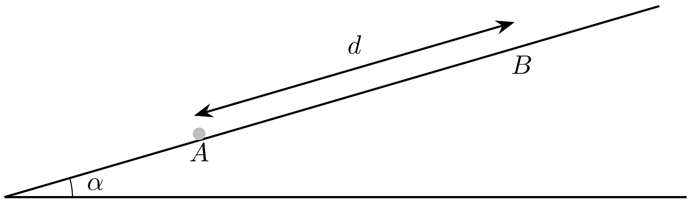
Questions



1. A particle of mass m is at a point A on a rough plane.

The plane is inclined at an angle α to the horizontal, where $\sin\alpha = \frac{7}{25}$.

The coefficient of friction between the particle and the plane is $\frac{1}{3}$.

The points A and B lie on a line of greatest slope of the plane, with B above A, and $AB = d$, as shown in the diagram.

(a) Show that the work done against friction as the particle moves from A to B is

$$\frac{8}{25}mgd$$

[3]

(b) The particle is projected up the line of greatest slope from A towards B with speed $\sqrt{gd}$. Use the work-energy principle to determine whether the particle reaches B. If it does not, find how far it travels from A before coming instantaneously to rest. **[4]**

Solution

(a) Since $\sin\alpha = \frac{7}{25}$, we have

$$\cos\alpha = \frac{24}{25}$$

The normal reaction is

$$R = mg\cos\alpha = mg\cdot\frac{24}{25} = \frac{24}{25}mg$$

So the friction force is

$$F = \mu R = \frac{1}{3}\cdot\frac{24}{25}mg = \frac{8}{25}mg$$

Hence the work done against friction in moving a distance d up the plane is

$$W = Fd = \frac{8}{25}mg\times d = \frac{8}{25}mgd$$

So the work done against friction is $\frac{8}{25}mgd$.

(b) Let the gravitational potential energy at A be zero.

First check whether the particle can reach B. Using the work-energy principle,

$$\text{KE}_A + \text{GPE}_A + \sum W = \text{KE}_B + \text{GPE}_B$$

If the particle reaches B with speed v, then

$$\text{KE}_A = \frac{1}{2}m(\sqrt{gd})^2 = \frac{1}{2}mgd$$

$$\text{GPE}_B = mg(d\sin\alpha) = mgd\cdot\frac{7}{25} = \frac{7}{25}mgd$$

and the work done by friction is negative:

$$\sum W = -\frac{8}{25}mgd$$

So

$$\begin{aligned}\frac{1}{2}mgd + 0 - \frac{8}{25}mgd &= \frac{1}{2}mv^2 + \frac{7}{25}mgd \\ \frac{1}{2}mv^2 &= \frac{1}{2}mgd - \frac{8}{25}mgd - \frac{7}{25}mgd \\ &= \left(\frac{1}{2} - \frac{15}{25}\right)mgd \\ &= -\frac{1}{10}mgd\end{aligned}$$

This is impossible, since kinetic energy cannot be negative.

Therefore the particle does not reach B.

Now let the distance travelled from A before coming instantaneously to rest be s.

Again using

$$\text{KE}_A + \text{GPE}_A + \sum W = \text{KE}_C + \text{GPE}_C$$

where C is the point where it stops:

$$\text{KE}_A = \frac{1}{2}mgd, \qquad \text{KE}_C = 0$$

$$\text{GPE}_C = mg(s\sin\alpha) = mgs \cdot \frac{7}{25} = \frac{7}{25}mgs$$

$$\sum W = -\frac{8}{25}mgs$$

Hence

$$\begin{aligned}\frac{1}{2}mgd + 0 - \frac{8}{25}mgs &= 0 + \frac{7}{25}mgs \\ \frac{1}{2}mgd &= \frac{15}{25}mgs \\ \frac{1}{2}mgd &= \frac{3}{5}mgs\end{aligned}$$

So

$$s = \frac{\frac{1}{2}}{\frac{3}{5}}d = \frac{5d}{6}$$

Hence the particle does not reach B, and it travels $\frac{5d}{6}$ from A before coming to rest.

2. A plane is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{5}{12}$.

A point B lies on the plane such that $AB = 39$ m.

A particle P is projected with speed 21 m s^{-1} from A, up a line of greatest slope of the plane.

In an initial model, the plane is modelled as being smooth and air resistance is modelled as being negligible.

Using this model and the principle of conservation of mechanical energy,

(a) find the speed of P when it reaches B. **[4]**

In a refined model, the plane is now modelled as being rough, with the coefficient of friction between P and the plane being $\frac{1}{3}$.

Air resistance is still modelled as being negligible.

