FURTHER MECHANICS / MISCELLANEOUS MECHANICS

Rigid Bodies, Sliding & Toppling

Worked solutions 15 questions · 180 marks

EXAM BOARDS Edexcel FM2 9FM0/4C · AQA 7367/3M · OCR A Y543 · OCR MEI Mech Minor Y431 OCR MEI Mech Major Y421 · CIE 9231 Paper 3

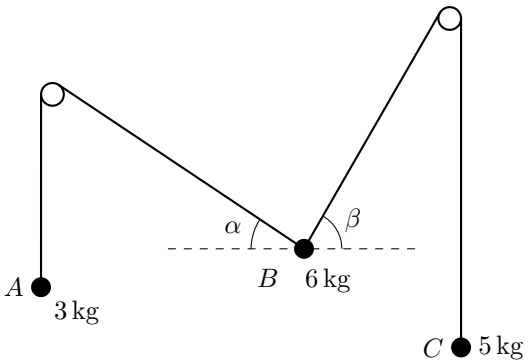
Instructions

Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question gives a value. Where a question leaves g unspecified and a final answer depends on it, the solutions also give that answer for $g = 10\,\mathrm{m\,s^{-2}}$.

CIE (9231 Paper 3): papers take $g = 10\,\mathrm{m\,s^{-2}}$; use the $g = 10$ final answers where the solutions give them.

Questions

Question 1	*6 marks*	2
Question 2	*9 marks*	4
Question 3	*17 marks*	6
Question 4	*13 marks*	8
Question 5	*9 marks*	11
Question 6	*10 marks*	13
Question 7	*8 marks*	15
Question 8	*12 marks*	17
Question 9	*12 marks*	19
Question 10	*14 marks*	22
Question 11	*16 marks*	25
Question 12	*15 marks*	28
Question 13	*14 marks*	30
Question 14	*14 marks*	33
Question 15	*11 marks*	36



1. One end of a light inextensible string is attached to a particle A of mass 3 kg. The other end of the string is attached to a particle B of mass 6 kg.
A second light inextensible string has one end attached to B and its other end attached to a particle C of mass 5 kg. Each string passes over a small smooth fixed pulley, and both strings are taut.
The system is in equilibrium with all three particles at rest in the same vertical plane, and with A and C hanging vertically below their pulleys. As shown in the diagram above, the portion of string between B and the pulley above A makes an angle α with the horizontal, and the portion of string between B and the pulley above C makes an angle β with the horizontal.

(a) Draw a closed figure to represent the three forces acting on B. **[1]**

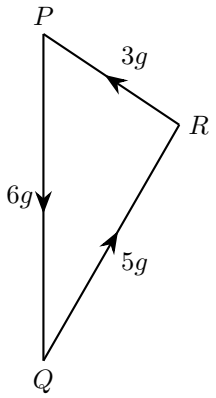
(b) Hence, or otherwise, find the following, giving each answer in degrees to 1 decimal place.

- The value of α
- The value of β

[5]

Solution

(a) The pulleys are smooth and A and C hang in equilibrium, so the tension in the string attached to A is $3g$ N and the tension in the string attached to C is $5g$ N. The forces on B are its weight $6g$ N vertically downwards, the tension $3g$ N along the string at α above the horizontal to the left, and the tension $5g$ N along the string at β above the horizontal to the right. Since B is in equilibrium, these form a closed triangle of forces, with the arrows following each other round the triangle:



(b) The side PQ is vertical and the side RP is parallel to the string at α to the horizontal, so $\angle QPR = 90° - \alpha$. Using the cosine rule in triangle PQR, where g cancels:

$$\cos(90° - \alpha) = \frac{6^2 + 3^2 - 5^2}{2 \times 6 \times 3} = \frac{20}{36} = \frac{5}{9}$$

Since $\cos(90° - \alpha) = \sin\alpha$:

$$\sin\alpha = \frac{5}{9} \quad \Longrightarrow \quad \alpha = 33.7° \qquad (1 \text{ d.p.})$$

The side QR is parallel to the string at β to the horizontal, so $\angle PQR = 90° - \beta$. Using the cosine rule again:

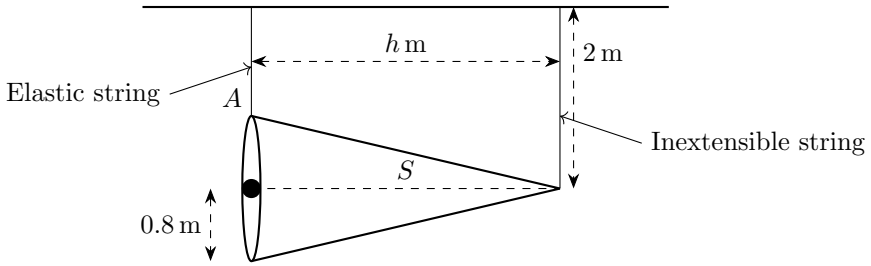
$$\cos(90° - \beta) = \frac{6^2 + 5^2 - 3^2}{2 \times 6 \times 5} = \frac{52}{60} = \frac{13}{15}$$

so

$$\sin\beta = \frac{13}{15} \quad \Longrightarrow \quad \beta = 60.1° \qquad (1 \text{ d.p.})$$

2. A shape S is formed by attaching a particle of mass $2m$ kg to the centre of the base of a uniform solid cone of mass $4m$ kg.
The height of the cone is h m and the radius of its base is 0.8 m.

(a) Explain why the centre of mass of S lies on the axis of the cone. **[1]**



Two light strings are attached to S. A light inextensible string of length 2 m is attached at the vertex of the cone, and a light elastic string of modulus of elasticity $15mg$ N is attached at the point A on the edge of the base.
The other ends of the strings are attached to a horizontal ceiling so that both strings are vertical.
Shape S hangs in equilibrium with the axis of the cone horizontal and A vertically above the centre of the base, as shown in the diagram above.

