OPTIONAL PURE / CONIC SECTIONS

Ellipses and Hyperbolas

Worked solutions 12 questions · 126 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

Question 1	*6 marks*	2
Question 2	*6 marks*	3
Question 3	*7 marks*	4
Question 4	*8 marks*	6
Question 5	*12 marks*	8
Question 6	*13 marks*	10
Question 7	*12 marks*	13
Question 8	*12 marks*	15
Question 9	*12 marks*	17
Question 10	*13 marks*	20
Question 11	*13 marks*	22
Question 12	*12 marks*	24

1. The ellipse E has equation $9x^2 + 5y^2 = 45$.

(a) Find the eccentricity of E and the coordinates of its foci. **[4]**

(b) Find the equations of the directrices of E. **[2]**

Solution

(a) Dividing the equation by 45 gives

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Since $9 > 5$, the major axis lies along the y-axis. The semi-major axis is $a = 3$ and the semi-minor axis is $b = \sqrt{5}$. For an ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ with $a > b$, the eccentricity e satisfies

$$b^2 = a^2\left(1 - e^2\right)$$

Substituting $a^2 = 9$ and $b^2 = 5$:

$$5 = 9\left(1 - e^2\right)$$

$$1 - e^2 = \frac{5}{9}$$

$$e^2 = \frac{4}{9}$$

The eccentricity is positive, so

$$e = \frac{2}{3}$$

The foci lie on the major axis, at $(0, \pm ae)$. Here $ae = 3 \times \frac{2}{3} = 2$, so the foci are

$$(0, 2) \quad \text{and} \quad (0, -2)$$

(b) The directrices are perpendicular to the major axis, with equations $y = \pm\frac{a}{e}$. Here

$$\frac{a}{e} = \frac{3}{\frac{2}{3}} = \frac{9}{2}$$

so the directrices are

$$y = \frac{9}{2} \quad \text{and} \quad y = -\frac{9}{2}$$

2. The hyperbola H has parametric equations

$$x = 3\sec\theta, \qquad y = 2\tan\theta$$

(a) Find a Cartesian equation of H and hence find the coordinates of the foci of H. **[4]**

(b) Find the equations of the directrices of H, giving your answers in exact simplified form. **[2]**

Solution

(a) From the parametric equations, $\sec\theta = \frac{x}{3}$ and $\tan\theta = \frac{y}{2}$. Substituting these into the identity

$$\sec^2\theta - \tan^2\theta = 1$$

gives

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{2}\right)^2 = 1 \quad \Longrightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = 1$$

This is the standard hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a = 3$ and $b = 2$. For this hyperbola, the eccentricity e satisfies

$$b^2 = a^2\left(e^2 - 1\right)$$

Substituting $a = 3$ and $b = 2$, with $e > 1$ for a hyperbola:

$$4 = 9\left(e^2 - 1\right)$$

$$e^2 = 1 + \frac{4}{9} = \frac{13}{9}$$

$$e = \frac{\sqrt{13}}{3}$$

The foci are at $(\pm ae, 0)$. Here $ae = 3 \times \frac{\sqrt{13}}{3} = \sqrt{13}$, so the foci are

$$\left(\sqrt{13},\, 0\right) \quad \text{and} \quad \left(-\sqrt{13},\, 0\right)$$

(b) The directrices of the hyperbola are

$$x = \pm\frac{a}{e}$$

Using $a = 3$ and $e = \frac{\sqrt{13}}{3}$ from part (a):

$$x = \pm\frac{3}{\frac{\sqrt{13}}{3}} = \pm\frac{9}{\sqrt{13}} = \pm\frac{9\sqrt{13}}{13}$$

3. The hyperbola H, with centre at the origin and foci on the x-axis, has

- asymptotes with equations $y = \pm\frac{15}{8}x$
- directrices with equations $x = \pm\frac{32}{17}$
- eccentricity e.

(a) Find the value of e. **[3]**

(b) Find an equation for H, giving your answer in the form $px^2 - qy^2 = r$, where p, q and r are integers, and state the coordinates of the foci of H. **[4]**

Solution

(a) The hyperbola has centre at the origin and foci on the x-axis, so its equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with asymptotes $\frac{x}{a} = \pm\frac{y}{b}$, that is $y = \pm\frac{b}{a}x$, and its eccentricity e satisfies

$$b^2 = a^2\left(e^2 - 1\right)$$

Comparing with the asymptotes $y = \pm\frac{15}{8}x$ gives $\frac{b}{a} = \frac{15}{8}$. Dividing the eccentricity relation by a^2:

$$e^2 - 1 = \frac{b^2}{a^2} = \left(\frac{15}{8}\right)^2 = \frac{225}{64}$$

$$e^2 = \frac{289}{64}$$

Since $e > 1$ for a hyperbola, taking the positive root gives

$$e = \frac{17}{8}$$

(b) The directrices of the hyperbola are $x = \pm\frac{a}{e}$, so

$$\frac{a}{e} = \frac{32}{17} \quad \Longrightarrow \quad a = \frac{32}{17} \times \frac{17}{8} = 4$$

Using $\frac{b}{a} = \frac{15}{8}$:

$$b = \frac{15}{8} \times 4 = \frac{15}{2} \quad \Longrightarrow \quad b^2 = \frac{225}{4}$$

Substituting $a^2 = 16$ and $b^2 = \frac{225}{4}$ into the equation of the hyperbola gives

$$\frac{x^2}{16} - \frac{4y^2}{225} = 1$$

Multiplying through by 3600, the lowest common multiple of 16 and 225:

$$225x^2 - 64y^2 = 3600$$

This is in the form $px^2 - qy^2 = r$ with $p = 225$, $q = 64$ and $r = 3600$.