Using this refined model and the work-energy principle,

(b) determine whether P reaches B. If it does not, find the distance travelled by P up the plane from A before it comes instantaneously to rest. **[8]**

Solution

(a) Since $\tan\alpha = \frac{5}{12}$, we use the 5-12-13 triangle, so

$$\sin\alpha = \frac{5}{13}$$

The vertical height gained from A to B is

$$39\sin\alpha = 39\cdot\frac{5}{13} = 15 \text{ m}$$

Taking gravitational potential energy to be 0 at A, and using conservation of mechanical energy,

$$\text{KE at } A + \text{GPE at } A = \text{KE at } B + \text{GPE at } B$$

so

$$\frac{1}{2}m(21)^2 + 0 = \frac{1}{2}mv^2 + mg(15)$$

With $g = 9.8$,

$$\begin{aligned}\frac{1}{2}m(441) &= \frac{1}{2}mv^2 + 15mg\\ 220.5m &= \frac{1}{2}mv^2 + 147m\\ \frac{1}{2}mv^2 &= 73.5m\\ v^2 &= 147\end{aligned}$$

Hence

$$v = \sqrt{147} = 7\sqrt{3} \text{ m s}^{-1}$$

So the speed of P at B is $7\sqrt{3}$ m s^{-1} ≈ 12.1 m s^{-1}

(b) Again, from $\tan\alpha = \frac{5}{12}$,

$$\sin\alpha = \frac{5}{13}, \qquad \cos\alpha = \frac{12}{13}$$

The normal reaction is

$$R = mg\cos\alpha = mg \cdot \frac{12}{13} = \frac{12mg}{13}$$

Since the coefficient of friction is $\frac{1}{3}$, the frictional force has magnitude

$$F = \mu R = \frac{1}{3} \cdot \frac{12mg}{13} = \frac{4mg}{13}$$

To test whether P reaches B, use the work-energy principle in the form

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

Between A and B, the only non-conservative work is the work done by friction, which is

$$-Fs = -\frac{4mg}{13} \cdot 39 = -12mg$$

Also, the gain in height is still 15 m, so the gain in GPE is $15mg$.
Therefore

$$\frac{1}{2}m(21)^2 + 0 - 12mg = \frac{1}{2}mv^2 + 15mg$$

Substituting $g = 9.8$,

$$\begin{aligned} 220.5m - 117.6m &= \frac{1}{2}mv^2 + 147m \\ 102.9m &= \frac{1}{2}mv^2 + 147m \\ \frac{1}{2}mv^2 &= -44.1m \end{aligned}$$

This is impossible, since kinetic energy cannot be negative. Therefore P does not reach B.

Now let the distance travelled up the plane before coming to rest be s m.

At the turning point, the speed is 0, so using

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

gives

$$\frac{1}{2}m(21)^2 + 0 - \frac{4mg}{13}s = 0 + mg(s\sin\alpha)$$

Since $\sin\alpha = \frac{5}{13}$,

$$220.5m - \frac{4mg}{13}s = \frac{5mg}{13}s$$

So

$$\begin{aligned} 220.5m &= \frac{9mg}{13}s \\ 220.5 &= \frac{9 \times 9.8}{13}s \\ s &= \frac{220.5 \times 13}{9 \times 9.8} \\ s &= 32.5 \end{aligned}$$

Since $32.5 < 39$, the particle does not reach B.

Therefore, P does not reach B, and it travels 32.5 m up the plane before coming instantaneously to rest.

3. A sledge of mass 40 kg is pulled up a rough slope inclined at 10° to the horizontal.

The coefficient of friction between the sledge and the slope is 0.25.

The sledge is at rest when a rope starts to pull it.

The tension in the rope is 220 N and the rope makes an angle of 15° above the slope.

After the sledge has moved 5 metres up the slope, the rope breaks.

(a) Use an energy method to find the maximum speed of the sledge. **[4]**

(b) Use an energy method to find the total distance the sledge moves up the slope before coming to rest. **[2]**

(c) A student claims that in reality the sledge is unlikely to move more than 6.8 metres up the slope in total. Comment on the validity of this claim. **[2]**

Solution

(a) While the rope is pulling, first find the normal reaction.

Resolving perpendicular to the slope,

$$R + 220\sin 15^\circ = 40g\cos 10^\circ$$

so

$$R = 40g\cos 10^\circ - 220\sin 15^\circ$$

With $g = 9.8$,

$$R = 40(9.8)\cos 10^\circ - 220\sin 15^\circ \approx 329.1 \text{ N}$$

Hence the friction force is

$$F = \mu R = 0.25R$$

Now apply the Work-Energy principle over the first 5 m:

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

The sledge starts from rest, so $\text{KE}_0 = 0$. Take the initial position as zero GPE, so $\text{GPE}_0 = 0$.

The work done by the tension is

$$W_T = 220 \times 5\cos 15^\circ$$

The work done against friction is

$$W_f = -0.25R \times 5$$

The increase in GPE is

$$40g \times 5\sin 10^\circ$$

So

$$0 + 0 + 220(5)\cos 15^\circ - 0.25R(5) = \frac{1}{2}(40)v^2 + 40g(5)\sin 10^\circ$$

Substituting $R = 40g\cos 10^\circ - 220\sin 15^\circ$,

$$220(5)\cos 15^\circ - 0.25\big(40g\cos 10^\circ - 220\sin 15^\circ\big)(5) = \frac{1}{2}(40)v^2 + 40g(5)\sin 10^\circ$$

Rearranging,

$$\begin{aligned}\frac{1}{2}(40)v^2 &= 220(5)\cos 15^\circ - 40g(5)\sin 10^\circ - 0.25\big(40g\cos 10^\circ - 220\sin 15^\circ\big)(5)\\ &\approx 310.8\end{aligned}$$

Hence

$$\begin{aligned}20v^2 &= 310.8\\ v^2 &= 15.54\\ v &\approx 3.94 \text{ m s}^{-1}\end{aligned}$$

While the rope is intact, all forces are constant, so the acceleration is constant. The resultant force is up the slope, so the speed increases until the rope breaks. Therefore this is the maximum speed.

The maximum speed is 3.94 m s^{-1}.