(b) Find, in terms of m and g, the tension in each string. **[5]**

(c) Determine the natural length of the elastic string. **[3]**

Solution

(a) The cone is uniform, so it is symmetric about its axis and its centre of mass lies on the axis. The particle is at the centre of the base, which is also on the axis, so the centre of mass of S lies on the axis of the cone.

(b) The centre of mass of a uniform solid cone lies on its axis, a quarter of the height from the base, so the cone's centre of mass is $\frac{h}{4}$ m from the base. Let $\bar{x}$ m be the distance of the centre of mass of S from the centre of the base, measured along the axis towards the vertex. The total mass is $4m + 2m = 6m$.
For a body made of pieces, the centre of mass is given by

$$M\bar{x} = \sum m_i x_i, \qquad \text{where } M = \sum m_i$$

The particle is at the centre of the base, so its distance is 0:

$$6m\bar{x} = 4m \times \frac{h}{4} + 2m \times 0$$

$$6m\bar{x} = mh$$

$$\bar{x} = \frac{h}{6}$$

Let the tension in the inextensible string be T_1 N and the tension in the elastic string be T_2 N. The point A is vertically above the centre of the base, so the weight $6mg$ acts at a horizontal distance $\frac{h}{6}$ from A, and the inextensible string acts at a horizontal distance h from A.

Taking moments about A:

$$T_1 \times h = 6mg \times \frac{h}{6}$$

$$T_1 = mg$$

Resolving vertically:

$$T_1 + T_2 = 6mg$$

$$mg + T_2 = 6mg$$

$$T_2 = 5mg$$

The tension in the inextensible string is mg N and the tension in the elastic string is $5mg$ N.

(c) The inextensible string is vertical with length 2 m, and the axis is horizontal, so the axis is 2 m below the ceiling. The point A is 0.8 m above the centre of the base, so it is $(2 - 0.8)$ m $= 1.2$ m below the ceiling, and the stretched elastic string has length 1.2 m.

Let the natural length be l m, so the extension is $1.2 - l$. Hooke's law is

$$T = \frac{\lambda e}{l}$$

where e is the extension and l the natural length. With $T = 5mg$, $\lambda = 15mg$ and $e = 1.2 - l$:

$$5mg = \frac{15mg(1.2 - l)}{l}$$

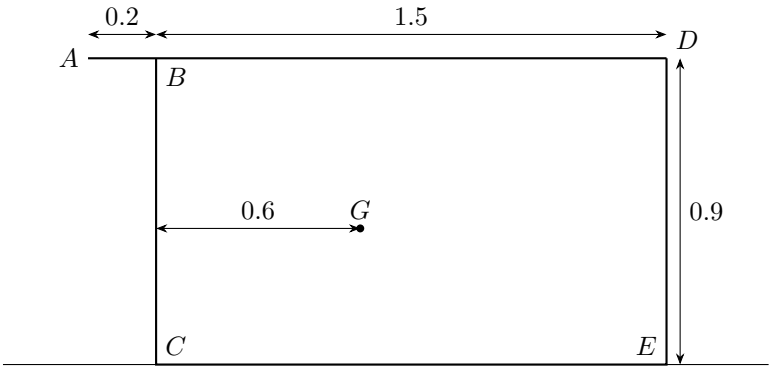
$$5l = 15(1.2 - l)$$

$$5l = 18 - 15l$$

$$20l = 18$$

$$l = 0.9$$

The natural length of the elastic string is 0.9 m.

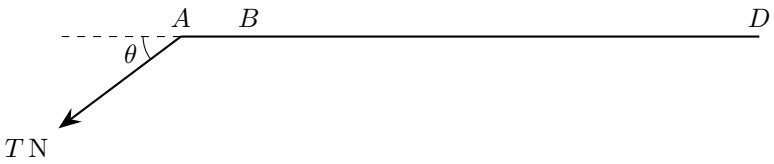


3. A side view of a free-standing cupboard on a horizontal floor is shown in the diagram above. The cupboard consists of a base CE, vertical ends BC and DE, and a horizontal top AD which overhangs the end BC. The dimensions of the cupboard, in metres, are shown in the diagram. The cupboard and its contents have a total weight of 350 N and centre of mass at G.

(a) Calculate the magnitude of the vertical force required for the cupboard to be on the point of tipping in each of the following cases.

(i) The force acts downwards at A. **[3]**

(ii) The force acts upwards at D. **[3]**



A force of magnitude T N is now applied at A, acting downwards and away from the cupboard at an angle θ to the horizontal, as shown in the diagram above, where $\tan\theta = \frac{3}{4}$.

(b) By considering the horizontal and vertical components of the force at A, show that the anticlockwise moment of this force about C is $\frac{21T}{25}$ N m. **[4]**

With this force acting, the cupboard is in equilibrium and is on the point of tipping but not on the point of sliding.

(c) Find the value of T, and find the range of possible values of the coefficient of friction between the base of the cupboard and the floor. **[7]**

Solution

(a) (i) Let the downward force at A have magnitude P. On the point of tipping, the cupboard is about to turn about C, with E about to lift off the floor, so the whole reaction from the floor acts at C.

A is 0.2 m to the left of C and G is 0.6 m to the right of C. Taking moments about C:

$$0.2P = 350 \times 0.6$$

$$0.2P = 210$$

$$P = 1050 \text{ N}$$

(ii) Let the upward force at D have magnitude Q. Again the cupboard is about to turn about C, with E about to lift off the floor, so the whole reaction from the floor acts at C.