The foci are at $(\pm ae, 0)$, where

$$ae = 4 \times \frac{17}{8} = \frac{17}{2}$$

so the foci of H are $\left(\frac{17}{2}, 0\right)$ and $\left(-\frac{17}{2}, 0\right)$.

4. A hyperbola has eccentricity e_1 and equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

An ellipse has eccentricity e_2 and equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b > 0$.
It is given that $e_1 e_2 = 1$.

(a) Show that $16a^2 = 25b^2$. **[4]**

It is given also that the foci of the ellipse lie at the points where the hyperbola meets the x-axis.

(b) Find the equation of the ellipse and the equations of its directrices. **[4]**

Solution

(a) For the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, the eccentricity satisfies $16 = 9\left(e_1^2 - 1\right)$, so

$$e_1^2 = 1 + \frac{16}{9} = \frac{25}{9}$$

$$e_1 = \frac{5}{3}$$

Since $e_1 e_2 = 1$, the eccentricity of the ellipse is

$$e_2 = \frac{3}{5}$$

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b > 0$, the eccentricity e_2 satisfies

$$b^2 = a^2\left(1 - e_2^2\right)$$

Substituting $e_2 = \frac{3}{5}$:

$$b^2 = a^2\left(1 - \frac{9}{25}\right)$$

$$b^2 = \frac{16a^2}{25}$$

$$16a^2 = 25b^2$$

as required.

(b) The hyperbola meets the x-axis where $y = 0$:

$$\frac{x^2}{9} = 1 \quad \Longrightarrow \quad x = \pm 3$$

so the foci of the ellipse are $(\pm 3, 0)$. The foci of the ellipse are at $(\pm a e_2, 0)$, so

$$\frac{3a}{5} = 3 \quad \Longrightarrow \quad a = 5$$

Using part (a):

$$25b^2 = 16 \times 25 \quad \Longrightarrow \quad b^2 = 16$$

so the equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

The directrices of the ellipse are $x = \pm\frac{a}{e_2}$, so their equations are

$$x = \pm\frac{5}{\frac{3}{5}} = \pm\frac{25}{3}$$

5. The ellipse E has equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

(a) Determine the exact value of the eccentricity of E. **[2]**

The point $P(5\cos\theta, 4\sin\theta)$ lies on E, where $0 < \theta < \frac{\pi}{2}$.
The line ℓ is the normal to E at the point P.

(b) Use calculus to show that an equation of ℓ is

$$5x\sin\theta - 4y\cos\theta = 9\sin\theta\cos\theta$$ **[4]**

The line ℓ meets the x-axis at the point A and the y-axis at the point B. The point M is the midpoint of AB.

(c) Find, in terms of θ, the coordinates of M. **[3]**

(d) Hence find a Cartesian equation of the locus of M as θ varies, and show that this locus is an ellipse which has the same eccentricity as E. **[3]**

Solution

(a) For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$, the eccentricity e satisfies

$$b^2 = a^2\left(1 - e^2\right)$$

Here $a = 5$ and $b = 4$, so

$$16 = 25\left(1 - e^2\right)$$

$$e^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$e = \frac{3}{5}$$

(b) Differentiating the parametric equations $x = 5\cos\theta$, $y = 4\sin\theta$ with respect to θ:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -5\sin\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 4\cos\theta$$

so the gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = -\frac{4\cos\theta}{5\sin\theta}$$

The normal is perpendicular to the tangent, so its gradient is $\frac{5\sin\theta}{4\cos\theta}$. The normal passes through $P(5\cos\theta, 4\sin\theta)$, so ℓ has equation

$$y - 4\sin\theta = \frac{5\sin\theta}{4\cos\theta}(x - 5\cos\theta)$$

Multiplying through by $4\cos\theta$:

$$4y\cos\theta - 16\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$$

$$5x\sin\theta - 4y\cos\theta = 9\sin\theta\cos\theta$$

as required.

(c) At A, $y = 0$, so

$$5x\sin\theta = 9\sin\theta\cos\theta$$

Since $0 < \theta < \frac{\pi}{2}$, $\sin\theta \neq 0$, so $x = \frac{9\cos\theta}{5}$ and A is $\left(\frac{9\cos\theta}{5}, 0\right)$.

