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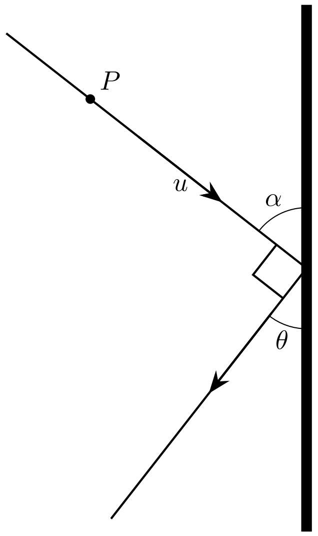


1. A particle P of mass m is moving with speed u on a fixed smooth horizontal surface. It collides with a fixed smooth vertical barrier. Before impact its path makes an angle α with the barrier, and after impact its path makes an angle θ with the barrier. The incident path and the outgoing path are perpendicular. The coefficient of restitution between P and the barrier is e. The particle loses 36% of its kinetic energy in the collision.

Find the value of e and the exact value of $\tan\alpha$. **[5]**

Solution

Let the speed of P immediately after impact be v.
The particle loses 36% of its kinetic energy, so it keeps 64% of it. Hence

$$\frac{1}{2}mv^2 = 0.64\left(\frac{1}{2}mu^2\right)$$

So

$$v^2 = 0.64u^2$$

hence

$$v = 0.8u = \frac{4}{5}u$$

since $v > 0$.
Since the incident and outgoing paths are perpendicular, and both angles are measured from the barrier,

$$\alpha + \theta = 90^\circ$$

So

$$\theta = 90^\circ - \alpha$$

The barrier is smooth, so the component of velocity parallel to the barrier is unchanged. Therefore

$$u\cos\alpha = v\cos\theta = v\sin\alpha$$

so

$$\tan\alpha = \frac{u}{v} = \frac{u}{\frac{4}{5}u} = \frac{5}{4}$$

For the coefficient of restitution, use the components perpendicular to the barrier:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{v\sin\theta}{u\sin\alpha} = \frac{v\cos\alpha}{u\sin\alpha}$$

Using $v = \frac{4}{5}u$ and $\tan\alpha = \frac{5}{4}$,

$$e = \frac{\frac{4}{5}u}{u\tan\alpha} = \frac{\frac{4}{5}}{\frac{5}{4}} = \frac{16}{25}$$

So the required values are

$$e = \frac{16}{25}, \qquad \tan\alpha = \frac{5}{4}$$

2. A hockey puck moving on a smooth horizontal rink collides with a fixed straight side-board. Let $\mathbf{i}$ be a unit vector parallel to the side-board and let $\mathbf{j}$ be a unit vector perpendicular to the side-board, directed away from it.

Immediately before the collision, the puck has velocity $(9\mathbf{i} - 12\mathbf{j})\,\mathrm{m\,s^{-1}}$.

Immediately after the collision, the puck moves with speed $10\,\mathrm{m\,s^{-1}}$ in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$.

Model the puck as a particle.

(a) Find the coefficient of restitution between the puck and the side-board. **[3]**

(b) Determine whether or not the side-board is smooth.
Fully justify your answer. **[3]**

Solution

(a) The direction after collision is along $3\mathbf{i} + 4\mathbf{j}$, whose magnitude is

$$\sqrt{3^2 + 4^2} = 5$$

So the velocity immediately after collision is

$$10 \times \frac{3\mathbf{i} + 4\mathbf{j}}{5} = 6\mathbf{i} + 8\mathbf{j}$$

For a collision with a fixed side-board, the coefficient of restitution is

$$e = \frac{\text{speed of separation perpendicular to board}}{\text{speed of approach perpendicular to board}}$$

Before collision, the perpendicular component is $-12\mathbf{j}$, so the speed of approach is 12.

After collision, the perpendicular component is $8\mathbf{j}$, so the speed of separation is 8.

Hence

$$e = \frac{8}{12} = \frac{2}{3}$$

So the coefficient of restitution is $\frac{2}{3}$.

(b) The component of velocity parallel to the side-board is the $\mathbf{i}$-component.

Before collision, this is

$$9 \text{ m s}^{-1}$$

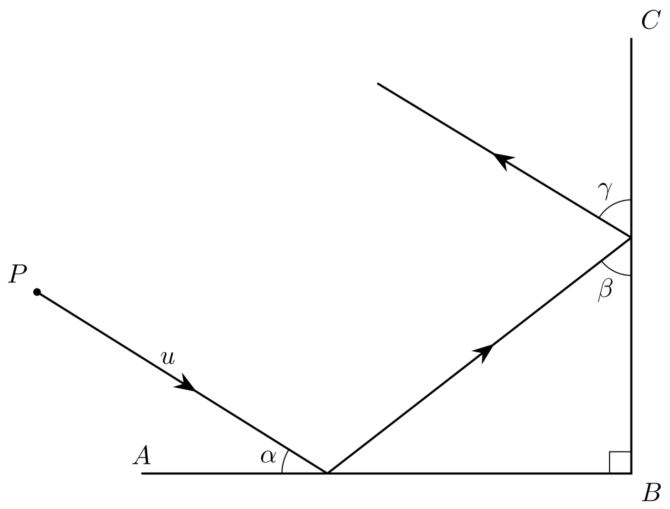
After collision, from part (a), the velocity is $6\mathbf{i} + 8\mathbf{j}$, so the parallel component is

$$6 \text{ m s}^{-1}$$

If the side-board were smooth, there would be no impulse parallel to the board, so the parallel component of velocity would be unchanged.

But here it changes from 9 to 6.

Therefore the side-board is not smooth.



3. The smooth walls AB and BC are at right angles to each other. A particle P moves on a smooth horizontal floor with speed u and strikes the wall AB at an angle α to AB. It rebounds, then strikes the wall BC at an angle β to BC. After the second impact it rebounds at an angle γ to BC, as shown in the diagram. The coefficient of restitution between P and each wall is e.

(a) Show that

$$\tan\alpha\tan\beta = \frac{1}{e}$$

[3]

(b) Express γ in terms of α and explain what this implies about the final direction of motion of P. **[4]**

During the first impact the particle loses four times as much kinetic energy as it loses during the second impact. After the second impact the kinetic energy of P is $\frac{9}{25}$ of its initial kinetic energy.

(c) Find the value of e and the value of $\tan\alpha$. **[4]**

Solution

(a) Take AB to be horizontal and BC vertical.

Before the first impact, since the particle strikes AB at angle α to the wall,

$$\text{component parallel to } AB = u\cos\alpha, \qquad \text{component perpendicular to } AB = u\sin\alpha$$

For impact with a smooth wall, the component parallel to the wall is unchanged, and the component perpendicular to the wall is reversed and multiplied by e.

So after rebounding from AB,

$$\text{horizontal component} = u\cos\alpha, \qquad \text{vertical component} = eu\sin\alpha$$

At BC, the angle β is measured with the wall BC, so

$$\tan\beta = \frac{\text{component perpendicular to } BC}{\text{component parallel to } BC} = \frac{u\cos\alpha}{eu\sin\alpha} = \frac{1}{e\tan\alpha}$$

Hence

$$\tan\alpha\tan\beta = \frac{1}{e}$$

(b) Just before the second impact with BC, the velocity components are

$$(u\cos\alpha,\ eu\sin\alpha)$$

On rebounding from the smooth wall BC:

- the component parallel to BC is unchanged,
- the component perpendicular to BC is reversed and multiplied by e.

So after the second impact, the velocity components are

$$(-eu\cos\alpha,\ eu\sin\alpha)$$

Therefore

$$\tan\gamma = \frac{\text{component perpendicular to } BC}{\text{component parallel to } BC} = \frac{eu\cos\alpha}{eu\sin\alpha} = \cot\alpha$$

Hence

$$\tan\gamma = \tan\left(\frac{\pi}{2} - \alpha\right)$$

Since γ is an acute angle,

$$\gamma = \frac{\pi}{2} - \alpha$$

Also, the initial velocity was

$$(u\cos\alpha,\ -u\sin\alpha)$$

and the final velocity is

$$(-eu\cos\alpha,\ eu\sin\alpha) = -e(u\cos\alpha,\ -u\sin\alpha)$$

So the final velocity is a negative multiple of the initial velocity. Therefore the final direction of motion is parallel to the initial direction, but in the opposite sense.

