}{3}y = 6\left(\frac{16}{81} - 1\right) = -\frac{130}{27}$$

$$8x - 18y = -130$$

$$4x - 9y = -65$$

On C, $x = \frac{y^2}{16}$. Substituting into the normal:

$$\frac{y^2}{4} - 9y + 65 = 0$$

$$y^2 - 36y + 260 = 0$$

$$(y - 10)(y - 26) = 0$$

so $y = 10$ or $y = 26$. Then $x = \frac{100}{16} = \frac{25}{4}$ or $x = \frac{676}{16} = \frac{169}{4}$. Since the x-coordinate of A is the smaller, the points are

$$A\left(\frac{25}{4}, 10\right), \qquad B\left(\frac{169}{4}, 26\right)$$

(c) The parabola $y^2 = 4ax$ has focus $(a, 0)$ and directrix $x = -a$. Here $4a = 16$, so $a = 4$, the focus is $R(4, 0)$ and the directrix is $x = -4$.

The gradient of AR is

$$\frac{10 - 0}{\frac{25}{4} - 4} = \frac{10}{\frac{9}{4}} = \frac{40}{9}$$

so the line AR is $y = \frac{40}{9}(x - 4)$, that is, $x = \frac{9y}{40} + 4$. Substituting into $y^2 = 16x$:

$$y^2 = \frac{18y}{5} + 64$$

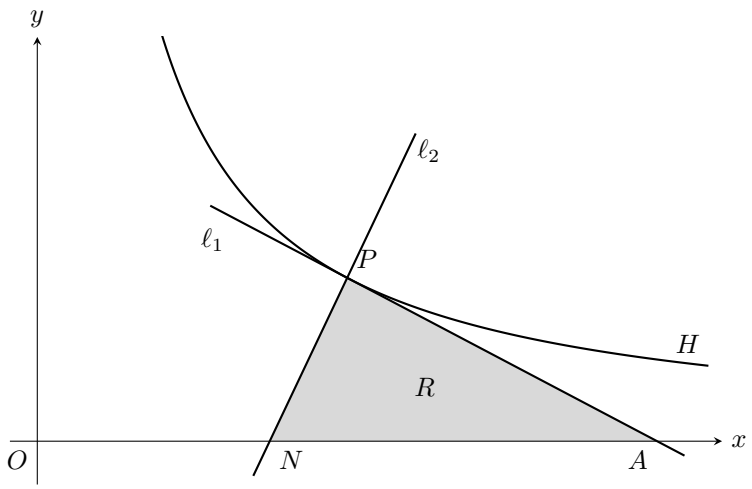
$$5y^2 - 18y - 320 = 0$$

$$(y - 10)(5y + 32) = 0$$

The root $y = 10$ gives A, so at T, $y = -\frac{32}{5}$ and $x = \frac{1}{16}\left(\frac{32}{5}\right)^2 = \frac{64}{25}$. Hence $T\left(\frac{64}{25}, -\frac{32}{5}\right)$.

Every point on C is the same distance from the focus as from the directrix $x = -4$, so the distance from a point on C with x-coordinate x to R is $x + 4$. The chord AT passes through R, with R between A and T, so

$$\begin{aligned} AT &= AR + RT \\ &= \left(\frac{25}{4} + 4\right) + \left(\frac{64}{25} + 4\right) \\ &= \frac{625 + 256 + 800}{100} \\ &= \frac{1681}{100} \end{aligned}$$



8. The diagram above shows a sketch of part of the rectangular hyperbola H with equation

$$xy = c^2 \qquad x > 0$$

where c is a positive constant.
The point $P\left(ct, \frac{c}{t}\right)$, where $t > 0$, lies on H.
The line ℓ_1 is the tangent to H at P, and it crosses the x-axis at the point A. The line ℓ_2 is the normal to H at P, and it crosses the x-axis at the point N.
The region R, shown shaded in the diagram above, is the triangle PNA.
Given that the length PA is twice the length PN, and that the area of R is 40, find the exact coordinates of the point P. **[11]**

Solution

On H, $x = ct$ and $y = \frac{c}{t}$, so

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{c}{t^2}$$

so the gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-\frac{c}{t^2}}{c} = -\frac{1}{t^2}$$

The tangent ℓ_1 at P has gradient $-\frac{1}{t^2}$, so its equation is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2 y - ct = -x + ct$$

$$x + t^2 y = 2ct$$

Setting $y = 0$ gives $x = 2ct$, so A is $(2ct, 0)$.