At B, $x = 0$, so

$$-4y\cos\theta = 9\sin\theta\cos\theta$$

Since $\cos\theta \neq 0$, $y = -\frac{9\sin\theta}{4}$ and B is $\left(0, -\frac{9\sin\theta}{4}\right)$.

The midpoint of AB is

$$M\left(\frac{9\cos\theta}{10},\ -\frac{9\sin\theta}{8}\right)$$

(d) Using part (c), M has coordinates $x = \frac{9\cos\theta}{10}$ and $y = -\frac{9\sin\theta}{8}$, so

$$\cos\theta = \frac{10x}{9}, \qquad \sin\theta = -\frac{8y}{9}$$

Substituting into $\cos^2\theta + \sin^2\theta = 1$:

$$\frac{100x^2}{81} + \frac{64y^2}{81} = 1$$

$$100x^2 + 64y^2 = 81$$

This can be written as

$$\frac{x^2}{\left(\frac{9}{10}\right)^2} + \frac{y^2}{\left(\frac{9}{8}\right)^2} = 1$$

which is an ellipse centred at the origin with semi-axes $\frac{9}{10}$ along the x-axis and $\frac{9}{8}$ along the y-axis. Since $\frac{9}{8} > \frac{9}{10}$, the major axis lies along the y-axis, with semi-major axis $\frac{9}{8}$ and semi-minor axis $\frac{9}{10}$. Its eccentricity e_M therefore satisfies

$$\left(\frac{9}{10}\right)^2 = \left(\frac{9}{8}\right)^2\left(1 - e_M^2\right)$$

$$1 - e_M^2 = \frac{81}{100} \times \frac{64}{81} = \frac{16}{25}$$

$$e_M^2 = \frac{9}{25}$$

$$e_M = \frac{3}{5}$$

This is the same as the eccentricity of E found in part (a).

6. The hyperbola H has equation $\frac{x^2}{36} - \frac{y^2}{b^2} = 1$, where b is a positive constant. The eccentricity of H is $\frac{5}{3}$.

(a) Show that $b = 8$, and find the coordinates of the foci S and S' of H, where S has positive x-coordinate. **[3]**

The point $P(6\sec\theta, 8\tan\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on H.

(b) Use calculus to show that an equation of the normal to H at P is

$$3x\sin\theta + 4y = 50\tan\theta$$ **[4]**

The tangent to H at P has equation $4x - 3y\sin\theta = 24\cos\theta$.
The tangent and the normal to H at P meet the x-axis at the points Q and R respectively.

(c) Write down the coordinates of Q. **[1]**

(d) Find the coordinates of R, giving your answer in terms of θ. **[1]**

(e) Hence show that

$$\frac{SQ}{QS'} = \frac{SR}{RS'}$$ **[4]**

Solution

(a) For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the eccentricity e satisfies

$$b^2 = a^2\left(e^2 - 1\right)$$

and the foci are $(\pm ae, 0)$. Here $a = 6$ and $e = \frac{5}{3}$, so

$$\begin{aligned} b^2 &= 36\left(\frac{25}{9} - 1\right) \\ &= 36 \times \frac{16}{9} \\ &= 64 \end{aligned}$$

Since $b > 0$, $b = 8$, as required.

Also $ae = 6 \times \frac{5}{3} = 10$, so the foci are $S(10, 0)$ and $S'(-10, 0)$.

(b) With $x = 6\sec\theta$ and $y = 8\tan\theta$:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 6\sec\theta\tan\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 8\sec^2\theta$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} \\ &= \frac{8\sec^2\theta}{6\sec\theta\tan\theta} \\ &= \frac{4\sec\theta}{3\tan\theta} \\ &= \frac{4}{3\cos\theta} \times \frac{\cos\theta}{\sin\theta} \\ &= \frac{4}{3\sin\theta}\end{aligned}$$

The normal is perpendicular to the tangent, so its gradient is $-\frac{3\sin\theta}{4}$, and its equation is

$$y - 8\tan\theta = -\frac{3\sin\theta}{4}(x - 6\sec\theta)$$

Multiplying by 4, and using $\sin\theta\sec\theta = \tan\theta$:

$$\begin{aligned}4y - 32\tan\theta &= -3x\sin\theta + 18\sin\theta\sec\theta \\ 4y - 32\tan\theta &= -3x\sin\theta + 18\tan\theta \\ 3x\sin\theta + 4y &= 50\tan\theta\end{aligned}$$

as required.

(c) Setting $y = 0$ in $4x - 3y\sin\theta = 24\cos\theta$ gives $4x = 24\cos\theta$, so $x = 6\cos\theta$ and Q is $(6\cos\theta, 0)$.

(d) Setting $y = 0$ in the equation of the normal gives

$$3x\sin\theta = 50\tan\theta \quad \Longrightarrow \quad x = \frac{50\tan\theta}{3\sin\theta} = \frac{50}{3\cos\theta}$$

so R is $\left(\frac{50}{3\cos\theta}, 0\right)$.