D is 1.5 m to the right of C. Taking moments about C:

$$1.5Q = 350 \times 0.6$$

$$1.5Q = 210$$

$$Q = 140\,\text{N}$$

(b) Since $\tan\theta = \frac{3}{4}$, a 3, 4, 5 right-angled triangle gives $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$. The components of the force at A are therefore

$$\text{horizontal (away from the cupboard)}: \quad T\cos\theta = \frac{4T}{5}$$

$$\text{vertical (downwards)}: \quad T\sin\theta = \frac{3T}{5}$$

A is 0.2 m to the left of C and 0.9 m above it. The downward component acts to the left of C, and the horizontal component acts to the left at a height of 0.9 m above C, so both turn the cupboard anticlockwise about C. The anticlockwise moment about C is

$$\frac{3T}{5} \times 0.2 + \frac{4T}{5} \times 0.9 = \frac{3T}{25} + \frac{18T}{25}$$

$$= \frac{21T}{25}\,\text{N m}$$

as required.

(c) On the point of tipping, the cupboard is about to turn about C, so the normal reaction R and the friction F from the floor both act at C. The weight acts 0.6 m to the right of C, giving a clockwise moment about C of $350 \times 0.6 = 210\,\text{N m}$.

Taking moments about C, using part (b):

$$\frac{21T}{25} = 210$$

$$T = 250$$

The horizontal component of the force at A acts away from the cupboard, so friction acts towards the cupboard. Resolving horizontally:

$$F = \frac{4T}{5} = \frac{4}{5} \times 250 = 200\,\text{N}$$

Resolving vertically:

$$R = 350 + \frac{3T}{5} = 350 + \frac{3}{5} \times 250 = 500\,\text{N}$$

The cupboard is not on the point of sliding, so friction has not reached its limiting value:

$$F < \mu R$$

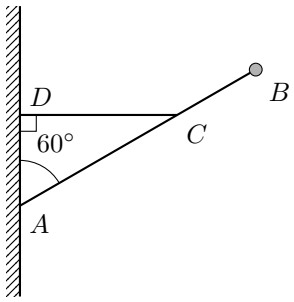
$$200 < 500\mu$$

$$\mu > 0.4$$

So $T = 250$, and the coefficient of friction satisfies $\mu > 0.4$.

4. A flagpole, AB, is 3 m long. The flagpole is modelled as a non-uniform rod so that, at a distance x metres from A, the mass per unit length of the flagpole, m kg m^{-1}, is given by $m = 20 - 4x$.

(a) Find the mass of the flagpole, and show that its centre of mass is at a distance of $\frac{9}{7}$ m from A. **[5]**



The end A of the flagpole is fixed to a point on a vertical wall. The flagpole lies in a vertical plane perpendicular to the wall and makes an angle of 60° with the upward vertical through A. A cable has one end attached to the point C on the flagpole, where $AC = 2$ m, and the other end attached to the point D on the wall, vertically above A. The cable is horizontal. A small ball of mass 5 kg is attached to the flagpole at B. The flagpole and ball rest in equilibrium, as shown in the diagram above.
The cable is modelled as a light inextensible string and the ball is modelled as a particle.

(b) Using the model, find the tension in the cable and the magnitude of the force exerted on the flagpole by the wall at A. **[7]**

(c) In reality the ball has a small but non-zero radius, and it is attached so that its centre lies on AB produced, a short distance beyond B. State, with a reason, whether the true tension in the cable is likely to be greater or less than your answer to part (b). **[1]**

Solution

(a) For a rod of length 3 m with mass per unit length m, the mass M and the distance $\bar{x}$ of the centre of mass from A are given by

$$M = \int_0^3 m \, \mathrm{d}x, \qquad M\bar{x} = \int_0^3 xm \, \mathrm{d}x$$

The mass is

$$\begin{aligned} M &= \int_0^3 (20 - 4x) \, \mathrm{d}x \\ &= \left[20x - 2x^2\right]_0^3 \\ &= 60 - 18 \\ &= 42 \end{aligned}$$

so the mass of the flagpole is 42 kg.

For the centre of mass:

$$\begin{aligned} M\bar{x} &= \int_0^3 (20x - 4x^2)\,\mathrm{d}x \\ &= \left[10x^2 - \tfrac{4}{3}x^3\right]_0^3 \\ &= 90 - 36 \\ &= 54 \end{aligned}$$

Hence

$$\bar{x} = \frac{54}{42} = \frac{9}{7}$$

so the centre of mass is $\frac{9}{7}$ m from A, as required.

(b) The flagpole makes $60°$ with the upward vertical, so a point a distance s along it from A is $s\cos 60°$ above A and $s\sin 60°$ horizontally from the wall. Hence C is $2\cos 60° = 1$ m above A, the centre of mass of the flagpole is $\frac{9}{7}\sin 60°$ m from the wall and B is $3\sin 60°$ m from the wall.

Let the tension in the cable be T N; it acts horizontally at C, towards the wall. The weights are $42g$ N at the centre of mass and $5g$ N at B.

Taking moments about A:

$$\begin{aligned} T \times 1 &= 42g \times \tfrac{9}{7}\sin 60° + 5g \times 3\sin 60° \\ &= (54g + 15g) \times \frac{\sqrt{3}}{2} \\ &= \frac{69\sqrt{3}\,g}{2} \end{aligned}$$

so

$$T = \frac{69\sqrt{3} \times 9.8}{2} \approx 586$$

so the tension in the cable is 586 N.

Let the force exerted by the wall at A have horizontal component H N, away from the wall, and vertical component V N, upwards.

Resolving horizontally:

$$H = T = \frac{69\sqrt{3}\,g}{2}$$

Resolving vertically:

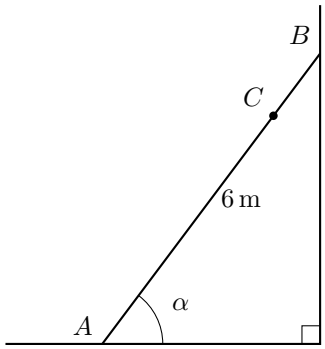
$$V = 42g + 5g = 47g$$

The magnitude of the force at A is

$$\begin{aligned} \sqrt{H^2 + V^2} &= \sqrt{\left(\frac{69\sqrt{3}\,g}{2}\right)^2 + (47g)^2} \\ &= \frac{g}{2}\sqrt{14\,283 + 8836} \\ &= \frac{9.8}{2}\sqrt{23\,119} \\ &\approx 745\,\text{N} \end{aligned}$$

(c) The true tension is likely to be greater. The centre of the ball lies further from A than B, so the moment of the ball's weight about A is larger, and the cable must provide a larger moment to balance it.