(c) After the second impact, the speed is

$$\sqrt{(eu\cos\alpha)^2 + (eu\sin\alpha)^2} = eu$$

So

$$\frac{\text{final KE}}{\text{initial KE}} = \frac{\frac{1}{2}m(eu)^2}{\frac{1}{2}mu^2} = e^2$$

Given that the final kinetic energy is $\frac{9}{25}$ of the initial kinetic energy,

$$e^2 = \frac{9}{25}$$

and since $0 < e < 1$,

$$e = \frac{3}{5}$$

Now find the losses in kinetic energy.

Before the first impact, KE $= \frac{1}{2}mu^2$.

After the first impact, speed squared is

$$(u\cos\alpha)^2 + (eu\sin\alpha)^2$$

so the loss in the first impact is

$$\begin{aligned}\Delta E_1 &= \frac{1}{2}mu^2 - \frac{1}{2}m\left((u\cos\alpha)^2 + (eu\sin\alpha)^2\right) \\ &= \frac{1}{2}mu^2\left(1 - \cos^2\alpha - e^2\sin^2\alpha\right) \\ &= \frac{1}{2}mu^2(1-e^2)\sin^2\alpha\end{aligned}$$

Just before the second impact, KE is

$$\frac{1}{2}m\left((u\cos\alpha)^2 + (eu\sin\alpha)^2\right)$$

and just after the second impact, KE is

$$\frac{1}{2}m\left((eu\cos\alpha)^2+(eu\sin\alpha)^2\right)$$

so the loss in the second impact is

$$\begin{aligned}\Delta E_2 &= \frac{1}{2}m\left((u\cos\alpha)^2+(eu\sin\alpha)^2-(eu\cos\alpha)^2-(eu\sin\alpha)^2\right)\\ &= \frac{1}{2}mu^2\left(\cos^2\alpha-e^2\cos^2\alpha\right)\\ &= \frac{1}{2}mu^2(1-e^2)\cos^2\alpha\end{aligned}$$

Given that the first loss is four times the second loss,

$$\Delta E_1 = 4\Delta E_2$$

so

$$\frac{1}{2}mu^2(1-e^2)\sin^2\alpha = 4\left(\frac{1}{2}mu^2(1-e^2)\cos^2\alpha\right)$$

Cancelling the common factor $\frac{1}{2}mu^2(1-e^2)$ gives

$$\sin^2\alpha = 4\cos^2\alpha$$

hence

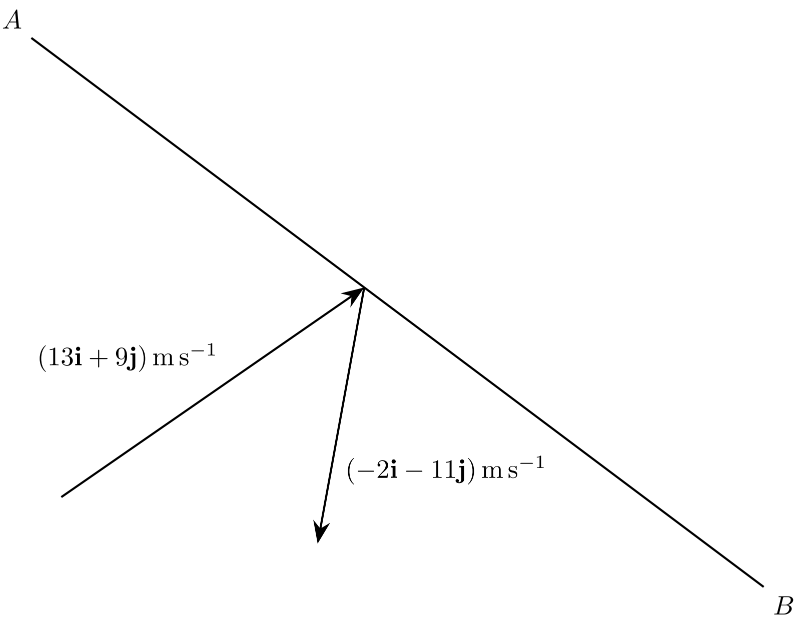
$$\tan^2\alpha = 4$$

Since α is acute,

$$\tan\alpha = 2$$

Therefore

$$e=\frac{3}{5},\qquad \tan\alpha = 2$$



4. The diagram above shows the plan view of part of a smooth horizontal table. The line segment AB represents a fixed smooth vertical wall.

A small ball of mass 0.4 kg moves across the table and rebounds after striking the wall.

Immediately before impact its velocity is $(13\mathbf{i} + 9\mathbf{j})\,\mathrm{m\,s^{-1}}$.

Immediately after impact its velocity is $(-2\mathbf{i} - 11\mathbf{j})\,\mathrm{m\,s^{-1}}$.

The coefficient of restitution between the ball and the wall is e.

(a) Show that AB is parallel to $(4\mathbf{i} - 3\mathbf{j})$. **[4]**

(b) Find the value of e. **[5]**

Solution

(a) Let the velocity before impact be

$$\mathbf{u} = 13\mathbf{i} + 9\mathbf{j}$$

and the velocity after impact be

$$\mathbf{v} = -2\mathbf{i} - 11\mathbf{j}$$

The change in velocity is

$$\begin{aligned}\mathbf{v} - \mathbf{u} &= (-2-13)\mathbf{i} + (-11-9)\mathbf{j}\\ &= -15\mathbf{i} - 20\mathbf{j}\\ &= -5(3\mathbf{i} + 4\mathbf{j})\end{aligned}$$

When a ball strikes a *smooth* wall, the impulse acts perpendicular to the wall, so the change in velocity is also perpendicular to the wall.
Hence the wall AB is perpendicular to

$$3\mathbf{i} + 4\mathbf{j}$$

A vector perpendicular to $3\mathbf{i} + 4\mathbf{j}$ is $4\mathbf{i} - 3\mathbf{j}$, since

$$(3\mathbf{i} + 4\mathbf{j}) \cdot (4\mathbf{i} - 3\mathbf{j}) = 12 - 12 = 0$$

Therefore

$$AB \parallel (4\mathbf{i} - 3\mathbf{j})$$

(b) From part (a), a direction parallel to the wall is $4\mathbf{i} - 3\mathbf{j}$, so a direction perpendicular to the wall is $3\mathbf{i} + 4\mathbf{j}$.

Take unit vectors

$$\hat{\mathbf{t}} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j}), \qquad \hat{\mathbf{n}} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j})$$

where $\hat{\mathbf{t}}$ is parallel to the wall and $\hat{\mathbf{n}}$ is perpendicular to the wall.

For the velocity before impact,

$$\begin{aligned}\mathbf{u} \cdot \hat{\mathbf{t}} &= (13\mathbf{i} + 9\mathbf{j}) \cdot \frac{1}{5}(4\mathbf{i} - 3\mathbf{j}) \\ &= \frac{1}{5}(52 - 27) \\ &= 5\end{aligned}$$

$$\begin{aligned}\mathbf{u} \cdot \hat{\mathbf{n}} &= (13\mathbf{i} + 9\mathbf{j}) \cdot \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) \\ &= \frac{1}{5}(39 + 36) \\ &= 15\end{aligned}$$

So before impact, the component parallel to the wall is 5, and the component perpendicular to the wall is 15 towards the wall.

For the velocity after impact,

$$\begin{aligned}\mathbf{v} \cdot \hat{\mathbf{t}} &= (-2\mathbf{i} - 11\mathbf{j}) \cdot \frac{1}{5}(4\mathbf{i} - 3\mathbf{j}) \\ &= \frac{1}{5}(-8 + 33) \\ &= 5\end{aligned}$$

$$\begin{aligned}\mathbf{v} \cdot \hat{\mathbf{n}} &= (-2\mathbf{i} - 11\mathbf{j}) \cdot \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) \\ &= \frac{1}{5}(-6 - 44) \\ &= -10\end{aligned}$$

So after impact, the component parallel to the wall is still 5, and the perpendicular component has reversed direction with speed 10 away from the wall.