(e) Since $0 < \theta < \frac{\pi}{2}$, $0 < \cos\theta < 1$. So $0 < 6\cos\theta < 6$, and Q lies between $S'(-10, 0)$ and $S(10, 0)$; also $\frac{50}{3\cos\theta} > \frac{50}{3} > 10$, so R lies beyond S. All four points are on the x-axis, so

$$SQ = 10 - 6\cos\theta, \qquad QS' = 6\cos\theta + 10$$

$$SR = \frac{50}{3\cos\theta} - 10, \qquad RS' = \frac{50}{3\cos\theta} + 10$$

Then

$$\frac{SQ}{QS'} = \frac{10 - 6\cos\theta}{10 + 6\cos\theta} = \frac{5 - 3\cos\theta}{5 + 3\cos\theta}$$

Multiplying the numerator and denominator by $3\cos\theta$:

$$\begin{aligned}\frac{SR}{RS'} &= \frac{\dfrac{50}{3\cos\theta} - 10}{\dfrac{50}{3\cos\theta} + 10} \\ &= \frac{50 - 30\cos\theta}{50 + 30\cos\theta} \\ &= \frac{5 - 3\cos\theta}{5 + 3\cos\theta}\end{aligned}$$

The two ratios are equal, so

$$\frac{SQ}{QS'} = \frac{SR}{RS'}$$

as required.

7. The hyperbola H has equation $4x^2 - 9y^2 = 36$.
The point $P(3\cosh t, 2\sinh t)$, where $t \in \mathbb{R}$, lies on H.

(a) Write down the equations of the two asymptotes of H. **[1]**

(b) Show that an equation of the tangent to H at P is

$$2x\cosh t - 3y\sinh t = 6$$ **[4]**

The tangent at P meets the asymptotes of H at the points Q and R, where Q lies on the asymptote with positive gradient. The point O is the origin.

(c) Find the coordinates of Q and R, giving your answers in terms of exponentials. **[3]**

(d) Show that $OQ \times OR$ is independent of t, and hence find the exact area of the triangle OQR. **[4]**

Solution

(a) Dividing by 36, the equation of H is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

This is a hyperbola with $a = 3$ and $b = 2$. The hyperbola has asymptotes $\frac{x}{a} = \pm\frac{y}{b}$, so the asymptotes of H are $\frac{x}{3} = \pm\frac{y}{2}$, that is

$$y = \frac{2}{3}x \qquad \text{and} \qquad y = -\frac{2}{3}x$$

(b) With $x = 3\cosh t$ and $y = 2\sinh t$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3\sinh t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2\cosh t$$

so the gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{2\cosh t}{3\sinh t}$$

The tangent at P is therefore

$$y - 2\sinh t = \frac{2\cosh t}{3\sinh t}(x - 3\cosh t)$$

Multiplying by $3\sinh t$ and rearranging:

$$3y\sinh t - 6\sinh^2 t = 2x\cosh t - 6\cosh^2 t$$

$$2x\cosh t - 3y\sinh t = 6\left(\cosh^2 t - \sinh^2 t\right)$$

Since $\cosh^2 t - \sinh^2 t = 1$, this gives

$$2x\cosh t - 3y\sinh t = 6$$

as required.

(c) Using $\cosh t = \frac{e^t+e^{-t}}{2}$ and $\sinh t = \frac{e^t-e^{-t}}{2}$ gives

$$\cosh t - \sinh t = e^{-t}, \qquad \cosh t + \sinh t = e^t$$

Q lies on $y = \frac{2}{3}x$. Substituting into the tangent:

$$\begin{aligned} 2x\cosh t - 2x\sinh t &= 6 \\ x(\cosh t - \sinh t) &= 3 \\ xe^{-t} = 3 \quad &\Longrightarrow \quad x = 3e^t \end{aligned}$$

Then $y = \frac{2}{3} \times 3e^t = 2e^t$, so Q is $(3e^t,\, 2e^t)$.

R lies on $y = -\frac{2}{3}x$. Substituting into the tangent:

$$\begin{aligned} 2x\cosh t + 2x\sinh t &= 6 \\ x(\cosh t + \sinh t) &= 3 \\ xe^{t} = 3 \quad &\Longrightarrow \quad x = 3e^{-t} \end{aligned}$$

Then $y = -\frac{2}{3} \times 3e^{-t} = -2e^{-t}$, so R is $(3e^{-t},\, -2e^{-t})$.

(d) Using part (c):

$$OQ = \sqrt{(3e^t)^2 + (2e^t)^2} = \sqrt{13e^{2t}} = \sqrt{13}\, e^t$$

$$OR = \sqrt{(3e^{-t})^2 + (-2e^{-t})^2} = \sqrt{13e^{-2t}} = \sqrt{13}\, e^{-t}$$

so

$$OQ \times OR = \sqrt{13}\, e^t \times \sqrt{13}\, e^{-t} = 13$$

which is independent of t.