Final answers with $g = 10$: (b) tension $598\,\mathrm{N}$, magnitude of the force at A $760\,\mathrm{N}$



5. A uniform ladder AB, of mass 20 kg and length 6 m, rests with end A on rough horizontal ground. The end B of the ladder rests against a rough vertical wall.
The ladder is inclined at an angle α to the ground, where $\tan\alpha = \frac{4}{3}$, as shown in the diagram above.
The coefficient of friction between the ladder and the ground is $\frac{1}{2}$.
The coefficient of friction between the ladder and the wall is $\frac{1}{3}$.
A man of mass 60 kg starts at A and slowly climbs the ladder. The ladder is on the point of slipping when the man reaches the point C on the ladder.
The ladder is modelled as a uniform rod which lies in a vertical plane perpendicular to the wall, and the man is modelled as a particle.

(a) Find the magnitude of the normal reaction between the ladder and the wall at B when the man is at C. **[5]**

(b) Find the distance AC. **[4]**

Solution

(a) Let the normal reactions at A and B be R and S, and the friction forces at A and B be F_A and F_B. The weights of the ladder and the man are $20g$ and $60g$.

The ladder is on the point of slipping, so friction is limiting at both ends. The foot A tends to slide away from the wall, so the friction at A acts towards the wall. The top B tends to slide down the wall, so the friction at B acts upwards. Hence

$$F_A = \frac{1}{2}R, \qquad F_B = \frac{1}{3}S$$

Resolving horizontally:

$$F_A = S \implies \frac{1}{2}R = S \implies R = 2S$$

Resolving vertically:

$$R + F_B = 20g + 60g$$

$$2S + \frac{1}{3}S = 80g$$

$$\frac{7}{3}S = 80g$$

$$S = \frac{240g}{7} = \frac{240 \times 9.8}{7} = 336\,\text{N}$$

(b) Let $AC = x$ m. Since $\tan\alpha = \frac{4}{3}$, $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$.

The weight of the ladder acts at its midpoint, 3 m from A. The normal reaction S at B is horizontal and acts at a height $6\sin\alpha$ above A, and the friction F_B at B is vertical and acts at a horizontal distance $6\cos\alpha$ from A.

Taking moments about A:

$$20g \times 3\cos\alpha + 60g \times x\cos\alpha = S \times 6\sin\alpha + F_B \times 6\cos\alpha$$

$$20g \times \frac{9}{5} + 60g \times \frac{3x}{5} = \frac{24}{5}S + \frac{1}{3}S \times \frac{18}{5}$$

$$36g + 36gx = 6S$$

Using $S = \frac{240g}{7}$ from part (a):

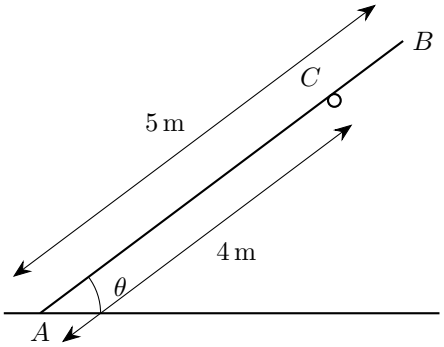
$$36g + 36gx = \frac{1440g}{7}$$

$$1 + x = \frac{40}{7}$$

$$x = \frac{33}{7}$$

So $AC = \frac{33}{7}\,\text{m} \approx 4.71\,\text{m}$.

Final answers with $g = 10$: (a) 343 N



6. A uniform rod AB has mass 6 kg and length 5 m. The end A of the rod is in contact with rough horizontal ground.
The rod rests in equilibrium on a smooth horizontal peg at the point C of the rod, where $AC = 4$ m, and is inclined at an angle θ to the ground, where $\tan\theta = \frac{3}{4}$, as shown in the diagram above.
The rod lies in a vertical plane perpendicular to the peg.

(a) Find the magnitude of the normal contact force between the peg and the rod. **[3]**

A particle of mass m kg is now attached to the rod at B, and the rod rests in the same position on the peg. The coefficient of friction between the rod and the ground is $\frac{3}{4}$.

(b) Find the greatest value of m for which the rod remains in equilibrium. **[7]**

Solution

(a) Since $\tan\theta = \frac{3}{4}$, a 3, 4, 5 right-angled triangle gives $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.

Let the normal contact force from the peg be S N. The peg is smooth, so this force acts at C perpendicular to the rod. The rod is uniform, so its weight $6g$ N acts at the midpoint of AB, 2.5 m from A.

Taking moments about A:

$$S \times 4 = 6g \times 2.5\cos\theta$$

$$4S = 6g \times 2.5 \times \frac{4}{5}$$

$$4S = 12g$$

$$S = 3g \approx 29.4$$

The magnitude of the normal contact force is $3g \approx 29.4$ N.

(b) Let the normal contact force from the peg now be S N, and let the normal reaction and the friction force at A be R N and F N. The particle's weight mg N acts at B, 5 m from A.

Taking moments about A:

$$S \times 4 = 6g \times 2.5\cos\theta + mg \times 5\cos\theta$$

$$4S = 6g \times 2.5 \times \frac{4}{5} + mg \times 5 \times \frac{4}{5}$$

$$4S = 12g + 4mg$$

$$S = (3+m)g$$

The force from the peg is perpendicular to the rod, so it has a horizontal component $S\sin\theta$ acting in the direction from B towards A. Friction at A therefore acts horizontally in the opposite direction, towards the side of B.