(b) After the rope breaks, the sledge still moves up the slope, then comes to rest.

Now the normal reaction is simply

$$R = 40g\cos 10^\circ$$

so the friction force is

$$0.25 \times 40g\cos 10^\circ$$

Let the further distance travelled be s m.

Using the Work-Energy principle from the instant the rope breaks until the sledge comes to rest,

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

From part (a), the kinetic energy at the break is 310.8 J. Also $\text{KE}_1 = 0$.

Therefore

$$310.8 - 0.25(40g\cos 10^\circ)s = 40g(\sin 10^\circ)s$$

So

$$310.8 = 40g(\sin 10^\circ + 0.25\cos 10^\circ)s$$

Hence

$$s = \frac{310.8}{40(9.8)(\sin 10^\circ + 0.25\cos 10^\circ)} \approx 1.89$$

So the total distance moved up the slope is

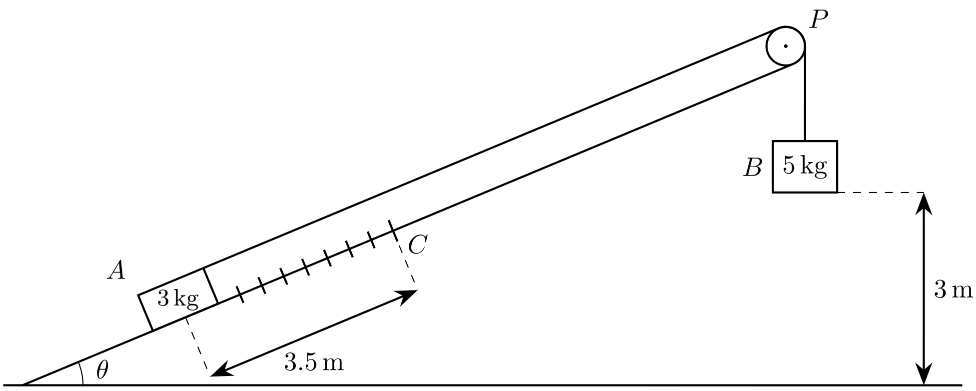
$$5 + 1.89 = 6.89 \text{ m}$$

The total distance is 6.89 m.

(c) The model gives a total distance of 6.89 m, so according to the model the sledge would move slightly more than 6.8 m.

So the claim is not proved by the model. However, 6.8 m is only 0.09 m less than the model value, and in reality quantities such as friction and tension may vary, so a slightly smaller distance is quite plausible.

Therefore the claim is reasonable, but it is not something we can conclude from the idealised model alone.



4. Two blocks, A and B, of masses 3 kg and 5 kg respectively are attached to the ends of a light inextensible string.

Initially A is held on a fixed plane. The plane is inclined to horizontal ground at an angle θ, where $\tan\theta = \frac{5}{12}$.

The string passes over a small smooth light pulley P that is fixed at the top of the plane. The part of the string from A to P is parallel to a line of greatest slope of the plane.

The section of the plane from the initial position of A to a point C, 3.5 m up the plane, is rough. The section from C to P is smooth.

Block B hangs freely below P at a distance of 3 m above the ground, as shown in the diagram. The coefficient of friction between A and the rough part of the plane is μ. Block A is released from rest with the string taut.

By modelling the blocks as particles,

(a) find the potential energy lost by the whole system as a result of B falling 3 m. **[3]**

Given that the speed of B at the instant it hits the ground is $4.8\,\text{m s}^{-1}$ and ignoring air resistance,

(b) use the work-energy principle to find the value of μ. **[6]**

After B hits the ground, A continues to move up the plane, passes C and does not reach the pulley in the subsequent motion.

Block A comes to instantaneous rest after moving a total distance of $(3.5 + d)$ m from its point of release.

Ignoring air resistance,

(c) use the work-energy principle to find the value of d. **[4]**

Solution

(a) From $\tan\theta = \frac{5}{12}$, using a 5-12-13 triangle,

$$\sin\theta = \frac{5}{13}$$

If B falls 3 m, then A moves 3 m up the plane, since the string is inextensible.
So the vertical rise of A is

$$3\sin\theta = 3 \times \frac{5}{13} = \frac{15}{13}\text{ m}$$

Potential energy lost by B:

$$5g \times 3 = 15g$$

Potential energy gained by A:

$$3g \times \frac{15}{13} = \frac{45g}{13}$$

Therefore the net potential energy lost by the whole system is

$$\begin{aligned} 15g - \frac{45g}{13} &= \frac{195g - 45g}{13} \\ &= \frac{150g}{13} \end{aligned}$$

With $g = 9.8$,

$$\frac{150g}{13} = \frac{1470}{13}\,\text{J} \approx 113\,\text{J}$$

So the potential energy lost is

$$\frac{1470}{13}\,\text{J} \approx 113\,\text{J}$$

(b) Consider the motion from release until B hits the ground.

For the whole two-block system, tension is internal, so the only non-conservative work is the work done by friction on A.