Let α be the acute angle that the asymptote $y = \frac{2}{3}x$ makes with the positive x-axis, so $\tan\alpha = \frac{2}{3}$, giving $\sin\alpha = \frac{2}{\sqrt{13}}$ and $\cos\alpha = \frac{3}{\sqrt{13}}$. Q lies above the positive x-axis on $y = \frac{2}{3}x$, and R lies below it on $y = -\frac{2}{3}x$, the reflection of that asymptote in the x-axis, so $\angle QOR = 2\alpha$. The double angle formula gives

$$\sin\angle QOR = \sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \frac{2}{\sqrt{13}} \times \frac{3}{\sqrt{13}} = \frac{12}{13}$$

The area of triangle OQR is then

$$\frac{1}{2} \times OQ \times OR \times \sin\angle QOR = \frac{1}{2} \times 13 \times \frac{12}{13} = 6$$

8. The ellipse E has equation

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

The line ℓ passes through the point $(4, 0)$ and has gradient k, where k is a constant.
Given that E and ℓ meet at two distinct points P and Q,

(a) show that the x-coordinates of P and Q are solutions of the equation

$$(4k^2 + 1)x^2 - 32k^2x + 64k^2 - 8 = 0 \quad \textbf{[2]}$$

(b) show that $-\frac{1}{2} < k < \frac{1}{2}$. **[2]**

The point M is the midpoint of PQ.

(c) Determine, in simplest form in terms of k, the coordinates of M. **[3]**

(d) Hence show that, as k varies, M lies on the curve with equation

$$x^2 + py^2 = qx$$

where p and q are constants to be determined. **[5]**

Solution

(a) The line ℓ passes through $(4, 0)$ with gradient k, so its equation is

$$y = k(x - 4)$$

Multiplying the equation of E by 8 gives $x^2 + 4y^2 = 8$. Substituting $y = k(x - 4)$:

$$x^2 + 4k^2(x - 4)^2 = 8$$

$$x^2 + 4k^2\big(x^2 - 8x + 16\big) - 8 = 0$$

$$x^2 + 4k^2x^2 - 32k^2x + 64k^2 - 8 = 0$$

$$(4k^2 + 1)x^2 - 32k^2x + 64k^2 - 8 = 0$$

The x-coordinates of P and Q are the solutions of this equation, as required.

(b) Since E and ℓ meet at two distinct points, the quadratic in part (a) has two distinct real roots, so its discriminant is positive:

$$b^2 - 4ac = (-32k^2)^2 - 4(4k^2 + 1)(64k^2 - 8) > 0$$

$$1024k^4 - 4\big(256k^4 + 32k^2 - 8\big) > 0$$

$$1024k^4 - 1024k^4 - 128k^2 + 32 > 0$$

$$32 - 128k^2 > 0$$

$$k^2 < \frac{1}{4}$$

Hence $-\frac{1}{2} < k < \frac{1}{2}$, as required.

(c) Let the x-coordinates of P and Q be α and β, the roots of the quadratic in part (a). The sum of the roots is

$$\alpha + \beta = \frac{32k^2}{4k^2+1}$$

so the x-coordinate of M is

$$x = \frac{\alpha+\beta}{2} = \frac{16k^2}{4k^2+1}$$

The point M lies on ℓ, so its y-coordinate is

$$\begin{aligned} y = k(x-4) &= k\left(\frac{16k^2}{4k^2+1} - 4\right) \\ &= k\left(\frac{16k^2 - 16k^2 - 4}{4k^2+1}\right) \\ &= -\frac{4k}{4k^2+1} \end{aligned}$$

So M is the point

$$\left(\frac{16k^2}{4k^2+1},\ -\frac{4k}{4k^2+1}\right)$$

(d) Using part (c), with $x = \frac{16k^2}{4k^2+1}$ and $y = -\frac{4k}{4k^2+1}$:

$$x^2 + py^2 = \frac{256k^4}{(4k^2+1)^2} + \frac{16pk^2}{(4k^2+1)^2} = \frac{256k^4 + 16pk^2}{(4k^2+1)^2}$$

$$qx = \frac{16qk^2}{4k^2+1} = \frac{16qk^2(4k^2+1)}{(4k^2+1)^2} = \frac{64qk^4 + 16qk^2}{(4k^2+1)^2}$$

For M to lie on the curve as k varies, these must be equal for all values of k, so the numerators are identical:

$$256k^4 + 16pk^2 = 64qk^4 + 16qk^2$$

Equating coefficients of k^4 and of k^2:

$$256 = 64q \implies q = 4, \qquad 16p = 16q \implies p = 4$$

With these values both sides equal $\frac{256k^4+64k^2}{(4k^2+1)^2}$ for every k, so M lies on the curve

$$x^2 + 4y^2 = 4x$$

where $p = 4$ and $q = 4$.

9. The ellipse E has equation

$$\frac{x^2}{b^2} + \frac{y^2}{36} = 1$$

where b is a constant and $0 < b < 6$.
The eccentricity of E is e.