Resolving horizontally:

$$F = S\sin\theta = \frac{3(3+m)g}{5}$$

Resolving vertically:

$$R + S\cos\theta = 6g + mg$$

$$R = (6+m)g - \frac{4(3+m)g}{5}$$

$$R = \frac{(18+m)g}{5}$$

For the rod to remain in equilibrium, $F \leq \frac{3}{4}R$, so

$$\frac{3(3+m)g}{5} \leq \frac{3}{4} \times \frac{(18+m)g}{5}$$

$$12(3+m) \leq 3(18+m)$$

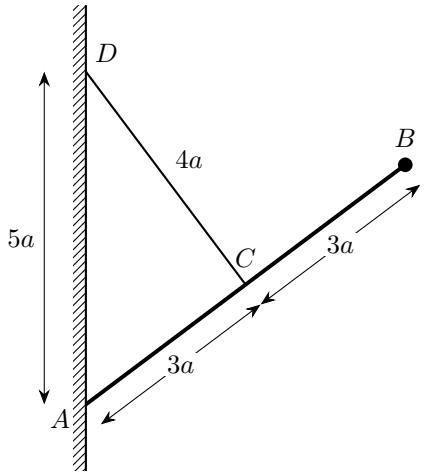
$$36 + 12m \leq 54 + 3m$$

$$9m \leq 18$$

$$m \leq 2$$

The greatest value of m is 2.

Final answers with $g = 10$: (a) 30 N



7. A uniform rod AB, of length $6a$ and weight W, has its end A in contact with a rough vertical wall, with B above the level of A. A light inextensible string of length $4a$ has one end attached to the midpoint C of the rod and the other end attached to a point D on the wall, vertically above A, where $AD = 5a$, as shown in the diagram above. A particle of weight kW is attached to the rod at B. The rod and the string lie in a vertical plane perpendicular to the wall.
The coefficient of friction between the rod and the wall is $\frac{1}{12}$. The rod is in limiting equilibrium and is about to slip up the wall.

(a) Find the value of k. **[6]**

(b) Find, in terms of W, the magnitude of the total force exerted by the wall on the rod. **[2]**

Solution

(a) In triangle ACD, $AC = 3a$, $CD = 4a$ and $AD = 5a$. Since

$$(3a)^2 + (4a)^2 = (5a)^2$$

the angle ACD is $90°$, so the string is perpendicular to the rod. Let θ be the angle between the rod and the wall, angle DAC. Then

$$\sin\theta = \frac{CD}{AD} = \frac{4}{5}, \qquad \cos\theta = \frac{AC}{AD} = \frac{3}{5}$$

Angle $ADC = 90° - \theta$, so the string makes an angle $90° - \theta$ with the vertical, and the tension T has horizontal component $T\cos\theta = \frac{3T}{5}$ towards the wall and vertical component $T\sin\theta = \frac{4T}{5}$ upwards.

Let R be the normal reaction and F the friction at A. The rod is about to slip up the wall, so friction on the rod acts down the wall. The equilibrium is limiting, so

$$F = \frac{1}{12}R$$

Resolving horizontally:

$$R = \frac{3T}{5} \quad \Longrightarrow \quad F = \frac{1}{12} \times \frac{3T}{5} = \frac{T}{20}$$

Resolving vertically:

$$\frac{4T}{5} - F = W + kW$$
$$\frac{4T}{5} - \frac{T}{20} = W(1+k)$$
$$\frac{3T}{4} = W(1+k)$$
$$T = \frac{4W(1+k)}{3}$$

The string is perpendicular to the rod, so its tension has moment $T \times 3a$ about A. The horizontal distances of C and B from the wall are $3a\sin\theta = \frac{12a}{5}$ and $6a\sin\theta = \frac{24a}{5}$. Taking moments about A:

$$T \times 3a = W \times \frac{12a}{5} + kW \times \frac{24a}{5}$$
$$T = \frac{4W(1+2k)}{5}$$

Equating the two expressions for T:

$$\frac{4W(1+k)}{3} = \frac{4W(1+2k)}{5}$$
$$5(1+k) = 3(1+2k)$$
$$5 + 5k = 3 + 6k$$
$$k = 2$$

(b) With $k = 2$, $T = \frac{4W\times 3}{3} = 4W$, so

$$R = \frac{3 \times 4W}{5} = \frac{12W}{5}, \qquad F = \frac{4W}{20} = \frac{W}{5}$$

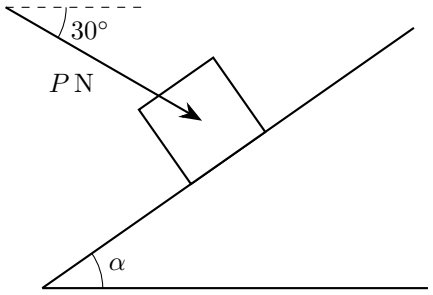
The total force exerted by the wall is the resultant of R and F, which are perpendicular, so its magnitude is

$$\sqrt{R^2 + F^2} = \sqrt{\left(\frac{12W}{5}\right)^2 + \left(\frac{W}{5}\right)^2} = \frac{\sqrt{145}}{5}W \approx 2.41W$$

8. A block B of mass m kg rests on a rough slope whose angle of inclination α to the horizontal can be adjusted. The coefficient of friction between B and the slope is $\frac{1}{2}$.

(a) The slope is set so that $\tan\alpha = \frac{3}{4}$, and B is projected up a line of greatest slope with speed $4\,\mathrm{m\,s^{-1}}$. Find the distance that B travels before it comes to instantaneous rest. **[4]**

(b) Show that B does not remain at rest, and find the magnitude of its acceleration as it slides back down the slope. **[3]**



The angle of the slope is now changed. A force of magnitude P N is applied to B in the vertical plane containing a line of greatest slope through B, directed towards the slope at 30° below the horizontal, as shown in the diagram above. P is first set at a value for which B is at rest, and is then gradually increased.