Since the wall is fixed,

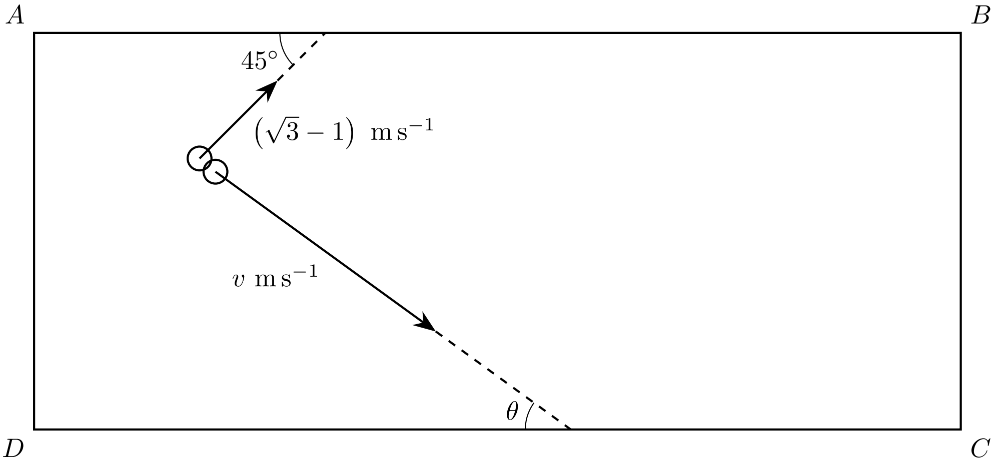
$$e = \frac{\text{speed of separation}}{\text{speed of approach}}$$

measured perpendicular to the wall. Therefore

$$e = \frac{10}{15} = \frac{2}{3}$$

Hence

$$e = \frac{2}{3}$$



5. Two snooker balls, one white and one red, have equal mass.
The balls are on a horizontal table $ABCD$.

The white ball is struck so that it moves at a speed of $2\,\text{m}\,\text{s}^{-1}$ in a direction making an angle of 15° with AB, towards CD.

The white ball hits a stationary red ball.

After the collision, the white ball moves at a speed of $(\sqrt{3}-1)\,\text{m}\,\text{s}^{-1}$ and at an angle of 45° to AB.

After the collision, the red ball moves at a speed $v\,\text{m}\,\text{s}^{-1}$ and at an angle θ to CD.

When the collision takes place, the point of impact is twice as far from CD as it is from AB.

The diagram below shows the velocities of the balls after the collision.

After the collision, the white ball hits AB and the red ball hits CD.

Model the balls as particles and neglect any resistance.

(a) Explain why the two balls hit the sides of the table at the same time. **[2]**

(b) Show that $\theta = 36.2°$ correct to one decimal place. **[4]**

(c) Find v. **[2]**

(d) Determine which ball travels the greater distance after the collision and before hitting the side of the table.
Fully justify your answer. **[2]**

(e) State one possible refinement to the model that you have used. **[1]**

Solution

(a) Let the perpendicular distance from the point of impact to AB be d. Then the perpendicular distance to CD is $2d$.

Since $AB \parallel CD$, the times to reach the sides depend on the velocity components perpendicular to these sides.

Take perpendicular to AB as positive towards CD. Since the masses are equal, conservation of momentum perpendicular to AB gives

$$2\sin 15° = v\sin\theta - (\sqrt{3}-1)\sin 45°$$

so

$$v\sin\theta = 2\sin 15^\circ + (\sqrt{3}-1)\sin 45^\circ$$

Now

$$2\sin 15^\circ = 2\cdot\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{2}$$
$$(\sqrt{3}-1)\sin 45^\circ = (\sqrt{3}-1)\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2}$$

Hence

$$v\sin\theta = \sqrt{6}-\sqrt{2}$$

So the red ball moves towards CD with speed component $\sqrt{6}-\sqrt{2}$, while the white ball moves towards AB with speed component

$$(\sqrt{3}-1)\sin 45^\circ = \frac{\sqrt{6}-\sqrt{2}}{2}$$

Therefore the red ball's perpendicular speed is twice the white ball's, and its perpendicular distance to travel is also twice as great.

$$t_{\text{white}} = \frac{d}{(\sqrt{6}-\sqrt{2})/2}, \qquad t_{\text{red}} = \frac{2d}{\sqrt{6}-\sqrt{2}}$$

These are equal, so the two balls hit the sides of the table at the same time.

(b) Since $AB \parallel CD$, the red ball makes the same angle θ with AB as with CD.

Resolving parallel to AB and using conservation of momentum:

$$2\cos 15^\circ = (\sqrt{3}-1)\cos 45^\circ + v\cos\theta$$

Hence

$$\begin{aligned} v\cos\theta &= 2\cos 15^\circ - (\sqrt{3}-1)\cos 45^\circ \\ &= 2\cdot\frac{\sqrt{6}+\sqrt{2}}{4} - (\sqrt{3}-1)\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{2} - \frac{\sqrt{6}-\sqrt{2}}{2} \\ &= \sqrt{2} \end{aligned}$$

From part (a),

$$v\sin\theta = \sqrt{6}-\sqrt{2}$$

Therefore

$$\begin{aligned} \tan\theta &= \frac{v\sin\theta}{v\cos\theta} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{2}} \\ &= \sqrt{3}-1 \end{aligned}$$

So

$$\theta = \tan^{-1}(\sqrt{3}-1) = 36.2^\circ \text{ (to 1 d.p.)}$$

Hence $\theta = 36.2^\circ$ correct to one decimal place.

(c) Using the results from part (b),

$$v\cos\theta = \sqrt{2}, \qquad v\sin\theta = \sqrt{6}-\sqrt{2}$$

So

$$\begin{aligned} v^2 &= (v\cos\theta)^2 + (v\sin\theta)^2 \\ &= (\sqrt{2})^2 + (\sqrt{6}-\sqrt{2})^2 \\ &= 2 + (6 + 2 - 2\sqrt{12}) \\ &= 2 + 8 - 4\sqrt{3} \\ &= 10 - 4\sqrt{3} \end{aligned}$$

Hence

$$v = \sqrt{10 - 4\sqrt{3}}\ \mathrm{m\,s^{-1}}$$

Since speed is positive,

$$v = \sqrt{10 - 4\sqrt{3}}\ \mathrm{m\,s^{-1}} \approx 1.75\ \mathrm{m\,s^{-1}}$$

(d) From part (a), both balls travel for the same time after the collision.

Let this common time be t. Then the distances travelled are

$$s_{\text{white}} = (\sqrt{3} - 1)t, \qquad s_{\text{red}} = vt$$

From part (c),

$$v = \sqrt{10 - 4\sqrt{3}} \approx 1.75$$

and

$$\sqrt{3} - 1 \approx 0.732$$

Since

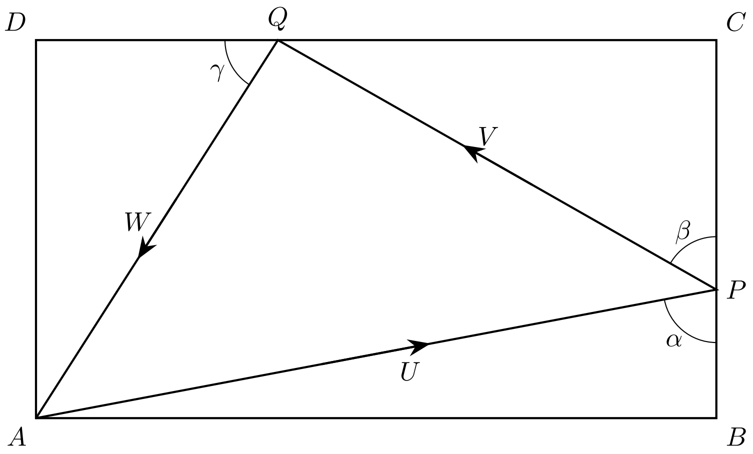
$$1.75 > 0.732$$

we have

$$s_{\text{red}} > s_{\text{white}}$$

Therefore the red ball travels the greater distance after the collision and before hitting the side of the table.