The normal ℓ_2 is perpendicular to the tangent, so its gradient is t^2 and its equation is

$$y - \frac{c}{t} = t^2(x - ct)$$

Setting $y = 0$:

$$-\frac{c}{t} = t^2 x - ct^3 \quad \Longrightarrow \quad x = ct - \frac{c}{t^3}$$

so N is $\left(ct - \frac{c}{t^3}, 0\right)$.

Now find the two lengths:

$$PA^2 = (2ct - ct)^2 + \left(0 - \frac{c}{t}\right)^2 = c^2t^2 + \frac{c^2}{t^2} = \frac{c^2\left(t^4+1\right)}{t^2}$$

$$PN^2 = \left(ct - \frac{c}{t^3} - ct\right)^2 + \left(0 - \frac{c}{t}\right)^2 = \frac{c^2}{t^6} + \frac{c^2}{t^2} = \frac{c^2\left(t^4+1\right)}{t^6}$$

Dividing:

$$\frac{PA^2}{PN^2} = \frac{c^2\left(t^4+1\right)}{t^2} \times \frac{t^6}{c^2(t^4+1)} = t^4$$

Since $PA = 2PN$, $\frac{PA^2}{PN^2} = 4$, so

$$t^4 = 4 \quad \Longrightarrow \quad t^2 = 2 \quad \Longrightarrow \quad t = \sqrt{2}$$

taking the positive root because $t > 0$.

The triangle PNA has base NA along the x-axis and perpendicular height equal to the y-coordinate of P, which is $\frac{c}{t}$. The base is

$$NA = 2ct - \left(ct - \frac{c}{t^3}\right) = ct + \frac{c}{t^3}$$

so the area of R is

$$\begin{aligned}\frac{1}{2}\left(ct + \frac{c}{t^3}\right) \times \frac{c}{t} &= \frac{c^2}{2}\left(1 + \frac{1}{t^4}\right) \\ &= \frac{c^2}{2}\left(1 + \frac{1}{4}\right) \\ &= \frac{5c^2}{8}\end{aligned}$$

The area is 40, so

$$\frac{5c^2}{8} = 40 \quad \Longrightarrow \quad c^2 = 64 \quad \Longrightarrow \quad c = 8$$

since $c > 0$. Then P is $\left(ct, \frac{c}{t}\right)$ with $c = 8$ and $t = \sqrt{2}$:

$$P = \left(8\sqrt{2}, \frac{8}{\sqrt{2}}\right) = \left(8\sqrt{2}, 4\sqrt{2}\right)$$

9. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. The point $P\left(cp, \frac{c}{p}\right)$, where $p \neq 0$, lies on H.

(a) Use calculus to show that an equation of the normal to H at P is

$$p^3x - py = c\left(p^4 - 1\right) \quad \textbf{[4]}$$

The normal to H at P meets H again at the point Q.

(b) Show that

$$PQ^2 = c^2\left(p^2 + \frac{1}{p^2}\right)^3 \quad \textbf{[5]}$$

(c) Hence find, in terms of c, the least possible length of PQ, and the coordinates of each point P for which this least length occurs. **[3]**

Solution

(a) Write the equation of H as $y = c^2x^{-1}$. Differentiating:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

At P, $x = cp$, so the gradient of the tangent is

$$-\frac{c^2}{c^2p^2} = -\frac{1}{p^2}$$

The normal is perpendicular to the tangent, so its gradient is p^2. The normal through $P\left(cp, \frac{c}{p}\right)$ is

$$y - \frac{c}{p} = p^2(x - cp)$$

Multiplying by p:

$$py - c = p^3x - cp^4$$

$$p^3x - py = cp^4 - c$$

$$p^3x - py = c(p^4 - 1)$$

as required.