The gain in kinetic energy of the system is

$$\begin{aligned} \frac{1}{2}(3+5)(4.8)^2 &= \frac{1}{2}(8)(4.8)^2 \\ &= 92.16\,\text{J} \end{aligned}$$

For block A, perpendicular to the plane,

$$R = 3g\cos\theta = 3g \times \frac{12}{13} = \frac{36g}{13}$$

Hence the frictional force is

$$\mu R = \mu \cdot \frac{36g}{13}$$

Since A moves 3 m along the rough plane, the work done against friction is

$$\mu \cdot \frac{36g}{13} \cdot 3 = \frac{108g\mu}{13}$$

Now use the work-energy principle

$$KE_0 + GPE_0 + \sum W = KE_1 + GPE_1$$

Here,

$$KE_0 = 0, \qquad KE_1 = 92.16, \qquad \sum W = -\frac{108g\mu}{13}$$

So

$$0 + GPE_0 - \frac{108g\mu}{13} = 92.16 + GPE_1$$

Rearranging,

$$GPE_0 - GPE_1 = 92.16 + \frac{108g\mu}{13}$$

From part (a),

$$GPE_0 - GPE_1 = \frac{150g}{13}$$

Hence

$$\frac{150g}{13} = 92.16 + \frac{108g\mu}{13}$$

Substituting $g = 9.8 = \frac{49}{5}$,

$$\frac{1470}{13} = \frac{2304}{25} + \frac{5292\mu}{65}$$

So

$$\begin{aligned}\frac{5292\mu}{65} &= \frac{1470}{13} - \frac{2304}{25} \\ &= \frac{36750 - 29952}{325} \\ &= \frac{6798}{325}\end{aligned}$$

Therefore

$$\begin{aligned}\mu &= \frac{6798}{325} \cdot \frac{65}{5292} \\ &= \frac{1133}{4410} \\ &\approx 0.257\end{aligned}$$

So

$$\mu = \frac{1133}{4410} \approx 0.257$$

(c) When B hits the ground, A is still moving at $4.8\,\text{m s}^{-1}$, and then continues up the plane alone. Since C is 3.5 m from the start, and A has already moved 3 m, it travels a further

$$3.5 - 3 = 0.5\,\text{m}$$

on the rough part, and then d m on the smooth part.
So after B hits the ground, the total distance moved by A is

$$0.5 + d$$

Initial kinetic energy of A:

$$\frac{1}{2} \cdot 3 \cdot (4.8)^2 = 34.56\,\text{J}$$

Increase in gravitational potential energy of A:

$$\begin{aligned}3g(0.5 + d)\sin\theta &= 3g(0.5 + d) \cdot \frac{5}{13} \\ &= \frac{147}{13}(0.5 + d)\end{aligned}$$

Friction acts only over the remaining 0.5 m of rough plane, so the work done against friction is

$$\begin{aligned}\mu R \times 0.5 &= \mu \cdot \frac{36g}{13} \cdot 0.5 \\ &= \frac{18g\mu}{13} \\ &= \frac{882\mu}{65}\end{aligned}$$

Now use the work-energy principle for block A:

$$KE_0 + GPE_0 + \sum W = KE_1 + GPE_1$$

Here,

$$KE_0 = 34.56, \qquad KE_1 = 0, \qquad \sum W = -\frac{882\mu}{65}$$

So

$$34.56 + GPE_0 - \frac{882\mu}{65} = 0 + GPE_1$$

Hence

$$34.56 = (GPE_1 - GPE_0) + \frac{882\mu}{65}$$

Therefore

$$34.56 = \frac{147}{13}(0.5 + d) + \frac{882\mu}{65}$$

Substitute $\mu = \frac{1133}{4410}$:

$$34.56 = \frac{147}{13}(0.5 + d) + \frac{882}{65} \cdot \frac{1133}{4410}$$

$$34.56 = \frac{147}{13}(0.5 + d) + \frac{1133}{325}$$

Write $34.56 = \frac{864}{25}$:

$$\frac{864}{25} = \frac{147}{13}(0.5 + d) + \frac{1133}{325}$$

So

$$\begin{aligned}\frac{147}{13}(0.5 + d) &= \frac{864}{25} - \frac{1133}{325} \\ &= \frac{11232 - 1133}{325} \\ &= \frac{10099}{325}\end{aligned}$$

Hence

$$\begin{aligned}0.5 + d &= \frac{13}{147} \cdot \frac{10099}{325} \\ &= \frac{10099}{3675}\end{aligned}$$

Therefore

$$\begin{aligned}d &= \frac{10099}{3675} - \frac{1}{2} \\ &= \frac{20198 - 3675}{7350} \\ &= \frac{16523}{7350} \\ &\approx 2.25\end{aligned}$$

So

$$d = \frac{16523}{7350}\,\mathrm{m} \approx 2.25\,\mathrm{m}$$

5. A crate, of mass 10 kg, is pulled up a rough plane inclined at 15° to the horizontal.

The crate is attached to a light string. The string lies in the vertical plane of greatest slope and is inclined at 20° above the plane.

The tension in the string is 80 newtons.

As the crate moves a distance of x metres up the plane, its speed increases from $1.5\,\text{m}\,\text{s}^{-1}$ to $4.5\,\text{m}\,\text{s}^{-1}$.

The coefficient of friction between the crate and the plane is 0.3.

(a) By using an energy method, find x. **[6]**

(b) Describe how the model could be refined to obtain a more realistic value of x and use an energy argument to explain whether this would increase or decrease the value of x. **[2]**

Solution

(a) Let the normal reaction be R N.