(a) Write down, in terms of e only,

(i) the coordinates of the foci of E, **[1]**

(ii) the equations of the directrices of E. **[1]**

Given that

- the point $P(x, y)$ lies on E, where $x > 0$ and $y > 0$
- the point S is the focus of E on the positive y-axis
- the line ℓ is the directrix of E which crosses the positive y-axis
- the point M lies on ℓ such that the line through P and M is parallel to the y-axis

(b) determine an expression for

(i) PS^2 in terms of e, x and y, **[1]**

(ii) PM^2 in terms of e and y. **[1]**

(c) Hence show that

$$b^2 = 36\left(1 - e^2\right)$$ **[2]**

Given that the distance between the two directrices of E is 18,

(d) determine the value of e and the exact value of b. **[2]**

Given also that $PS = 3$,

(e) determine the exact area of triangle PSM. **[4]**

Solution

(a) Since $0 < b < 6$, the major axis of E lies along the y-axis, with semi-major axis 6.

(i) The foci are

$$(0,\, 6e) \quad \text{and} \quad (0,\, -6e)$$

(ii) The directrices are

$$y = \frac{6}{e} \quad \text{and} \quad y = -\frac{6}{e}$$

(b) From part (a), $S = (0,\, 6e)$ and ℓ is the line $y = \frac{6}{e}$.

(i) With $P(x, y)$ and $S(0, 6e)$:

$$PS^2 = x^2 + (y - 6e)^2$$

(ii) Since PM is parallel to the y-axis and M lies on ℓ, $M = \left(x,\, \frac{6}{e}\right)$, so

$$PM^2 = \left(\frac{6}{e} - y\right)^2$$

(c) By the focus–directrix property of the ellipse, every point P on E satisfies $PS = e\,PM$, so $PS^2 = e^2PM^2$. Using part (b):

$$x^2 + (y - 6e)^2 = e^2\left(\frac{6}{e} - y\right)^2$$

$$x^2 + y^2 - 12ey + 36e^2 = (6 - ey)^2$$

$$x^2 + y^2 - 12ey + 36e^2 = 36 - 12ey + e^2y^2$$

$$x^2 + \left(1 - e^2\right)y^2 = 36\left(1 - e^2\right)$$

Dividing by $36\left(1 - e^2\right)$ gives

$$\frac{x^2}{36(1 - e^2)} + \frac{y^2}{36} = 1$$

This holds for every point P on E, so comparing with $\frac{x^2}{b^2} + \frac{y^2}{36} = 1$ gives

$$b^2 = 36\left(1 - e^2\right)$$

as required.

(d) The directrices are $y = \pm\frac{6}{e}$, so the distance between them is

$$\frac{12}{e} = 18 \quad \Longrightarrow \quad e = \frac{2}{3}$$

Using part (c):

$$b^2 = 36\left(1 - \frac{4}{9}\right) = 20$$

Since $b > 0$, $b = 2\sqrt{5}$.

(e) Now $S = (0,\, 4)$ and ℓ is the line $y = 9$. By the focus–directrix property, $PS = e\,PM$, so

$$PM = \frac{PS}{e} = \frac{3}{\frac{2}{3}} = \frac{9}{2}$$

Every point on E has $y \leq 6 < 9$, so $PM = 9 - y$, giving

$$9 - y = \frac{9}{2} \quad \Longrightarrow \quad y = \frac{9}{2}$$

Substituting into $\frac{x^2}{20} + \frac{y^2}{36} = 1$:

$$\frac{x^2}{20} = 1 - \frac{81}{144} = \frac{7}{16}$$

$$x^2 = \frac{35}{4}$$

Since $x > 0$, $x = \frac{\sqrt{35}}{2}$, so $P = \left(\frac{\sqrt{35}}{2},\, \frac{9}{2}\right)$ and $M = \left(\frac{\sqrt{35}}{2},\, 9\right)$.

Take PM as the base of triangle PSM. It lies on the line $x = \frac{\sqrt{35}}{2}$, so the perpendicular height from $S(0, 4)$ is $\frac{\sqrt{35}}{2}$. The area is

$$\frac{1}{2} \times \frac{9}{2} \times \frac{\sqrt{35}}{2} = \frac{9\sqrt{35}}{8}$$

10. The hyperbola H has equation

$$9x^2 - 16y^2 = 144$$

The line ℓ with equation $y = mx + c$, where m and c are constants, is a tangent to H.

(a) Show that $c^2 = 16m^2 - 9$. **[4]**

The point P has coordinates $(4, 1)$. Exactly two tangents to H pass through P.

(b) Hence find the equations of both of these tangents, explaining why only one of them can be found using the result of part (a). **[6]**

(c) Find the coordinates of the point at which each of these tangents touches H. **[3]**

Solution

(a) Substituting $y = mx + c$ into the equation of H:

$$\begin{aligned} 9x^2 - 16(mx + c)^2 &= 144 \\ 9x^2 - 16\big(m^2x^2 + 2mcx + c^2\big) &= 144 \\ \big(9 - 16m^2\big)x^2 - 32mcx - \big(16c^2 + 144\big) &= 0 \end{aligned}$$

Since ℓ is a tangent to H, this quadratic in x has a repeated root, so its discriminant is zero:

$$\begin{aligned} (-32mc)^2 - 4\big(9 - 16m^2\big)\big(-16c^2 - 144\big) &= 0 \\ 1024m^2c^2 + 64\big(9 - 16m^2\big)\big(c^2 + 9\big) &= 0 \\ 16m^2c^2 + \big(9 - 16m^2\big)\big(c^2 + 9\big) &= 0 \\ 16m^2c^2 + 9c^2 + 81 - 16m^2c^2 - 144m^2 &= 0 \\ 9c^2 &= 144m^2 - 81 \end{aligned}$$

Dividing by 9 gives

$$c^2 = 16m^2 - 9$$

as required.