(b) Let Q be the point $\left(cq, \frac{c}{q}\right)$ on H, where $q \neq p$. Substituting into the equation of the normal:

$$p^3cq - \frac{pc}{q} = c(p^4 - 1)$$

Dividing by c, multiplying by q and rearranging gives

$$p^3q^2 + (1 - p^4)q - p = 0$$

Since P lies on the normal, $q = p$ is a root, so $q - p$ is a factor:

$$(q - p)(p^3q + 1) = 0$$

As $q \neq p$, $q = -\frac{1}{p^3}$, so Q is the point $\left(-\frac{c}{p^3}, -cp^3\right)$. The differences in the coordinates of P and Q are

$$cp - \left(-\frac{c}{p^3}\right) = c\left(p + \frac{1}{p^3}\right) = \frac{c\left(1+p^4\right)}{p^3}$$

$$\frac{c}{p} - \left(-cp^3\right) = c\left(\frac{1}{p} + p^3\right) = \frac{c\left(1+p^4\right)}{p}$$

Hence

$$\begin{aligned} PQ^2 &= \frac{c^2\left(1+p^4\right)^2}{p^6} + \frac{c^2\left(1+p^4\right)^2}{p^2} \\ &= \frac{c^2\left(1+p^4\right)^2}{p^6}\left(1+p^4\right) \\ &= \frac{c^2\left(1+p^4\right)^3}{p^6} \\ &= c^2\left(\frac{1+p^4}{p^2}\right)^3 \\ &= c^2\left(p^2 + \frac{1}{p^2}\right)^3 \end{aligned}$$

as required.

(c) Completing the square:

$$p^2 + \frac{1}{p^2} = \left(p - \frac{1}{p}\right)^2 + 2 \geq 2$$

with equality when $p = \frac{1}{p}$, that is $p^2 = 1$, so $p = 1$ or $p = -1$. Since $p^2 + \frac{1}{p^2} > 0$, PQ^2 is least when $p^2 + \frac{1}{p^2}$ is least. Using part (b):

$$PQ^2 \geq c^2 \times 2^3 = 8c^2 \quad \Longrightarrow \quad PQ \geq 2\sqrt{2}\,c$$

The least length of PQ is $2\sqrt{2}\,c$. It occurs when $p = 1$, giving $P(c, c)$, and when $p = -1$, giving $P(-c, -c)$.

10. The parabola C has equation $y^2 = 12x$.
The point $P(3p^2, 6p)$, where $p \neq 0$, lies on C. The normal to C at P meets C again at the point Q.
The tangent to C at P and the tangent to C at Q meet at the point R, whose y-coordinate is -3.
Show that the area of triangle PQR is $\frac{1125}{2}$. **[9]**

Solution

Differentiating $y^2 = 12x$ implicitly:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{y}$$

At $P(3p^2, 6p)$ the gradient of the tangent is $\frac{6}{6p} = \frac{1}{p}$, so the tangent at P is

$$y - 6p = \frac{1}{p}\left(x - 3p^2\right)$$

$$py - 6p^2 = x - 3p^2$$

$$py = x + 3p^2$$

The normal is perpendicular to the tangent, so its gradient is $-p$, and the normal at P is

$$y - 6p = -p\left(x - 3p^2\right)$$

Let Q be the point $(3q^2, 6q)$, where $q \neq p$. Substituting Q into the normal:

$$6q - 6p = -p\left(3q^2 - 3p^2\right)$$

$$6(q - p) = -3p(q - p)(q + p)$$

Since $q \neq p$, dividing by $3(q - p)$ gives

$$2 = -p(q + p) \quad \Longrightarrow \quad p + q = -\frac{2}{p}$$

By the same working as at P, the tangent at Q is $qy = x + 3q^2$. Subtracting this from the tangent at P:

$$(p - q)y = 3\left(p^2 - q^2\right)$$

$$(p - q)y = 3(p - q)(p + q)$$

$$y = 3(p + q)$$

Then $x = py - 3p^2 = 3p^2 + 3pq - 3p^2 = 3pq$, so R is the point $(3pq,\ 3(p + q))$.

The y-coordinate of R is -3, so

$$3(p + q) = -3 \quad \Longrightarrow \quad p + q = -1$$

Combining with $p + q = -\frac{2}{p}$:

$$-\frac{2}{p} = -1 \quad \Longrightarrow \quad p = 2, \qquad q = -1 - 2 = -3$$

Hence the three points are

$$P(12,\ 12), \qquad Q(27,\ -18), \qquad R(-18,\ -3)$$

The line PQ is the normal at P and the line PR is the tangent at P, so they are perpendicular and angle QPR is a right angle. The two shorter sides are

$$PQ = \sqrt{(27-12)^2 + (-18-12)^2} = \sqrt{225+900} = \sqrt{1125}$$

$$PR = \sqrt{(-18-12)^2 + (-3-12)^2} = \sqrt{900+225} = \sqrt{1125}$$

So the area of triangle PQR is

$$\frac{1}{2} \times PQ \times PR = \frac{1}{2} \times \sqrt{1125} \times \sqrt{1125} = \frac{1125}{2}$$

as required.