First resolve perpendicular to the plane. There is no motion perpendicular to the plane, so the forces balance in that direction:

$$\begin{aligned} R + 80\sin 20^\circ &= 10g\cos 15^\circ \\ R &= 98\cos 15^\circ - 80\sin 20^\circ \end{aligned}$$

Hence

$$\begin{aligned} R &\approx 98(0.9659) - 80(0.3420) \\ &\approx 67.3 \text{ N} \end{aligned}$$

So the friction force is

$$\begin{aligned} F &= \mu R \\ &= 0.3R \\ &\approx 0.3(67.3) \\ &\approx 20.2 \text{ N} \end{aligned}$$

Now apply the work-energy principle:

$$\text{KE}_0 + \text{GPE}_0 + \sum W_k = \text{KE}_1 + \text{GPE}_1$$

Take the initial position as zero gravitational potential energy.

- Initial kinetic energy:
$$\text{KE}_0 = \frac{1}{2} \times 10 \times 1.5^2$$
- Final kinetic energy:
$$\text{KE}_1 = \frac{1}{2} \times 10 \times 4.5^2$$
- Gain in height after moving x m up a plane at 15°:
$$x\sin 15^\circ$$
so
$$\text{GPE}_1 = 98x\sin 15^\circ$$
- Work done by the tension:
$$80x\cos 20^\circ$$

- Work done against friction:

$$-Fx = -0.3Rx$$

Substitute into the work-energy equation:

$$\frac{1}{2} \times 10 \times 1.5^2 + 80x \cos 20^\circ - 0.3Rx = \frac{1}{2} \times 10 \times 4.5^2 + 98x \sin 15^\circ$$

Rearranging,

$$\begin{aligned} 80x \cos 20^\circ - 98x \sin 15^\circ - 0.3Rx &= \frac{1}{2} \times 10 \times (4.5^2 - 1.5^2) \\ 80x \cos 20^\circ - 98x \sin 15^\circ - 0.3(98 \cos 15^\circ - 80 \sin 20^\circ)x &= 90 \end{aligned}$$

Therefore

$$\begin{aligned} x &= \frac{90}{80 \cos 20^\circ - 98 \sin 15^\circ - 0.3(98 \cos 15^\circ - 80 \sin 20^\circ)} \\ &\approx 3.04 \end{aligned}$$

Hence

$$x \approx 3.04 \text{ m}$$

(b) One refinement would be to include air resistance.

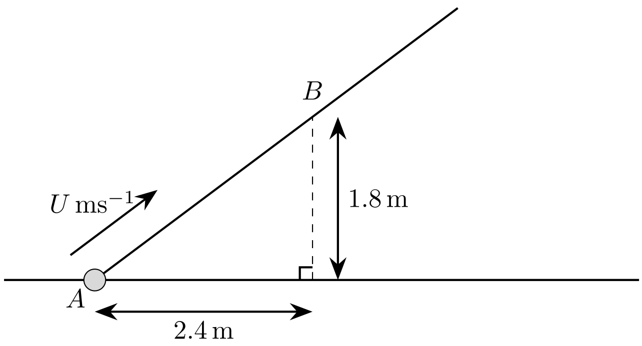
Air resistance would do additional negative work on the crate. So in

$$\text{KE}_0 + \text{GPE}_0 + \sum W_k = \text{KE}_1 + \text{GPE}_1$$

the total work term $\sum W_k$ would be smaller for each metre travelled.

Therefore, with the same tension of 80 N, a greater distance is needed to produce the same increase in kinetic energy and gravitational potential energy.

So a more realistic model including air resistance would make x increase.



6. A rough loading ramp is fixed to horizontal ground. Point A is the lower end of the ramp and point B lies on the ramp above A, as shown in the diagram above.

The point vertically below B is 2.4 m horizontally from A, and B is 1.8 m above the ground.

A crate of mass 2.5 kg is projected up the ramp from A with speed U m s^{-1} and first comes to instantaneous rest at B.

The coefficient of friction between the crate and the ramp is $\frac{1}{4}$.

The crate is modelled as a particle.

(a) Find the work done against friction as the crate moves from A to B. **[3]**

(b) Use the work-energy principle to find the value of U. **[4]**

After coming to instantaneous rest at B, the crate slides back down the ramp.

(c) Use the work-energy principle to find the speed of the crate at the instant it returns to A. **[3]**

Solution

(a) First find the length of the ramp.

$$AB = \sqrt{2.4^2 + 1.8^2} = \sqrt{5.76 + 3.24} = \sqrt{9} = 3 \text{ m}$$

Let the angle of the ramp to the horizontal be θ. Then

$$\cos\theta = \frac{2.4}{3} = \frac{4}{5}$$

The normal reaction is

$$R = mg\cos\theta = 2.5g \times \frac{4}{5} = 2g$$

So the friction force is

$$F = \mu R = \frac{1}{4}(2g) = \frac{g}{2}$$

Hence the work done against friction from A to B is

$$W = Fs = \frac{g}{2} \times 3 = \frac{3g}{2} = 14.7 \text{ J}$$

Therefore, the work done against friction is 14.7 J.

(b) Take A as the zero level for gravitational potential energy.

From part (a), the work done against friction is 14.7 J, so the work done by friction is -14.7 J.