(b) A tangent $y = mx + c$ through $P(4, 1)$ satisfies

$$1 = 4m + c \quad \Longrightarrow \quad c = 1 - 4m$$

Substituting into the result of part (a):

$$\begin{aligned} (1 - 4m)^2 &= 16m^2 - 9 \\ 1 - 8m + 16m^2 &= 16m^2 - 9 \\ 8m &= 10 \\ m &= \frac{5}{4} \end{aligned}$$

Then $c = 1 - 4 \times \frac{5}{4} = -4$, so one tangent is

$$y = \frac{5}{4}x - 4$$

The vertical line through P is $x = 4$. Substituting $x = 4$ into the equation of H:

$$9 \times 16 - 16y^2 = 144 \quad \Longrightarrow \quad 16y^2 = 0$$

This has the repeated root $y = 0$, so $x = 4$ meets H at one point only and is a tangent to H. The two tangents are therefore

$$y = \frac{5}{4}x - 4 \qquad \text{and} \qquad x = 4$$

The line $x = 4$ is vertical, so it has no gradient and cannot be written in the form $y = mx + c$. The result of part (a) applies only to lines of that form, which is why the m^2 terms cancel and the equation for m has only one solution: it finds only $y = \frac{5}{4}x - 4$.

(c) For $y = \frac{5}{4}x - 4$, substitute $m = \frac{5}{4}$ and $c = -4$ into the quadratic from part (a). Here $9 - 16m^2 = 9 - 25 = -16$, $32mc = 32 \times \frac{5}{4} \times (-4) = -160$ and $16c^2 + 144 = 400$, so

$$-16x^2 + 160x - 400 = 0$$

$$x^2 - 10x + 25 = 0$$

$$(x - 5)^2 = 0$$

So $x = 5$ and $y = \frac{5}{4} \times 5 - 4 = \frac{9}{4}$. This tangent touches H at $\left(5, \frac{9}{4}\right)$.

From part (b), $x = 4$ meets H only where $y = 0$, so it touches H at $(4, 0)$.

11. The hyperbola H has equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

The line ℓ is the tangent to H at the point $P(3\cosh\theta, 4\sinh\theta)$, where $\theta \neq 0$.
The line ℓ meets the x-axis at the point A and the y-axis at the point C. The midpoint of AC is the point M.

(a) Show that, as θ varies, a Cartesian equation for the locus of M is

$$y^2 = \frac{16x^2}{9 - 4x^2}, \qquad p < x < q$$

where p and q are constants to be determined. **[10]**

(b) Show that every point (x, y) on the locus of M satisfies $|y| > \frac{4}{3}x$, and interpret this result geometrically in relation to the asymptotes of H. **[3]**

Solution

(a) At P, $x = 3\cosh\theta$ and $y = 4\sinh\theta$, so

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sinh\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 4\cosh\theta$$

Since $\theta \neq 0$, $\sinh\theta \neq 0$, and the gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{4\cosh\theta}{3\sinh\theta}$$

The equation of ℓ is

$$\begin{aligned} y - 4\sinh\theta &= \frac{4\cosh\theta}{3\sinh\theta}(x - 3\cosh\theta) \\ 3y\sinh\theta - 12\sinh^2\theta &= 4x\cosh\theta - 12\cosh^2\theta \\ 4x\cosh\theta - 3y\sinh\theta &= 12\left(\cosh^2\theta - \sinh^2\theta\right) \\ 4x\cosh\theta - 3y\sinh\theta &= 12 \end{aligned}$$

using $\cosh^2\theta - \sinh^2\theta = 1$.

Setting $y = 0$ gives $x = \frac{3}{\cosh\theta}$, and setting $x = 0$ gives $y = -\frac{4}{\sinh\theta}$, so

$$A\left(\frac{3}{\cosh\theta}, 0\right), \qquad C\left(0, -\frac{4}{\sinh\theta}\right)$$

The midpoint of AC is

$$M\left(\frac{3}{2\cosh\theta}, -\frac{2}{\sinh\theta}\right)$$

At M, the coordinates give

$$x = \frac{3}{2\cosh\theta} \implies \cosh\theta = \frac{3}{2x}, \qquad y = -\frac{2}{\sinh\theta} \implies \sinh\theta = -\frac{2}{y}$$

Substituting into $\cosh^2\theta - \sinh^2\theta = 1$:

$$\frac{9}{4x^2} - \frac{4}{y^2} = 1$$

$$\frac{4}{y^2} = \frac{9}{4x^2} - 1 = \frac{9 - 4x^2}{4x^2}$$

$$y^2 = \frac{16x^2}{9 - 4x^2}$$

Since $\theta \neq 0$, $\cosh\theta$ takes every value greater than 1, so $x = \frac{3}{2\cosh\theta}$ takes every value with

$$0 < x < \frac{3}{2}$$

so $p = 0$ and $q = \frac{3}{2}$.