11. The point $P(3p^2, 6p)$, where $p \neq 0$, lies on the parabola with equation $y^2 = 12x$.

(a) Show that the point $Q\left(\frac{48}{p^2}, -\frac{24}{p}\right)$ also lies on the parabola. **[1]**

(b) Show that the chord PQ passes through the point $(12, 0)$. **[4]**

The normal to the parabola at P and the normal to the parabola at Q meet at the point N.

(c) Determine, in simplest form, the coordinates of N in terms of p. **[9]**

Solution

(a) Substituting the coordinates of Q into each side of $y^2 = 12x$:

$$y^2 = \left(-\frac{24}{p}\right)^2 = \frac{576}{p^2}, \qquad 12x = 12 \times \frac{48}{p^2} = \frac{576}{p^2}$$

The two sides are equal, so Q lies on the parabola, as required.

(b) If $p^2 = 4$, then P and Q both have x-coordinate $3p^2 = \frac{48}{p^2} = 12$, so the chord PQ is the line $x = 12$, which passes through $(12, 0)$.

Otherwise, the gradient of PQ is

$$\begin{aligned} m_{PQ} &= \frac{6p + \frac{24}{p}}{3p^2 - \frac{48}{p^2}} \\ &= \frac{6p^3 + 24p}{3p^4 - 48} \\ &= \frac{6p(p^2 + 4)}{3(p^2 - 4)(p^2 + 4)} \\ &= \frac{2p}{p^2 - 4} \end{aligned}$$

The equation of the chord PQ is then

$$y - 6p = \frac{2p}{p^2 - 4}\left(x - 3p^2\right)$$

Setting $y = 0$ and multiplying by $p^2 - 4$:

$$\begin{aligned} -6p(p^2 - 4) &= 2p\left(x - 3p^2\right) \\ -3(p^2 - 4) &= x - 3p^2 \\ x &= 12 \end{aligned}$$

So the chord PQ meets the x-axis at $(12, 0)$, as required.

(c) Differentiating $y^2 = 12x$ implicitly with respect to x:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{y}$$

At P, $y = 6p$, so the gradient of the tangent is $\frac{6}{6p} = \frac{1}{p}$. The normal is perpendicular to the tangent, so its gradient is $-p$, and the normal at P is

$$y - 6p = -p\left(x - 3p^2\right)$$

$$y = -px + 6p + 3p^3$$

At Q, $y = -\frac{24}{p}$, so the gradient of the tangent is $6 \times \left(-\frac{p}{24}\right) = -\frac{p}{4}$ and the gradient of the normal is $\frac{4}{p}$. The normal at Q is

$$y + \frac{24}{p} = \frac{4}{p}\left(x - \frac{48}{p^2}\right)$$

$$y = \frac{4x}{p} - \frac{192}{p^3} - \frac{24}{p}$$

At N the two values of y are equal:

$$-px + 6p + 3p^3 = \frac{4x}{p} - \frac{192}{p^3} - \frac{24}{p}$$

Multiplying through by p^3 and collecting the terms in x:

$$-p^4x + 6p^4 + 3p^6 = 4p^2x - 192 - 24p^2$$

$$p^2\left(p^2 + 4\right)x = 3p^6 + 6p^4 + 24p^2 + 192$$

The right-hand side has a factor of $p^2 + 4$:

$$3p^6 + 6p^4 + 24p^2 + 192 = 3\left(p^2 + 4\right)\left(p^4 - 2p^2 + 16\right)$$

Since $p \neq 0$ and $p^2 + 4 > 0$, dividing by $p^2\left(p^2 + 4\right)$ gives

$$x = \frac{3\left(p^4 - 2p^2 + 16\right)}{p^2} = 3p^2 - 6 + \frac{48}{p^2}$$

Substituting into the normal at P:

$$y = -p\left(3p^2 - 6 + \frac{48}{p^2}\right) + 6p + 3p^3$$

$$= -3p^3 + 6p - \frac{48}{p} + 6p + 3p^3$$

$$= 12p - \frac{48}{p}$$

Hence

$$N\left(3p^2 - 6 + \frac{48}{p^2},\ 12p - \frac{48}{p}\right)$$