Using the work-energy principle

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

for the motion from A to B,

$$\begin{aligned} \frac{1}{2}(2.5)U^2 + 0 - 14.7 &= 0 + 2.5g(1.8) \\ 1.25U^2 - 14.7 &= 44.1 \\ 1.25U^2 &= 58.8 \\ U^2 &= 47.04 \end{aligned}$$

Taking the positive root,

$$U = \sqrt{47.04} = 6.86 \text{ m s}^{-1}$$

Also,

$$U = \frac{14\sqrt{6}}{5} \text{ m s}^{-1}$$

Hence

$$U = \frac{14\sqrt{6}}{5} \text{ m s}^{-1} \approx 6.86 \text{ m s}^{-1}$$

(c) Let the speed of the crate when it returns to A be v m s^{-1}.

The work done against friction is again 14.7 J, so the work done by friction is −14.7 J.

Using the work-energy principle

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

for the motion from B to A,

$$\begin{aligned} 0 + 2.5g(1.8) - 14.7 &= \frac{1}{2}(2.5)v^2 + 0 \\ 44.1 - 14.7 &= 1.25v^2 \\ 29.4 &= 1.25v^2 \\ v^2 &= 23.52 \end{aligned}$$

Taking the positive root,

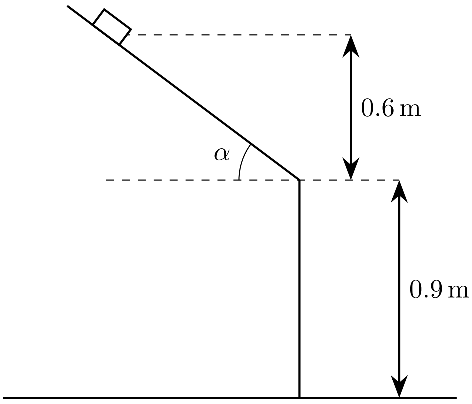
$$v = \sqrt{23.52} = 4.85 \text{ m s}^{-1}$$

Also,

$$v = \frac{14\sqrt{3}}{5} \text{ m s}^{-1}$$

So the speed as it returns to A is

$$\frac{14\sqrt{3}}{5} \text{ m s}^{-1} \approx 4.85 \text{ m s}^{-1}$$



7. A parcel of mass m is held on a rough straight roof which is inclined at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. The parcel is released from rest at a point on the roof that is 0.6 m vertically above the eaves, as shown in the diagram above. The eaves are 0.9 m above the ground. The coefficient of friction between the parcel and the roof is 0.25.

The parcel is modelled as a particle which, after leaving the roof, is assumed to move freely under gravity.

(a) Find, in terms of m and g, the magnitude of the normal reaction on the parcel as it slides down the roof. **[2]**

(b) Use the work-energy principle to find the speed of the parcel as it hits the ground. **[5]**

Solution

(a) Resolving perpendicular to the roof, the parcel has no acceleration in this direction, so

$$R = mg\cos\alpha$$

Given $\sin\alpha = \frac{3}{5}$ and α is acute,

$$\cos\alpha = \frac{4}{5}$$

Hence

$$R = mg \cdot \frac{4}{5} = \frac{4mg}{5}$$

So the magnitude of the normal reaction is $\frac{4mg}{5}$.

(b) Let the distance travelled along the roof be s m.

Since the vertical drop from the release point to the eaves is 0.6 m,

$$s\sin\alpha = 0.6$$

so

$$s = \frac{0.6}{3/5} = 1$$

Therefore the parcel slides 1 m along the roof.

From part (a), the normal reaction is $\frac{4mg}{5}$, so the friction force has magnitude

$$F = \mu R = 0.25 \times \frac{4mg}{5} = \frac{mg}{5}$$

Hence the work done by friction on the roof is

$$-Fs = -\frac{mg}{5} \times 1 = -\frac{mg}{5}$$

The normal reaction does no work, since it is perpendicular to the motion. After the parcel leaves the roof, it moves freely under gravity, so there is no further friction.

Take the ground as the zero level for GPE. The parcel starts at height

$$0.6 + 0.9 = 1.5 \text{ m}$$

above the ground.

Now use the work-energy principle

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

Applying this from release to impact with the ground,

$$\begin{aligned} 0 + mg(1.5) - \frac{mg}{5} &= \frac{1}{2}mv^2 + 0 \\ \frac{3mg}{2} - \frac{mg}{5} &= \frac{1}{2}mv^2 \\ \frac{15mg - 2mg}{10} &= \frac{1}{2}mv^2 \\ \frac{13mg}{10} &= \frac{1}{2}mv^2 \end{aligned}$$

So

$$v^2 = \frac{13g}{5}$$

and therefore

$$v = \sqrt{\frac{13g}{5}}$$

So the speed of the parcel as it hits the ground is $\sqrt{\frac{13g}{5}}$ m s^{-1}.

8. A rough plane is inclined to the horizontal at an angle θ, where $\tan\theta = \frac{5}{12}$.

A particle P of mass m is at rest at a point O on the plane.

The coefficient of friction between P and the plane is $\frac{1}{4}$.

The particle is projected up the plane with speed $4\sqrt{ag}$.

The particle moves up a line of greatest slope of the plane, comes to instantaneous rest, and then slides back down to O.

(a) Show that the magnitude of the frictional force acting on P is

$$\frac{3mg}{13}$$

[3]

Air resistance is assumed to be negligible.