(e) One possible refinement is to include friction (or rolling resistance) between the balls and the table.



6. A small smooth billiard ball is projected from the corner A of a horizontal rectangular table $ABCD$.

The ball first hits the side BC at the point P, then hits the side CD at the point Q, and then returns to A.

Angle $APB = \alpha$, angle $CPQ = \beta$ and angle $DQA = \gamma$.

The ball moves along AP with speed U, along PQ with speed V and along QA with speed W, as shown in the diagram.

The coefficient of restitution between the ball and side BC is e_1.

The coefficient of restitution between the ball and side CD is e_2.

The ball is modelled as a particle.

(a) Show that $\tan\beta = e_1 \tan\alpha$. **[4]**

(b) Hence show that $e_1 \tan\alpha \tan\gamma = e_2$. **[3]**

(c) By considering $\angle PAQ$ or otherwise, show that it is possible for the ball to return to A only if $e_2 > e_1$. **[6]**

If instead $e_1 = e_2$, the ball would not return to A.

(d) Given that $e_1 = e_2$, use the result from part (b) to describe the path of the ball after it hits CD at Q, explaining your answer. **[1]**

Solution

(a) At the first impact, the ball hits the side BC.

For a smooth side:

- the component of velocity *parallel* to the side is unchanged,
- the component of velocity *perpendicular* to the side is reversed and multiplied by the coefficient of restitution.

Resolve the velocity relative to BC.

Before impact, along AP with speed U:

$$\text{parallel to } BC = U\cos\alpha, \qquad \text{perpendicular to } BC = U\sin\alpha$$

After impact, along PQ with speed V:

$$\text{parallel to } BC = V\cos\beta, \qquad \text{perpendicular to } BC = V\sin\beta$$

So

$$\begin{aligned} V\cos\beta &= U\cos\alpha \\ V\sin\beta &= e_1 U\sin\alpha \end{aligned}$$

Dividing the second equation by the first,

$$\begin{aligned} \frac{V\sin\beta}{V\cos\beta} &= \frac{e_1 U\sin\alpha}{U\cos\alpha} \\ \tan\beta &= e_1\tan\alpha \end{aligned}$$

Hence

$$\tan\beta = e_1\tan\alpha$$

(b) At the second impact, the ball hits the side CD.

Resolve the velocity relative to CD.

Before impact, along PQ with speed V:

$$\text{parallel to } CD = V\sin\beta, \qquad \text{perpendicular to } CD = V\cos\beta$$

After impact, along QA with speed W:

$$\text{parallel to } CD = W\cos\gamma, \qquad \text{perpendicular to } CD = W\sin\gamma$$

Therefore

$$\begin{aligned} W\cos\gamma &= V\sin\beta \\ W\sin\gamma &= e_2 V\cos\beta \end{aligned}$$

Dividing,

$$\begin{aligned} \tan\gamma &= \frac{e_2 V\cos\beta}{V\sin\beta} \\ &= \frac{e_2}{\tan\beta} \end{aligned}$$

So

$$e_2 = \tan\beta\tan\gamma$$

Using part (a), $\tan\beta = e_1\tan\alpha$, hence

$$\begin{aligned} e_2 &= (e_1\tan\alpha)\tan\gamma \\ &= e_1\tan\alpha\tan\gamma \end{aligned}$$

Hence

$$e_1\tan\alpha\tan\gamma = e_2$$

(c) Since the ball returns to A, the lines AP and AQ form the angle $\angle PAQ$, so we must have

$$\angle PAQ > 0$$

Now

$$\angle PAB = 90^\circ - \alpha$$

because AB is perpendicular to BC.

Also $AB \parallel CD$, so

$$\angle QAB = \angle DQA = \gamma$$

Therefore

$$\begin{aligned}\angle PAQ &= \angle QAB - \angle PAB \\ &= \gamma - (90^\circ - \alpha) \\ &= \alpha + \gamma - 90^\circ\end{aligned}$$

Since $\angle PAQ > 0$,

$$\begin{aligned}\alpha + \gamma - 90^\circ &> 0 \\ \alpha + \gamma &> 90^\circ\end{aligned}$$

Both α and γ are acute, so

$$\alpha > 90^\circ - \gamma$$

On $0^\circ < \theta < 90^\circ$, $\tan\theta$ is increasing, so

$$\begin{aligned}\tan\alpha &> \tan(90^\circ - \gamma) \\ &= \cot\gamma \\ &= \frac{1}{\tan\gamma}\end{aligned}$$

Hence

$$\tan\alpha\tan\gamma > 1$$

From part (b),

$$e_2 = e_1 \tan\alpha\tan\gamma$$

Therefore

$$e_2 > e_1$$

So it is possible for the ball to return to A only if $e_2 > e_1$.

(d) If $e_1 = e_2$, then part (b) gives

$$\tan\alpha\tan\gamma = 1$$

Since α and γ are acute,

$$\gamma = 90^\circ - \alpha$$

So after hitting CD at Q, the ball leaves making the same angle with CD as AP makes with AB. Since $AB \parallel CD$, the outgoing path from Q is parallel to AP, so the ball will hit AD, not return to A.

7. A particle P is moving with velocity $(7\mathbf{i} - 4\mathbf{j})\,\mathrm{m\,s^{-1}}$ on a smooth horizontal plane. It collides with a smooth vertical wall which is parallel to the direction vector $2\mathbf{i}+\mathbf{j}$ and rebounds with velocity $(2\mathbf{i}+6\mathbf{j})\,\mathrm{m\,s^{-1}}$.

The coefficient of restitution between P and this wall is e.

(a) Find the value of e. **[5]**

After this collision, P goes on to hit a second smooth vertical wall which is parallel to the direction vector $\mathbf{i} - 2\mathbf{j}$. The coefficient of restitution between P and this second wall is $\frac{1}{2}$.

The angle through which the direction of motion of P is deflected by its collision with this second wall is $\alpha°$.

(b) Find the value of α, giving your answer to the nearest whole number. **[5]**

Solution

(a) Let the velocity before the first collision be

$$\mathbf{u} = 7\mathbf{i} - 4\mathbf{j}$$

and the velocity after the collision be

$$\mathbf{v} = 2\mathbf{i} + 6\mathbf{j}$$

A direction vector for the wall is $2\mathbf{i} + \mathbf{j}$, so a unit vector parallel to the wall is

$$\hat{\mathbf{d}} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

A unit vector perpendicular to the wall is therefore

$$\hat{\mathbf{n}} = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

For any velocity $\mathbf{w}$, its scalar component in the direction of a unit vector $\hat{\mathbf{a}}$ is

$$\mathbf{w} \cdot \hat{\mathbf{a}}$$

Since the wall is smooth, the component parallel to the wall is unchanged.
Before the collision,

$$\mathbf{u} \cdot \hat{\mathbf{d}} = (7\mathbf{i} - 4\mathbf{j}) \cdot \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{14 - 4}{\sqrt{5}} = 2\sqrt{5}$$

After the collision,

$$\mathbf{v} \cdot \hat{\mathbf{d}} = (2\mathbf{i} + 6\mathbf{j}) \cdot \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{4 + 6}{\sqrt{5}} = 2\sqrt{5}$$

So the component parallel to the wall is indeed unchanged.
Now consider the components perpendicular to the wall.
Before the collision,

$$\mathbf{u} \cdot \hat{\mathbf{n}} = (7\mathbf{i} - 4\mathbf{j}) \cdot \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}} = \frac{7 + 8}{\sqrt{5}} = 3\sqrt{5}$$