(b) For $0 < x < \frac{3}{2}$, $0 < 4x^2 < 9$, so $0 < 9 - 4x^2 < 9$ and $\frac{1}{9-4x^2} > \frac{1}{9}$. As $16x^2 > 0$, it follows that

$$y^2 = \frac{16x^2}{9 - 4x^2} > \frac{16x^2}{9}$$

Taking square roots, with $x > 0$, gives $|y| > \frac{4}{3}x$, as required.

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has asymptotes $\frac{x}{a} = \pm\frac{y}{b}$, so with $a = 3$ and $b = 4$, the asymptotes of H are $y = \pm\frac{4}{3}x$. The right-hand branch of H lies between these asymptotes, in the region $|y| < \frac{4}{3}x$. The locus of M lies in $x > 0$ above the asymptote $y = \frac{4}{3}x$ or below the asymptote $y = -\frac{4}{3}x$: that is, between the asymptotes and the y-axis, on the opposite side of each asymptote from H, and it never meets either asymptote.

12. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive constants. The point $P(a\cos\theta,\, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on E.

(a) Show that an equation of the tangent to E at P is

$$bx\cos\theta + ay\sin\theta = ab \qquad \textbf{[4]}$$

The tangent to E at P meets the x-axis at the point A and the y-axis at the point B.

(b) Show that, as θ varies, the minimum length of AB is $a + b$, and find, in terms of a and b, the coordinates of P when AB has this minimum length. **[8]**

Solution

(a) Differentiating $x = a\cos\theta$ and $y = b\sin\theta$ with respect to θ:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = b\cos\theta$$

The gradient of the tangent at P is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = -\frac{b\cos\theta}{a\sin\theta}$$

The tangent passes through $P(a\cos\theta,\, b\sin\theta)$, so its equation is

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

Multiplying through by $a\sin\theta$, which is non-zero since $0 < \theta < \frac{\pi}{2}$:

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab\left(\sin^2\theta + \cos^2\theta\right)$$

$$bx\cos\theta + ay\sin\theta = ab$$

as required.

(b) At A, $y = 0$, so $bx\cos\theta = ab$, giving $x = \frac{a}{\cos\theta} = a\sec\theta$. At B, $x = 0$, so $ay\sin\theta = ab$, giving $y = \frac{b}{\sin\theta} = b\operatorname{cosec}\theta$. So A is $(a\sec\theta,\, 0)$ and B is $(0,\, b\operatorname{cosec}\theta)$, and by Pythagoras:

$$AB^2 = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta$$

$$= a^2\left(1 + \tan^2\theta\right) + b^2\left(1 + \cot^2\theta\right)$$

$$= a^2 + b^2 + a^2\tan^2\theta + b^2\cot^2\theta$$

Since $\tan\theta\cot\theta = 1$, the cross term of $(a\tan\theta - b\cot\theta)^2$ is $-2ab\tan\theta\cot\theta = -2ab$. Adding and subtracting $2ab$ to complete the square:

$$AB^2 = a^2 + 2ab + b^2 + \left(a^2\tan^2\theta - 2ab + b^2\cot^2\theta\right)$$

$$= (a + b)^2 + (a\tan\theta - b\cot\theta)^2$$

The squared term is never negative, so $AB^2 \geq (a+b)^2$, with equality exactly when $a\tan\theta = b\cot\theta = \frac{b}{\tan\theta}$, that is

$$\tan^2\theta = \frac{b}{a}$$

Since a and b are positive, this has a solution with $0 < \theta < \frac{\pi}{2}$, namely $\tan\theta = \sqrt{\frac{b}{a}}$. Hence the least value of AB^2 is $(a+b)^2$, and since $AB > 0$ and $a+b > 0$, the minimum length of AB is $a+b$.

At this value of θ:

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+\frac{b}{a}} = \frac{a}{a+b}$$

$$\sin^2\theta = 1 - \cos^2\theta = \frac{b}{a+b}$$

Since $0 < \theta < \frac{\pi}{2}$, both $\cos\theta$ and $\sin\theta$ are positive, so

$$\cos\theta = \sqrt{\frac{a}{a+b}}, \qquad \sin\theta = \sqrt{\frac{b}{a+b}}$$

Therefore, when AB has its minimum length, P is the point

$$\left(a\sqrt{\frac{a}{a+b}},\ b\sqrt{\frac{b}{a+b}}\right) = \left(\frac{a\sqrt{a}}{\sqrt{a+b}},\ \frac{b\sqrt{b}}{\sqrt{a+b}}\right)$$