Using the work-energy principle,

(b) find the speed of P when it returns to O. **[5]**

Solution

(a) Since

$$\tan\theta = \frac{5}{12},$$

we use the 5-12-13 triangle, so

$$\sin\theta = \frac{5}{13}, \qquad \cos\theta = \frac{12}{13}.$$

Perpendicular to the plane, the particle is in equilibrium, so the normal reaction is

$$R = mg\cos\theta = mg \cdot \frac{12}{13} = \frac{12mg}{13}.$$

The coefficient of friction is $\mu = \frac{1}{4}$, so the magnitude of the frictional force is

$$F = \mu R = \frac{1}{4} \cdot \frac{12mg}{13} = \frac{3mg}{13}.$$

Therefore, the magnitude of the frictional force is

$$\frac{3mg}{13}.$$

(b) Let the particle travel a distance s up the plane before coming to rest at the highest point A.

From part (a), the frictional force has magnitude

$$\frac{3mg}{13}.$$

Also,

$$\sin\theta = \frac{5}{13}.$$

We use the work-energy principle in the form

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1.$$

Take the gravitational potential energy at O to be 0.

Journey up the plane: from O to A

Initial kinetic energy:

$$\text{KE}_O = \frac{1}{2}m\left(4\sqrt{ag}\right)^2 = \frac{1}{2}m(16ag) = 8mag.$$

Final kinetic energy at A:

$$\text{KE}_A = 0.$$

Gain in gravitational potential energy:

$$\text{GPE}_A = mg(s\sin\theta) = mg\left(s\cdot\frac{5}{13}\right) = \frac{5mgs}{13}.$$

Work done by friction is negative, since friction opposes the motion:

$$W_f = -\frac{3mg}{13}s.$$

So

$$\text{KE}_O + \text{GPE}_O + \sum W = \text{KE}_A + \text{GPE}_A$$
$$8mag + 0 - \frac{3mgs}{13} = 0 + \frac{5mgs}{13}.$$

Hence

$$8mag = \frac{8mgs}{13}$$
$$s = 13a.$$

Journey down the plane: from A back to O

At A, the particle starts from rest, so

$$\text{KE}_A = 0.$$

Its gravitational potential energy at A is

$$\text{GPE}_A = mg(s\sin\theta) = mg\left(13a\cdot\frac{5}{13}\right) = 5mga.$$

Friction again opposes the motion, so over distance $13a$ the work done by friction is

$$W_f = -\frac{3mg}{13}(13a) = -3mga.$$

Let the speed when it returns to O be v. Then

$$\text{KE}_O = \frac{1}{2}mv^2, \qquad \text{GPE}_O = 0.$$

Applying the work-energy principle:

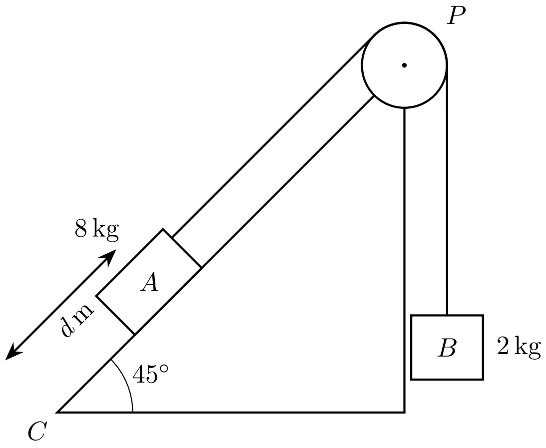
$$\text{KE}_A + \text{GPE}_A + \sum W = \text{KE}_O + \text{GPE}_O$$
$$0 + 5mga - 3mga = \frac{1}{2}mv^2 + 0.$$

So

$$2mga = \frac{1}{2}mv^2$$
$$4ag = v^2$$
$$v = 2\sqrt{ag}.$$

Therefore, the speed when P returns to O is

$$2\sqrt{ag}.$$



9. One end of a rope is attached to a block A of mass 8 kg. The other end of the rope is attached to a second block B of mass 2 kg. Block A is held at rest on a fixed rough ramp inclined at 45° to the horizontal. The lower end of the ramp is C. The rope is taut and passes over a small smooth pulley P which is fixed at the top of the ramp. The part of the rope from A to P is parallel to a line of greatest slope of the ramp. Block A is initially d m from C, as shown in the diagram.

Block B is more than d m below P. The blocks are released from rest and A moves down the ramp towards C.

The coefficient of friction between A and the ramp is $\frac{1}{4}$.

The blocks are modelled as particles, the rope is modelled as light and inextensible, and air resistance can be ignored.

(a) Determine, in terms of g and d, the work done against friction as A moves d m down the ramp. **[3]**

(b) Given that the speed of A immediately before it reaches C is $2.1\,\mathrm{m\,s^{-1}}$, use the work-energy principle to determine the value of d. **[5]**

(c) Suggest one improvement, apart from including air resistance, that could be made to the model to make it more realistic. **[1]**

Solution

(a) Since A moves down the ramp, friction acts up the ramp.

The normal reaction on A is

$$R = 8g\cos 45° = 8g \cdot \frac{\sqrt{2}}{2} = 4\sqrt{2}g$$

So the frictional force is

$$F = \mu R = \frac{1}{4}(4\sqrt{2}g) = \sqrt{2}g$$

Hence the work done against friction as A moves distance d is

$$W = Fd = (\sqrt{2}g)(d) = \sqrt{2}gd$$

So the work done against friction is $\sqrt{2}gd$ J.

(b) Consider the system consisting of both blocks A and B.

Since the rope is light and inextensible, both blocks have the same speed. Immediately before A reaches C, this speed is $2.1\,\mathrm{m\,s^{-1}}$.