After the collision,

$$\mathbf{v} \cdot \hat{\mathbf{n}} = (2\mathbf{i} + 6\mathbf{j}) \cdot \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}} = \frac{2 - 12}{\sqrt{5}} = -2\sqrt{5}$$

The negative sign shows that the perpendicular component has reversed direction after the collision.
Therefore

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{2\sqrt{5}}{3\sqrt{5}} = \frac{2}{3}$$

So

$$e = \frac{2}{3}$$

(b) Before the second collision, the particle has velocity

$$\mathbf{u} = 2\mathbf{i} + 6\mathbf{j}$$

A direction vector for the second wall is $\mathbf{i} - 2\mathbf{j}$, so a unit vector parallel to the wall is

$$\hat{\mathbf{d}} = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

A unit vector perpendicular to the wall is therefore

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

The scalar component of $\mathbf{u}$ parallel to the wall is

$$\mathbf{u} \cdot \hat{\mathbf{d}} = (2\mathbf{i} + 6\mathbf{j}) \cdot \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}} = \frac{2 - 12}{\sqrt{5}} = -2\sqrt{5}$$

Hence the vector component of $\mathbf{u}$ parallel to the wall is

$$(-2\sqrt{5})\hat{\mathbf{d}} = (-2\sqrt{5})\frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}} = -2\mathbf{i} + 4\mathbf{j}$$

The scalar component of $\mathbf{u}$ perpendicular to the wall is

$$\mathbf{u} \cdot \hat{\mathbf{n}} = (2\mathbf{i} + 6\mathbf{j}) \cdot \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{4 + 6}{\sqrt{5}} = 2\sqrt{5}$$

Hence the vector component of $\mathbf{u}$ perpendicular to the wall is

$$(2\sqrt{5})\hat{\mathbf{n}} = (2\sqrt{5})\frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = 4\mathbf{i} + 2\mathbf{j}$$

Since the wall is smooth, the parallel component is unchanged.

The coefficient of restitution for the second wall is $\frac{1}{2}$, so the perpendicular component is reversed and multiplied by $\frac{1}{2}$.

Thus the perpendicular component after impact is

$$-\frac{1}{2}(4\mathbf{i} + 2\mathbf{j}) = -2\mathbf{i} - \mathbf{j}$$

Hence the velocity after the second collision is

$$\mathbf{v} = (-2\mathbf{i} + 4\mathbf{j}) + (-2\mathbf{i} - \mathbf{j}) = -4\mathbf{i} + 3\mathbf{j}$$

The deflection angle α is the angle between the velocity before and after the second collision, so

$$\cos\alpha = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

Now

$$\mathbf{u} \cdot \mathbf{v} = (2\mathbf{i} + 6\mathbf{j}) \cdot (-4\mathbf{i} + 3\mathbf{j}) = -8 + 18 = 10$$

Also

$$|\mathbf{u}| = \sqrt{2^2 + 6^2} = \sqrt{40}$$

and

$$|\mathbf{v}| = \sqrt{(-4)^2 + 3^2} = 5$$

Therefore

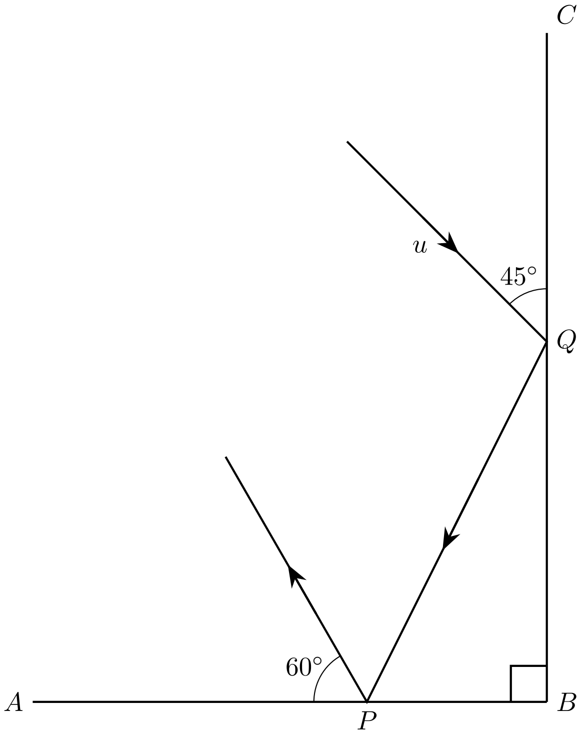
$$\cos\alpha = \frac{10}{5\sqrt{40}} = \frac{1}{\sqrt{10}}$$

So

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 71.6^\circ$$

Therefore, to the nearest whole number,

$$\alpha = 72^\circ$$



8. The diagram above represents the plan view of part of a horizontal floor, where AB and BC are perpendicular vertical walls.
The floor and the walls are modelled as smooth.

A ball is projected along the floor towards BC with speed $u\,\text{m}\,\text{s}^{-1}$ on a path at an angle of 45° to BC. The ball hits BC at Q and then hits AB at P. After striking AB, the ball moves away on a path at an angle of 60° to AB.

The ball is modelled as a particle.

The coefficient of restitution between the ball and wall BC is $\frac{1}{2}$.

(a) Find the coefficient of restitution between the ball and wall AB and hence show that, using this model, the final kinetic energy of the ball is 50% of the initial kinetic energy of the ball. **[8]**

(b) In practice the walls and the floor may not be smooth. Explain how this model is likely to affect the percentage of kinetic energy remaining. **[1]**

Solution

(a) Since the walls are smooth, in each impact the component of velocity *parallel* to the wall is unchanged, and the component *perpendicular* to the wall is reversed and multiplied by the coefficient of restitution.

For the collision with the vertical wall BC:

The ball approaches BC at 45° to BC, so its velocity components are equal in magnitude:

$$\text{perpendicular to } BC = \frac{u}{\sqrt{2}}, \qquad \text{parallel to } BC = \frac{u}{\sqrt{2}}.$$

Given $e_{BC} = \frac{1}{2}$, after striking BC,

$$\text{perpendicular component} = \frac{1}{2} \cdot \frac{u}{\sqrt{2}} = \frac{u}{2\sqrt{2}}$$

in the opposite direction, while

$$\text{parallel component} = \frac{u}{\sqrt{2}}$$

is unchanged.

So, on the path from Q to P, the components of velocity are:

$$\frac{u}{2\sqrt{2}} \text{ parallel to } AB, \qquad \frac{u}{\sqrt{2}} \text{ perpendicular to } AB.$$

Let the coefficient of restitution with wall AB be e.

At AB, the component parallel to AB is unchanged, so after the impact it is still

$$\frac{u}{2\sqrt{2}}.$$

The perpendicular component is reversed and multiplied by e, so after the impact its magnitude is

$$e \cdot \frac{u}{\sqrt{2}}.$$

After striking AB, the ball moves away at 60° to AB. Therefore

$$\tan 60^\circ = \frac{\text{perpendicular component}}{\text{parallel component}} = \frac{e\frac{u}{\sqrt{2}}}{\frac{u}{2\sqrt{2}}} = 2e.$$

Since $\tan 60^\circ = \sqrt{3}$,

$$\sqrt{3} = 2e$$

and so

$$e = \frac{\sqrt{3}}{2}.$$

Hence the coefficient of restitution between the ball and wall AB is

$$\frac{\sqrt{3}}{2}.$$

Now find the final speed.