(c) Show that, for B to slide up the slope,

$$P\,(2\cos(\alpha + 30^\circ) - \sin(\alpha + 30^\circ)) > mg\,(2\sin\alpha + \cos\alpha)$$ **[3]**

(d) Find, in degrees, the least value of α for which B cannot be made to slide up the slope, however large P becomes. **[2]**

Solution

(a) Since $\tan\alpha = \frac{3}{4}$, $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.

Resolving perpendicular to the slope:

$$R = mg\cos\alpha = \frac{4}{5}mg$$

B is moving, so friction is limiting, $F = \frac{1}{2}R = \frac{2}{5}mg$, acting down the slope.

Take the direction of motion, up the slope, as positive. Using Newton's second law:

$$-mg\sin\alpha - \frac{2}{5}mg = ma_1$$

$$-\frac{3}{5}mg - \frac{2}{5}mg = ma_1$$

$$a_1 = -g$$

The constant acceleration formula used is

$$v^2 = u^2 + 2as$$

With $u = 4$, $v = 0$ and $a = -g$:

$$0 = 4^2 - 2gs$$

$$s = \frac{8}{g} = \frac{8}{9.8} \approx 0.816\,\text{m}$$

(b) At instantaneous rest, the component of the weight down the slope is $mg\sin\alpha = \frac{3}{5}mg$, while the greatest friction available is $\frac{1}{2}R = \frac{2}{5}mg$. Since $\frac{3}{5}mg > \frac{2}{5}mg$, friction cannot hold B, so B does not remain at rest.

As B slides down, friction $\frac{2}{5}mg$ acts up the slope. Take the direction of motion, down the slope, as positive. Using Newton's second law:

$$\frac{3}{5}mg - \frac{2}{5}mg = ma_2$$

$$a_2 = \frac{g}{5} = \frac{9.8}{5} = 1.96\,\text{m}\,\text{s}^{-2}$$

(c) The force is directed $30°$ below the horizontal and the slope rises at α above the horizontal, so the force makes an angle $\alpha + 30°$ with the line of greatest slope pointing up the slope. Its component up the slope is $P\cos(\alpha + 30°)$, and its component into the slope is $P\sin(\alpha + 30°)$.

Resolving perpendicular to the slope:

$$R = mg\cos\alpha + P\sin(\alpha + 30°)$$

When B is on the point of sliding up the slope, friction is limiting and acts down the slope, with magnitude $\frac{1}{2}R$. For B to slide up, the force up the slope must exceed the weight component and the limiting friction down the slope:

$$P\cos(\alpha + 30°) > mg\sin\alpha + \frac{1}{2}(mg\cos\alpha + P\sin(\alpha + 30°))$$

$$2P\cos(\alpha + 30°) > 2mg\sin\alpha + mg\cos\alpha + P\sin(\alpha + 30°)$$

$$P(2\cos(\alpha + 30°) - \sin(\alpha + 30°)) > mg(2\sin\alpha + \cos\alpha)$$

as required.

(d) For $0° < \alpha < 90°$, the right-hand side $mg(2\sin\alpha + \cos\alpha)$ is positive. If $2\cos(\alpha + 30°) - \sin(\alpha + 30°) \leq 0$, the left-hand side is never positive, so the inequality fails however large P becomes; otherwise a large enough P satisfies it. So B cannot be made to slide up when

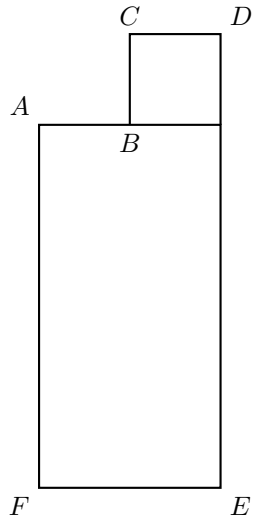
$$\sin(\alpha + 30°) \geq 2\cos(\alpha + 30°) \quad \Longrightarrow \quad \tan(\alpha + 30°) \geq 2$$

The least value of α is therefore given by

$$\alpha + 30° = \tan^{-1} 2 \quad \Longrightarrow \quad \alpha = \tan^{-1} 2 - 30° \approx 33.4°$$

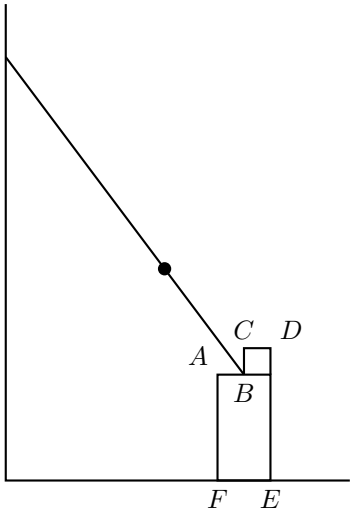
Final answers with $g = 10$: (a) $0.8\,\text{m}$; (b) $2\,\text{m}\,\text{s}^{-2}$

9. Two blocks have rectangular cross sections.
One block has mass 8 kg and its cross section is a rectangle of width 20 cm and height 40 cm.
The other block has mass 2 kg and its cross section is a square with sides of length 10 cm.
The blocks are fixed together to form the composite body shown in the diagram below, in which $FE = 20\,\text{cm}$, $AF = 40\,\text{cm}$ and $BC = CD = 10\,\text{cm}$.



(a) Find the distance of the centre of mass of the composite body from AF. **[2]**

(b) A uniform rod has mass 8 kg and length 1.5 m.
One end of the rod rests against a smooth vertical wall, and the other end rests on the composite body at the point B.
The composite body stands with FE on a rough horizontal surface, and the coefficient of friction between the composite body and the surface is 0.2.
The rod lies in a vertical plane perpendicular to the wall, and the angle between the rod and AB is α, where $\tan\alpha = \frac{4}{3}$.
A particle of mass 6 kg is fixed to the rod at a distance of x m from B.
The rod, particle and composite body are shown in the diagram below.