Initially the system is at rest, so

$$\mathrm{KE}_0 = 0$$

The final kinetic energy is

$$\text{KE}_1 = \frac{1}{2}(8+2)(2.1)^2 = 22.05$$

For block A, the vertical drop is $d \sin 45°$, so the loss in GPE of A is

$$8g(d \sin 45°) = 8gd \cdot \frac{\sqrt{2}}{2} = 4\sqrt{2}gd$$

For block B, it rises by d, so the gain in GPE of B is

$$2gd$$

Therefore the overall loss in GPE of the system is

$$\text{GPE}_0 - \text{GPE}_1 = 4\sqrt{2}gd - 2gd$$

From part (a), the work done against friction is $\sqrt{2}gd$, so the work done by friction is

$$-\sqrt{2}gd$$

Now use the work-energy principle

$$\text{KE}_0 + \text{GPE}_0 + \sum W = \text{KE}_1 + \text{GPE}_1$$

Substituting in,

$$\begin{aligned} 0 + \text{GPE}_0 - \sqrt{2}gd &= 22.05 + \text{GPE}_1 \\ \text{GPE}_0 - \text{GPE}_1 &= 22.05 + \sqrt{2}gd \end{aligned}$$

But

$$\text{GPE}_0 - \text{GPE}_1 = 4\sqrt{2}gd - 2gd$$

So

$$\begin{aligned} 4\sqrt{2}gd - 2gd &= 22.05 + \sqrt{2}gd \\ 3\sqrt{2}gd - 2gd &= 22.05 \\ (3\sqrt{2} - 2)gd &= 22.05 \end{aligned}$$

Hence

$$d = \frac{22.05}{g(3\sqrt{2} - 2)}$$

Using $g = 9.8$,

$$\begin{aligned} d &= \frac{22.05}{9.8(3\sqrt{2} - 2)} \\ &= \frac{9}{4(3\sqrt{2} - 2)} \\ &= \frac{9(3\sqrt{2} + 2)}{56} \\ &\approx 1.00 \end{aligned}$$

So

$$d \approx 1.00 \text{ m}$$

(c) One possible improvement is to model the pulley as not perfectly smooth, so friction at the pulley is taken into account.

10. A small block of mass 0.5 kg is sliding down a rough slope which is inclined at 30° to the horizontal. At the instant that its speed is $5\,\mathrm{m\,s^{-1}}$ directly down the slope, it is struck by a mallet. Immediately after the impact its speed is $12\,\mathrm{m\,s^{-1}}$ directly up the slope.

(a) Find the magnitude of the impulse exerted by the mallet on the block. **[2]**

After the impact, the block moves up the slope until it comes to instantaneous rest. It then slides back down the slope and passes through the point where it was struck with speed $4\,\mathrm{m\,s^{-1}}$. Assume that, whenever the block is moving, the resistance to its motion has constant magnitude R N.

(b) Use an energy method to determine the value of R. **[5]**

Solution

(a) Take up the slope as positive. Then

$$u = -5, \qquad v = 12, \qquad m = 0.5$$

Impulse = change in momentum, so

$$I = m(v - u) = 0.5\big(12 - (-5)\big) = 0.5(17) = 8.5$$

Hence the magnitude of the impulse exerted by the mallet is 8.5 N s.

(b) Let the distance travelled up the slope before the block comes to rest be s m.

Take the point where the block was struck as the zero of gravitational potential energy.

For the upward motion, the initial speed is $12\,\mathrm{m\,s^{-1}}$ and the final speed is 0. The resistance and gravity both oppose the motion.

Using the work-energy principle

$$\mathrm{KE}_0 + \mathrm{GPE}_0 + \sum W = \mathrm{KE}_1 + \mathrm{GPE}_1$$

we have

$$\begin{aligned}\frac{1}{2}(0.5)(12^2) + 0 + (-Rs) &= 0 + (0.5)gs\sin 30^\circ \\ 36 - Rs &= 0.25gs \\ 36 &= (0.25g + R)s\end{aligned}$$

Now consider the motion back down the slope from rest to the striking point, where the speed is $4\,\mathrm{m\,s^{-1}}$.

Again using

$$\mathrm{KE}_0 + \mathrm{GPE}_0 + \sum W = \mathrm{KE}_1 + \mathrm{GPE}_1$$

gives

$$\begin{aligned}0 + (0.5)gs\sin 30^\circ + (-Rs) &= \frac{1}{2}(0.5)(4^2) + 0 \\ 0.25gs - Rs &= 4 \\ (0.25g - R)s &= 4\end{aligned}$$

So

$$(0.25g + R)s = 36, \qquad (0.25g - R)s = 4$$

Adding the equations,

$$\begin{aligned}0.5gs &= 40 \\ s &= \frac{80}{g}\end{aligned}$$

Substitute into $36 = (0.25g + R)s$

$$\begin{aligned} 36 &= (0.25g + R)\,\frac{80}{g} \\ 36 &= 20 + \frac{80R}{g} \\ 16 &= \frac{80R}{g} \\ R &= \frac{g}{5} \end{aligned}$$

With $g = 9.8\,\mathrm{m\,s^{-2}}$,

$$R = \frac{9.8}{5} = 1.96\,\mathrm{N}$$

Therefore

$$R = \frac{g}{5}\,\mathrm{N} = 1.96\,\mathrm{N}$$