After the second impact, the components are

$$\frac{u}{2\sqrt{2}} \quad \text{and} \quad \frac{\sqrt{3}}{2} \cdot \frac{u}{\sqrt{2}} = \frac{\sqrt{3}\,u}{2\sqrt{2}}.$$

Therefore

$$\begin{aligned} v^2 &= \left(\frac{u}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}\,u}{2\sqrt{2}}\right)^2 \\ &= \frac{u^2}{8} + \frac{3u^2}{8} \\ &= \frac{u^2}{2}. \end{aligned}$$

Initial kinetic energy:

$$\frac{1}{2}mu^2$$

Final kinetic energy:

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{u^2}{2}\right) = \frac{1}{4}mu^2.$$

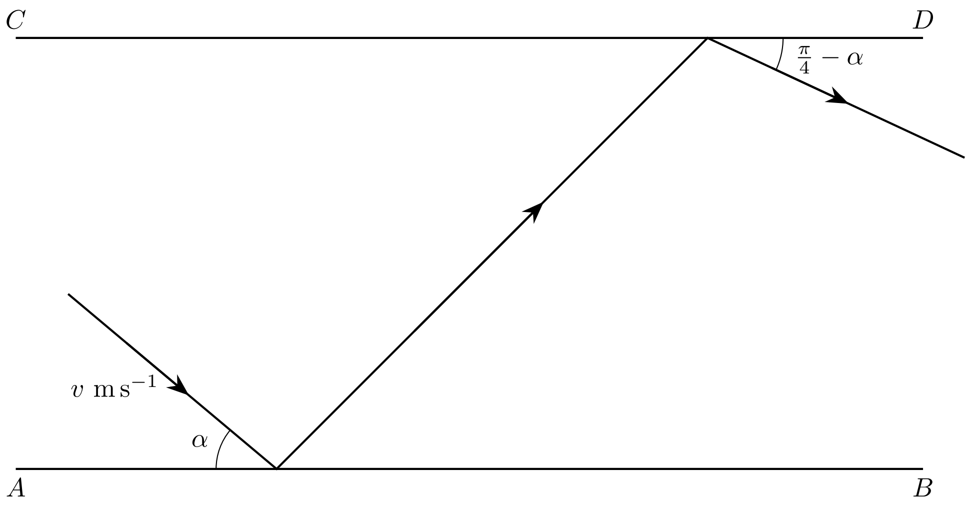
So

$$\frac{\text{final KE}}{\text{initial KE}} = \frac{\frac{1}{4}mu^2}{\frac{1}{2}mu^2} = \frac{1}{2}.$$

Hence the final kinetic energy is 50% of the initial kinetic energy.

(b) If the walls or floor are rough, friction may act during the motion or impacts, so there would be extra loss of kinetic energy and the tangential components might also change.

Therefore the actual percentage of kinetic energy remaining would be less than 50%.



9. The diagram above represents the plan view of part of a horizontal floor, where AB and CD represent fixed vertical walls, with AB parallel to CD.

A small ball is projected along the floor towards wall AB. Immediately before hitting wall AB, the ball is moving with speed $v \text{ m s}^{-1}$ at an angle α to AB, where $0 < \alpha < \frac{\pi}{4}$.

The ball hits wall AB and then hits wall CD.

Immediately after the impact with wall CD, the ball is moving at an angle $\frac{\pi}{4} - \alpha$ to CD.

The coefficient of restitution between the ball and wall AB is $\frac{3}{5}$.

The coefficient of restitution between the ball and wall CD is $\frac{1}{2}$.

The floor and the walls are modelled as being smooth. The ball is modelled as a particle.

(a) Show that

$$\tan\alpha = \frac{2}{3}$$

[7]

(b) Find the percentage of the ball's initial kinetic energy that is lost in the two impacts. **[4]**

Solution

(a) Take components of velocity parallel and perpendicular to the walls.

Immediately before hitting wall AB, the speed is v and the angle to the wall is α, so the components are

$$\text{parallel to } AB: \ v\cos\alpha, \qquad \text{perpendicular to } AB: \ v\sin\alpha$$

Since the wall is smooth, the component parallel to the wall is unchanged in the impact.

The coefficient of restitution with wall AB is $\frac{3}{5}$, so the perpendicular component after impact has magnitude

$$\frac{3}{5}v\sin\alpha$$

Therefore after the first impact, the velocity components are

$$\text{parallel to the walls: } v\cos\alpha, \qquad \text{perpendicular to the walls: } \frac{3}{5}v\sin\alpha$$

As $AB \parallel CD$, these are also the components immediately before striking wall CD.

Now the coefficient of restitution with wall CD is $\frac{1}{2}$, so after impact with CD, the perpendicular component becomes

$$\frac{1}{2}\left(\frac{3}{5}v\sin\alpha\right) = \frac{3}{10}v\sin\alpha$$

The component parallel to the wall is still unchanged, so after the second impact the components are

$$\text{parallel to } CD: \ v\cos\alpha, \qquad \text{perpendicular to } CD: \ \frac{3}{10}v\sin\alpha$$

We are told that after this impact the ball moves at angle $\frac{\pi}{4} - \alpha$ to CD, so

$$\begin{aligned}\tan\left(\frac{\pi}{4} - \alpha\right) &= \frac{\frac{3}{10}v\sin\alpha}{v\cos\alpha} \\ &= \frac{3}{10}\tan\alpha\end{aligned}$$

Let $t = \tan\alpha$. Using

$$\tan\left(\frac{\pi}{4} - \alpha\right) = \frac{1-t}{1+t}$$

we get

$$\begin{aligned}\frac{1-t}{1+t} &= \frac{3t}{10} \\ 10(1-t) &= 3t(1+t) \\ 10 - 10t &= 3t + 3t^2 \\ 3t^2 + 13t - 10 &= 0 \\ (3t-2)(t+5) &= 0\end{aligned}$$

So

$$t = \frac{2}{3} \quad \text{or} \quad t = -5$$

But $0 < \alpha < \frac{\pi}{4}$, so $0 < \tan\alpha < 1$. Hence $t = -5$ is impossible.

Therefore

$$\tan\alpha = \frac{2}{3}$$

(b) The floor is smooth, so kinetic energy changes only in the two impacts. Hence we can compare the initial and final kinetic energies.

Let the mass of the ball be m.

Initially,

$$\text{KE}_{\text{initial}} = \frac{1}{2}mv^2$$

After the second impact, the velocity components are

$$v\cos\alpha \qquad \text{and} \qquad \frac{3}{10}v\sin\alpha$$

So the final speed squared is

$$\begin{aligned}(v\cos\alpha)^2 + \left(\frac{3}{10}v\sin\alpha\right)^2 &= v^2\cos^2\alpha + \frac{9}{100}v^2\sin^2\alpha \\ &= v^2\left(\cos^2\alpha + \frac{9}{100}\sin^2\alpha\right)\end{aligned}$$

Hence

$$\frac{\text{KE}_{\text{final}}}{\text{KE}_{\text{initial}}} = \cos^2\alpha + \frac{9}{100}\sin^2\alpha$$

So the fraction of kinetic energy lost is

$$\begin{aligned} 1 - \left(\cos^2\alpha + \frac{9}{100}\sin^2\alpha\right) &= \sin^2\alpha - \frac{9}{100}\sin^2\alpha \\ &= \frac{91}{100}\sin^2\alpha \end{aligned}$$

From part (a), $\tan\alpha = \dfrac{2}{3}$. Therefore

$$\begin{aligned} \sin^2\alpha &= \frac{\tan^2\alpha}{1+\tan^2\alpha} \\ &= \frac{\left(\frac{2}{3}\right)^2}{1+\left(\frac{2}{3}\right)^2} \\ &= \frac{4/9}{13/9} \\ &= \frac{4}{13} \end{aligned}$$

Hence the fraction of kinetic energy lost is

$$\begin{aligned} \frac{91}{100}\cdot\frac{4}{13} &= \frac{364}{1300} \\ &= \frac{7}{25} \end{aligned}$$

Therefore the percentage of the ball's initial kinetic energy lost is

$$28\%$$

10. A particle P of mass m is moving downwards and to the right in a direction making an angle 45° with the horizontal when it strikes a fixed smooth inclined plane. The plane is inclined to the horizontal at an angle α, where $0 < \alpha < 45°$.

At the instant immediately before the impact, the speed of P is u.

At the instant immediately after the impact, P is moving upwards and to the right in a direction making an angle 45° with the horizontal, with speed v.

(a) Show that the magnitude of the impulse exerted on the plane by P is $mu\sec(45° - \alpha)$. **[5]**

The coefficient of restitution between P and the plane is e, where $0 < e < 1$.