(i) Write down the magnitude of the vertical reaction force acting on the rod at B, in terms of g. **[1]**

(ii) Show that the magnitude of the horizontal reaction force acting on the rod at B is $3g(1 + x)$. **[3]**

(iii) Find the greatest value of x for which the composite body neither slides nor topples. Fully justify your answer. **[6]**

Solution

(a) Each block is uniform, so its centre of mass is at the centre of its cross section. The centre of mass of the 8 kg block is 10 cm from AF, and that of the 2 kg block is $20 - 5 = 15$ cm from AF. For a body made of pieces, the centre of mass is given by

$$M\bar{x} = \sum m_i x_i, \qquad \text{where } M = \sum m_i$$

Taking distances from AF, with $M = 8 + 2 = 10$:

$$10\bar{x} = 8 \times 10 + 2 \times 15$$

$$10\bar{x} = 110$$

$$\bar{x} = 11$$

The centre of mass is 11 cm from AF.

(b) Since $\tan\alpha = \frac{4}{3}$, $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$. The line AB is horizontal, so the rod makes angle α with the horizontal. Let the force of the wall on the rod be S N, and the vertical and horizontal components of the reaction on the rod at B be V N and H N.

(i) The wall is smooth, so S is horizontal. Resolving vertically for the rod and particle:

$$V = 8g + 6g = 14g$$

The vertical reaction at B is $14g$ N.

(ii) The top of the rod is $1.5\sin\alpha = 1.2$ m above B. The rod's weight acts at its midpoint, a horizontal distance $0.75\cos\alpha = 0.45$ m from B, and the particle is a horizontal distance $x\cos\alpha = 0.6x$ m from B.

Taking moments about B for the rod and particle:

$$S \times 1.2 = 8g \times 0.45 + 6g \times 0.6x$$

$$1.2S = 3.6g + 3.6gx$$

$$S = 3g(1 + x)$$

Resolving horizontally, the only horizontal forces on the rod and particle are S and H, so

$$H = S = 3g(1 + x)$$

as required.

(iii) The wall pushes the rod away from the wall, so H acts on the rod towards the wall. By Newton's third law, the rod exerts on the composite body at B a downward force $14g$ N and a horizontal force $3g(1 + x)$ N away from the wall, at a height of 0.4 m above the surface. Let the normal reaction from the surface be R N and the friction be Q N, acting towards the wall.

Sliding. Resolving vertically for the composite body:

$$R = 10g + 14g = 24g$$

Resolving horizontally:

$$Q = 3g(1 + x)$$

The body does not slide provided $Q \leq 0.2R$:

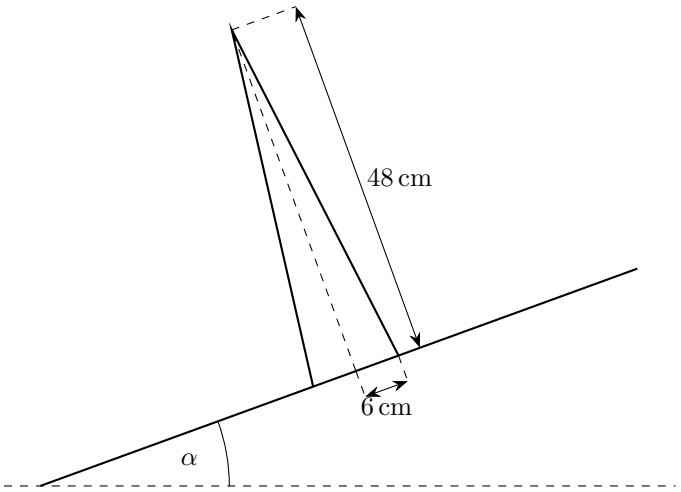
$$\begin{aligned} 3g(1+x) &\leq 0.2 \times 24g \\ 1+x &\leq 1.6 \\ x &\leq 0.6 \end{aligned}$$

Toppling. The horizontal force at B tends to turn the body about the edge through E, the edge further from the wall. Measured horizontally from E, the weight $10g$ acts at $20 - 11 = 9\,\text{cm} = 0.09\,\text{m}$, and the force $14g$ at B acts at $CD = 10\,\text{cm} = 0.1\,\text{m}$, since B is directly below C and D is directly above E. On the point of toppling, R acts at E, and Q acts along the surface through E, so neither has a moment about E. The body does not topple provided the moment of the horizontal force about E does not exceed the moment of the two vertical forces.

Taking moments about E:

$$\begin{aligned} 3g(1+x) \times 0.4 &\leq 10g \times 0.09 + 14g \times 0.1 \\ 1.2(1+x) &\leq 2.3 \\ 1+x &\leq \frac{23}{12} \\ x &\leq \frac{11}{12} \end{aligned}$$

Since $0.6 < \frac{11}{12}$, the body would slide before it could topple, so sliding gives the limiting condition. Also $0.6 < 1.5$, so the particle lies on the rod. The greatest value is $x = 0.6$.



10. A uniform solid right circular cone L has base radius 6 cm, height 48 cm and mass 6 kg. L is placed with its base on a rough plane inclined at an angle α to the horizontal, as shown in the diagram above.

(a) On a copy of the diagram, mark the forces acting on L. **[1]**

The coefficient of friction between L and the plane is 0.6. Initially $\alpha = 20°$.

(b) Show that L rests in equilibrium on the plane. **[4]**

A couple is applied to L, acting in the vertical plane of the diagram. It is given that L will topple if the couple is applied in an anticlockwise sense, but L will not topple if the couple is applied in a clockwise sense.

