OPTIONAL PURE / CONIC SECTIONS

Ellipses and Hyperbolas

Questions by topic 12 questions · 126 marks

EXAM BOARDS Edexcel FP1 9FM0/3A

Questions

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1. The ellipse E has equation $9x^2 + 5y^2 = 45$.

(a) Find the eccentricity of E and the coordinates of its foci. **[4]**

(b) Find the equations of the directrices of E. **[2]**

2. The hyperbola H has parametric equations

$$x = 3\sec\theta, \qquad y = 2\tan\theta$$

(a) Find a Cartesian equation of H and hence find the coordinates of the foci of H. **[4]**

(b) Find the equations of the directrices of H, giving your answers in exact simplified form. **[2]**

3. The hyperbola H, with centre at the origin and foci on the x-axis, has

- asymptotes with equations $y = \pm\frac{15}{8}x$
- directrices with equations $x = \pm\frac{32}{17}$
- eccentricity e.

(a) Find the value of e. **[3]**

(b) Find an equation for H, giving your answer in the form $px^2 - qy^2 = r$, where p, q and r are integers, and state the coordinates of the foci of H. **[4]**

4. A hyperbola has eccentricity e_1 and equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

An ellipse has eccentricity e_2 and equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b > 0$.
It is given that $e_1 e_2 = 1$.

(a) Show that $16a^2 = 25b^2$. **[4]**

It is given also that the foci of the ellipse lie at the points where the hyperbola meets the x-axis.

(b) Find the equation of the ellipse and the equations of its directrices. **[4]**

5. The ellipse E has equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

(a) Determine the exact value of the eccentricity of E. **[2]**

The point $P(5\cos\theta, 4\sin\theta)$ lies on E, where $0 < \theta < \frac{\pi}{2}$.
The line ℓ is the normal to E at the point P.

(b) Use calculus to show that an equation of ℓ is

$$5x\sin\theta - 4y\cos\theta = 9\sin\theta\cos\theta$$ **[4]**

The line ℓ meets the x-axis at the point A and the y-axis at the point B. The point M is the midpoint of AB.

(c) Find, in terms of θ, the coordinates of M. **[3]**

(d) Hence find a Cartesian equation of the locus of M as θ varies, and show that this locus is an ellipse which has the same eccentricity as E. **[3]**

6. The hyperbola H has equation $\frac{x^2}{36} - \frac{y^2}{b^2} = 1$, where b is a positive constant. The eccentricity of H is $\frac{5}{3}$.

(a) Show that $b = 8$, and find the coordinates of the foci S and S' of H, where S has positive x-coordinate. **[3]**

The point $P(6\sec\theta, 8\tan\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on H.

(b) Use calculus to show that an equation of the normal to H at P is

$$3x\sin\theta + 4y = 50\tan\theta$$ **[4]**

The tangent to H at P has equation $4x - 3y\sin\theta = 24\cos\theta$.
The tangent and the normal to H at P meet the x-axis at the points Q and R respectively.

(c) Write down the coordinates of Q. **[1]**

(d) Find the coordinates of R, giving your answer in terms of θ. **[1]**

(e) Hence show that

$$\frac{SQ}{QS'} = \frac{SR}{RS'}$$ **[4]**

7. The hyperbola H has equation $4x^2 - 9y^2 = 36$.
The point $P(3\cosh t, 2\sinh t)$, where $t \in \mathbb{R}$, lies on H.

(a) Write down the equations of the two asymptotes of H. **[1]**

(b) Show that an equation of the tangent to H at P is

$$2x\cosh t - 3y\sinh t = 6$$ **[4]**

The tangent at P meets the asymptotes of H at the points Q and R, where Q lies on the asymptote with positive gradient. The point O is the origin.

(c) Find the coordinates of Q and R, giving your answers in terms of exponentials. **[3]**

(d) Show that $OQ \times OR$ is independent of t, and hence find the exact area of the triangle OQR. **[4]**

8. The ellipse E has equation

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

The line ℓ passes through the point $(4, 0)$ and has gradient k, where k is a constant.
Given that E and ℓ meet at two distinct points P and Q,

(a) show that the x-coordinates of P and Q are solutions of the equation

$$(4k^2 + 1)x^2 - 32k^2x + 64k^2 - 8 = 0$$ **[2]**

(b) show that $-\frac{1}{2} < k < \frac{1}{2}$. **[2]**

The point M is the midpoint of PQ.

(c) Determine, in simplest form in terms of k, the coordinates of M. **[3]**

(d) Hence show that, as k varies, M lies on the curve with equation

$$x^2 + py^2 = qx$$

where p and q are constants to be determined. **[5]**

9. The ellipse E has equation

$$\frac{x^2}{b^2} + \frac{y^2}{36} = 1$$

where b is a constant and $0 < b < 6$.
The eccentricity of E is e.

(a) Write down, in terms of e only,

(i) the coordinates of the foci of E, **[1]**

(ii) the equations of the directrices of E. **[1]**

Given that

- the point $P(x, y)$ lies on E, where $x > 0$ and $y > 0$
- the point S is the focus of E on the positive y-axis
- the line ℓ is the directrix of E which crosses the positive y-axis
- the point M lies on ℓ such that the line through P and M is parallel to the y-axis

(b) determine an expression for

(i) PS^2 in terms of e, x and y, **[1]**

(ii) PM^2 in terms of e and y. **[1]**

(c) Hence show that

$$b^2 = 36\left(1 - e^2\right)$$ **[2]**

Given that the distance between the two directrices of E is 18,

(d) determine the value of e and the exact value of b. **[2]**

Given also that $PS = 3$,

(e) determine the exact area of triangle PSM. **[4]**

10. The hyperbola H has equation

$$9x^2 - 16y^2 = 144$$

The line ℓ with equation $y = mx + c$, where m and c are constants, is a tangent to H.

(a) Show that $c^2 = 16m^2 - 9$. **[4]**

The point P has coordinates $(4, 1)$. Exactly two tangents to H pass through P.

(b) Hence find the equations of both of these tangents, explaining why only one of them can be found using the result of part (a). **[6]**

(c) Find the coordinates of the point at which each of these tangents touches H. **[3]**

11. The hyperbola H has equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

The line ℓ is the tangent to H at the point $P(3\cosh\theta, 4\sinh\theta)$, where $\theta \neq 0$.
The line ℓ meets the x-axis at the point A and the y-axis at the point C. The midpoint of AC is the point M.

(a) Show that, as θ varies, a Cartesian equation for the locus of M is

$$y^2 = \frac{16x^2}{9 - 4x^2}, \qquad p < x < q$$

where p and q are constants to be determined. **[10]**

(b) Show that every point (x, y) on the locus of M satisfies $|y| > \frac{4}{3}x$, and interpret this result geometrically in relation to the asymptotes of H. **[3]**

12. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive constants. The point $P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on E.

(a) Show that an equation of the tangent to E at P is

$$bx\cos\theta + ay\sin\theta = ab$$ **[4]**

The tangent to E at P meets the x-axis at the point A and the y-axis at the point B.

(b) Show that, as θ varies, the minimum length of AB is $a + b$, and find, in terms of a and b, the coordinates of P when AB has this minimum length. **[8]**