(b) Show that **[3]**

$$v^2 = u^2\left(\cos^2(45° + \alpha) + e^2\sin^2(45° + \alpha)\right)$$

(c) Show that the kinetic energy lost by P in the impact is

$$\frac{1}{2}mu^2(1 - e^2)\sin^2(45° + \alpha)$$ **[2]**

(d) Hence find, in terms of m, u and e only, the kinetic energy lost by P in the impact. **[2]**

Solution

(a) Resolve parallel and perpendicular to the plane.

The angle between the initial direction and the plane is $45° + \alpha$, and the angle between the final direction and the plane is $45° - \alpha$.

So, taking positive parallel to the plane upwards, and positive perpendicular to the plane away from the plane:

$$\text{before impact:}\begin{cases}\text{parallel component} &= u\cos(45° + \alpha)\\ \text{normal component} &= -u\sin(45° + \alpha)\end{cases}$$

$$\text{after impact:}\begin{cases}\text{parallel component} &= v\cos(45° - \alpha)\\ \text{normal component} &= v\sin(45° - \alpha)\end{cases}$$

Since the plane is smooth, there is no impulse parallel to the plane, so the parallel component of velocity is unchanged:

$$u\cos(45° + \alpha) = v\cos(45° - \alpha)$$

Hence

$$v = \frac{u\cos(45° + \alpha)}{\cos(45° - \alpha)}$$

Using $\cos(45° + \alpha) = \sin(45° - \alpha)$,

$$v = u\tan(45° - \alpha)$$

The impulse on P is perpendicular to the plane, so its magnitude is the change in normal momentum:

$$J = m\left(v\sin(45° - \alpha) - (-u\sin(45° + \alpha))\right)$$

$$J = m\left(v\sin(45^\circ - \alpha) + u\sin(45^\circ + \alpha)\right)$$

Now use $\sin(45^\circ + \alpha) = \cos(45^\circ - \alpha)$ and $v = u\tan(45^\circ - \alpha)$:

$$\begin{aligned}
J &= m\left(u\tan(45^\circ - \alpha)\sin(45^\circ - \alpha) + u\cos(45^\circ - \alpha)\right)\\
&= mu\left(\frac{\sin^2(45^\circ - \alpha)}{\cos(45^\circ - \alpha)} + \cos(45^\circ - \alpha)\right)\\
&= mu\left(\frac{\sin^2(45^\circ - \alpha) + \cos^2(45^\circ - \alpha)}{\cos(45^\circ - \alpha)}\right)\\
&= mu\sec(45^\circ - \alpha)
\end{aligned}$$

By Newton's third law, the impulse exerted on the plane by P has the same magnitude. Hence the magnitude of the impulse exerted on the plane by P is

$$mu\sec(45^\circ - \alpha)$$

(b) The coefficient of restitution is

$$e = \frac{\text{speed of separation perpendicular to plane}}{\text{speed of approach perpendicular to plane}}$$

So

$$e = \frac{v\sin(45^\circ - \alpha)}{u\sin(45^\circ + \alpha)}$$

Hence

$$v\sin(45^\circ - \alpha) = eu\sin(45^\circ + \alpha)$$

Also, from the smoothness of the plane,

$$v\cos(45^\circ - \alpha) = u\cos(45^\circ + \alpha)$$

Now square both equations and add:

$$v^2\sin^2(45^\circ - \alpha) + v^2\cos^2(45^\circ - \alpha) = e^2u^2\sin^2(45^\circ + \alpha) + u^2\cos^2(45^\circ + \alpha)$$

So

$$v^2\left(\sin^2(45^\circ - \alpha) + \cos^2(45^\circ - \alpha)\right) = u^2\left(\cos^2(45^\circ + \alpha) + e^2\sin^2(45^\circ + \alpha)\right)$$

Since $\sin^2\theta + \cos^2\theta = 1$,

$$v^2 = u^2\left(\cos^2(45^\circ + \alpha) + e^2\sin^2(45^\circ + \alpha)\right)$$

as required.

(c) The kinetic energy lost is

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(u^2 - v^2)$$

Using part (b),

$$\begin{aligned} u^2 - v^2 &= u^2 - u^2\left(\cos^2(45^\circ + \alpha) + e^2\sin^2(45^\circ + \alpha)\right) \\ &= u^2\left(1 - \cos^2(45^\circ + \alpha) - e^2\sin^2(45^\circ + \alpha)\right) \\ &= u^2\left(\sin^2(45^\circ + \alpha) - e^2\sin^2(45^\circ + \alpha)\right) \\ &= u^2(1 - e^2)\sin^2(45^\circ + \alpha) \end{aligned}$$

Therefore the kinetic energy lost is

$$\frac{1}{2}mu^2(1 - e^2)\sin^2(45^\circ + \alpha)$$

(d) From part (a),

$$v = u\tan(45^\circ - \alpha)$$

Using the restitution equation from part (b),

$$e = \frac{v\sin(45^\circ - \alpha)}{u\sin(45^\circ + \alpha)}$$

Let $\theta = 45^\circ - \alpha$. Then $\sin(45^\circ + \alpha) = \cos\theta$, so

$$\begin{aligned} e &= \frac{u\tan\theta\sin\theta}{u\cos\theta} \\ &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta \end{aligned}$$

Hence

$$e = \tan^2(45^\circ - \alpha)$$

Also,

$$\sin^2(45^\circ + \alpha) = \cos^2(45^\circ - \alpha)$$

and since $1 + \tan^2\theta = \sec^2\theta$,

$$\cos^2\theta = \frac{1}{1 + \tan^2\theta}$$

So

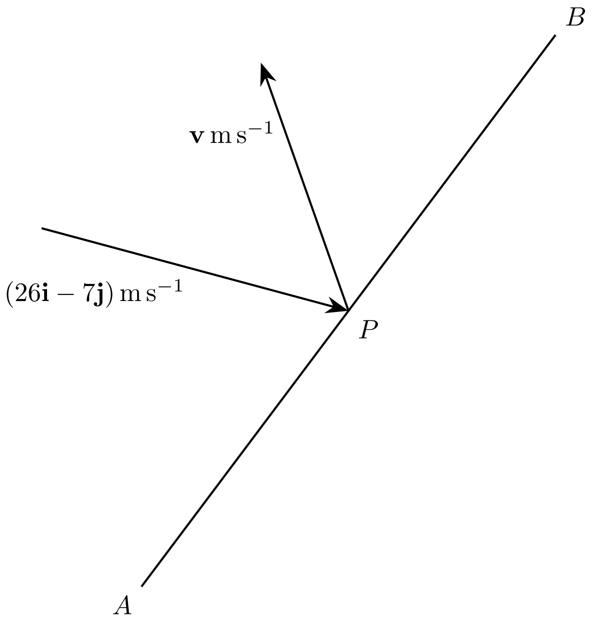
$$\sin^2(45^\circ + \alpha) = \frac{1}{1 + e}$$

Substitute this into the result from part (c):

$$\begin{aligned} \text{KE lost} &= \frac{1}{2}mu^2(1 - e^2)\sin^2(45^\circ + \alpha) \\ &= \frac{1}{2}mu^2(1 - e^2)\cdot\frac{1}{1 + e} \\ &= \frac{1}{2}mu^2(1 - e) \end{aligned}$$

Hence the kinetic energy lost is

$$\frac{1}{2}mu^2(1 - e)$$



11. The diagram above represents the plan view of part of a smooth horizontal billiards table, where AB is a fixed smooth cushion.

The direction of $\overrightarrow{AB}$ is parallel to the vector $(3\mathbf{i} + 4\mathbf{j})$.

A small disc of mass 0.2 kg is moving on the table when it strikes the cushion AB.

Immediately before impact with AB, the velocity of the disc is $(26\mathbf{i} - 7\mathbf{j})\,\mathrm{m\,s^{-1}}$.

Immediately after impact with AB, the velocity of the disc is $\mathbf{v}\,\mathrm{m\,s^{-1}}$.

The coefficient of restitution between the disc and the cushion is $\frac{3}{5}$.

By modelling the disc as a particle,

(a) show that $\mathbf{v} = -6\mathbf{i} + 17\mathbf{j}$ **[6]**

(b) Find the magnitude of the impulse exerted by the cushion on the disc in the impact. **[3]**

Solution

(a) Let the velocity before impact be

$$\mathbf{u} = 26\mathbf{i} - 7\mathbf{j}.$$

Since the direction of the cushion AB is parallel to $3\mathbf{i} + 4\mathbf{j}$, a unit vector along the cushion is

$$\hat{\mathbf{t}} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}).$$

A unit vector perpendicular to the cushion is

$$\hat{\mathbf{n}} = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j}).$$

Resolve $\mathbf{u}$ parallel and perpendicular to the cushion.

Parallel component:

$$\mathbf{u} \cdot \hat{\mathbf{t}} = (26\mathbf{i} - 7\mathbf{j}) \cdot \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) = \frac{1}{5}(78 - 28) = 10.$$

Perpendicular component:

$$\mathbf{u}\cdot\hat{\mathbf{n}} = (26\mathbf{i}-7\mathbf{j})\cdot\frac{1}{5}(4\mathbf{i}-3\mathbf{j}) = \frac{1}{5}(104+21) = 25.$$

So

$$\mathbf{u} = 10\hat{\mathbf{t}} + 25\hat{\mathbf{n}}.$$

Because the cushion is smooth, the component parallel to the cushion is unchanged in the impact. Hence after impact, the tangential component is still $10\hat{\mathbf{t}}$.

The coefficient of restitution is $\frac{3}{5}$, so

$$\text{speed of separation along the normal} = \frac{3}{5}\times\text{speed of approach along the normal} = \frac{3}{5}\times 25 = 15.$$

Since the normal component reverses direction on impact, the normal component after impact is

$$-15\hat{\mathbf{n}}.$$

Therefore

$$\mathbf{v} = 10\hat{\mathbf{t}} - 15\hat{\mathbf{n}}.$$

Substituting for $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$:

$$\begin{aligned}\mathbf{v} &= 10\left(\frac{3\mathbf{i}+4\mathbf{j}}{5}\right) - 15\left(\frac{4\mathbf{i}-3\mathbf{j}}{5}\right)\\ &= 2(3\mathbf{i}+4\mathbf{j}) - 3(4\mathbf{i}-3\mathbf{j})\\ &= (6\mathbf{i}+8\mathbf{j}) - (12\mathbf{i}-9\mathbf{j})\\ &= -6\mathbf{i}+17\mathbf{j}.\end{aligned}$$

Hence

$$\mathbf{v} = -6\mathbf{i}+17\mathbf{j}.$$

(b) The impulse exerted by the cushion on the disc is the change in momentum:

$$\mathbf{J} = m(\mathbf{v}-\mathbf{u}).$$

Here $m = 0.2$, $\mathbf{v} = -6\mathbf{i}+17\mathbf{j}$ and $\mathbf{u} = 26\mathbf{i}-7\mathbf{j}$, so

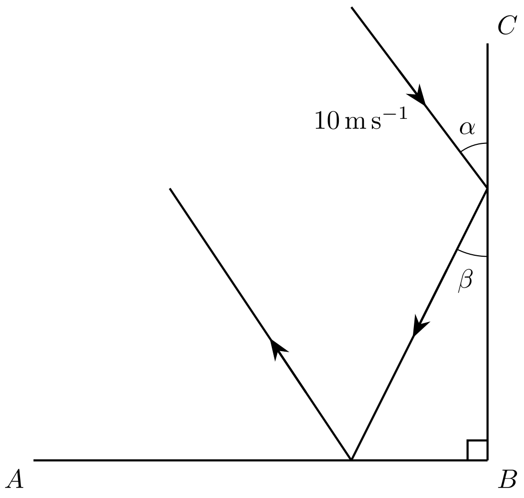
$$\begin{aligned}\mathbf{J} &= 0.2\big[(-6-26)\mathbf{i} + (17-(-7))\mathbf{j}\big]\\ &= 0.2(-32\mathbf{i}+24\mathbf{j}).\end{aligned}$$

Its magnitude is

$$\begin{aligned}|\mathbf{J}| &= 0.2\sqrt{(-32)^2+24^2}\\ &= 0.2\sqrt{1024+576}\\ &= 0.2\sqrt{1600}\\ &= 0.2(40) = 8.\end{aligned}$$

So the magnitude of the impulse is

$$8\ \mathrm{N\,s}.$$



12. The diagram above represents the plan view of one corner of a smooth air-hockey table, where AB and BC are fixed cushions with AB perpendicular to BC.

A small puck is struck along the table towards BC with speed $10\,\mathrm{m\,s^{-1}}$ on a path that makes an angle α with BC, where $\tan\alpha = \frac{3}{4}$. The puck hits BC and then hits AB.

Immediately after hitting BC, the puck is moving at an angle β to BC, where $\tan\beta = \frac{1}{2}$.

The coefficient of restitution between the puck and BC is e.

The coefficient of restitution between the puck and AB is $\frac{3}{4}$.

By modelling the puck as a particle and the table and cushions as being smooth,

(a) show that $e = \frac{2}{3}$ **[5]**

(b) find the speed of the puck immediately after it hits AB. **[4]**

(c) Suggest two ways in which the model could be refined to make it more realistic. **[2]**

Solution

(a) Resolve the velocity into components *perpendicular* and *parallel* to the cushion BC.

Since $\tan\alpha = \frac{3}{4}$, we can use the 3-4-5 triangle:

$$\sin\alpha = \frac{3}{5}, \qquad \cos\alpha = \frac{4}{5}$$

Before hitting BC, the speed is $10\,\mathrm{m\,s^{-1}}$, so

$$\text{component perpendicular to } BC = 10\sin\alpha = 10\cdot\frac{3}{5} = 6$$
$$\text{component parallel to } BC = 10\cos\alpha = 10\cdot\frac{4}{5} = 8$$

Because the cushion is smooth, the component parallel to BC is unchanged by the impact, so after hitting BC it is still 8.

Let the coefficient of restitution with BC be e. Then

$$\text{speed of separation perpendicular to } BC = e \times \text{speed of approach perpendicular to } BC$$

so the perpendicular component after impact is

$$6e$$

After impact, the puck moves at angle β to BC, with $\tan\beta = \frac{1}{2}$. Therefore

$$\tan\beta = \frac{\text{perpendicular component}}{\text{parallel component}} = \frac{6e}{8}$$

Hence

$$\frac{6e}{8} = \frac{1}{2}$$

$$\begin{aligned} 12e &= 8 \\ e &= \frac{8}{12} = \frac{2}{3} \end{aligned}$$

So

$$e = \frac{2}{3}$$

(b) From part (a), after hitting BC the perpendicular component is

$$6e = 6 \cdot \frac{2}{3} = 4$$

and the parallel component is unchanged at

$$8$$

So, just before hitting AB, the puck has velocity components:

- parallel to AB: $4\,\mathrm{m\,s^{-1}}$,
- perpendicular to AB: $8\,\mathrm{m\,s^{-1}}$.

Now consider the impact with AB, where the coefficient of restitution is $\frac{3}{4}$.

Since AB is smooth, the component parallel to AB is unchanged, so it remains

$$4\,\mathrm{m\,s^{-1}}$$

The perpendicular component reverses direction and is multiplied by $\frac{3}{4}$, so after impact its magnitude is

$$\frac{3}{4} \times 8 = 6\,\mathrm{m\,s^{-1}}$$

Therefore the speed immediately after hitting AB is

$$\begin{aligned} \sqrt{4^2 + 6^2} &= \sqrt{16 + 36} \\ &= \sqrt{52} \\ &= 2\sqrt{13} \end{aligned}$$

So the speed is

$$2\sqrt{13}\,\mathrm{m\,s^{-1}}$$

which is approximately

$$7.21\,\mathrm{m\,s^{-1}}$$

(c) Two possible refinements are:

- include friction or air resistance, rather than treating the table as perfectly smooth;
- allow the puck to have size and rotation/spin, rather than modelling it as a particle.