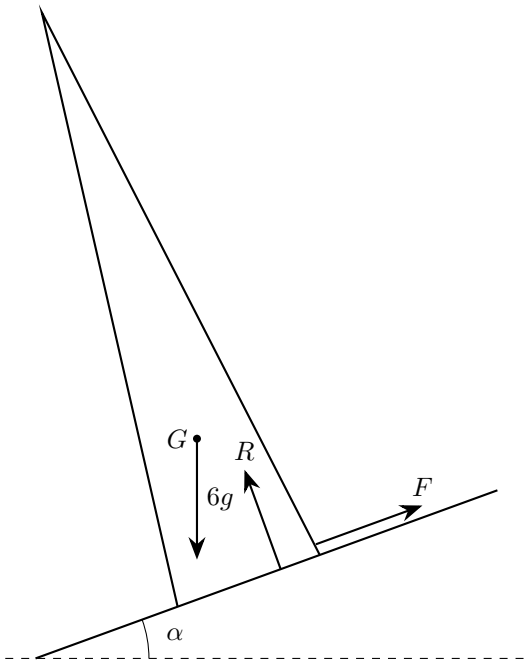
(c) Find the range of possible values of the magnitude of the couple. **[4]**

The couple is now removed and the plane is slowly tilted so that α increases from 20°.

(d) Determine whether L topples first without sliding or slides first without toppling, and find the value of α at which L first ceases to be in equilibrium. **[5]**

Solution

(a) The forces on L are its weight $6g$ N acting vertically downwards at its centre of mass G, the normal reaction R N perpendicular to the plane, and friction F N acting up the plane along the base:



(b) Resolving perpendicular to the plane and parallel to the plane:

$$R = 6g\cos 20^\circ$$

$$F = 6g\sin 20^\circ$$

L does not slide provided $F \leq 0.6R$, that is

$$6g\sin 20^\circ \leq 0.6 \times 6g\cos 20^\circ \quad \Longrightarrow \quad \tan 20^\circ \leq 0.6$$

Since $\tan 20^\circ \approx 0.364 < 0.6$, friction can hold L, so L does not slide.

The centre of mass G of a uniform solid cone lies on its axis, a quarter of the height from the base, so G is $\frac{48}{4} = 12$ cm from the centre O of the base. The axis is perpendicular to the plane, so the vertical through G makes an angle of 20° with the axis. It therefore meets the plane down the slope from O, at a distance

$$12\tan 20^\circ \approx 4.37 \text{ cm}$$

This is less than the base radius of 6 cm, so the vertical through G passes through the base and L does not topple. As L neither slides nor topples, it rests in equilibrium on the plane.

(c) Let the magnitude of the couple be C N m. A couple adds no resultant force, so resolving still gives $F = 6g\sin 20^\circ$ and $R = 6g\cos 20^\circ$, and L does not slide. Let P be the lowest point of the base and Q the highest point of the base, in the vertical plane of the diagram. Relative to either edge, G is 0.06 m along the plane and 0.12 m perpendicular to it. Split the weight into $6g\cos 20^\circ$ perpendicular to the plane and $6g\sin 20^\circ$ down the plane.

An anticlockwise couple turns L about P. When L is about to turn, the contact forces act at P. Taking moments about P, the weight has a clockwise moment of

$$6g(0.06\cos 20^\circ - 0.12\sin 20^\circ) \approx 0.902 \text{ N m}$$

L topples when the couple exceeds this, so $C > 0.902$.

A clockwise couple turns L about Q. Taking moments about Q, both components of the weight have an anticlockwise moment, giving a total of

$$6g(0.06\cos 20^\circ + 0.12\sin 20^\circ) \approx 5.73\,\text{N m}$$

L does not topple provided the couple does not exceed this, so $C \leq 5.73$. Therefore

$$0.902 < C \leq 5.73 \qquad (3\text{ s.f.})$$

(d) By the working in part (b), with 20° replaced by α, L does not slide while $\tan\alpha \leq 0.6$, that is while

$$\alpha \leq \tan^{-1} 0.6 \approx 31.0^\circ$$

The vertical through G meets the plane $12\tan\alpha$ cm from O, so L does not topple while $12\tan\alpha \leq 6$, that is while

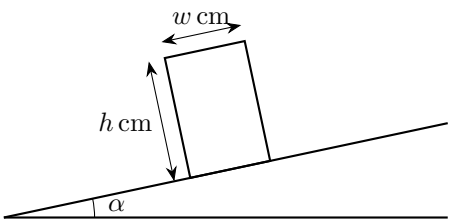
$$\alpha \leq \tan^{-1}\tfrac{1}{2} \approx 26.6^\circ$$

Since $26.6^\circ < 31.0^\circ$, the toppling condition fails first, and at that angle $\tan\alpha = 0.5 < 0.6$, so friction still prevents sliding. Hence L topples first without sliding, and it first ceases to be in equilibrium when

$$\alpha = \tan^{-1}\tfrac{1}{2} \approx 26.6^\circ$$

Final answers with $g = 10$: (c) $0.920 < C \leq 5.85$

11. A block of mass m kg is modelled as a uniform rectangular prism whose cross section has width w cm and height h cm. The block rests on a rough plane inclined at an angle α to the horizontal, with the cross section containing a line of greatest slope, as shown in the diagram below.



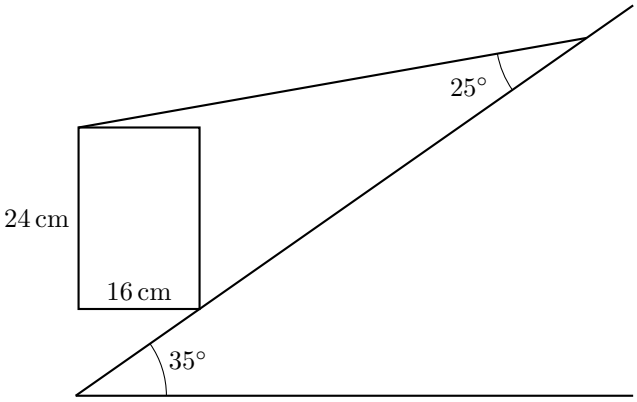
At first the block is in equilibrium, and then α is gradually increased.

(a) Assuming that the block does not slide first, show that it is on the point of toppling when $\tan\alpha = \frac{w}{h}$. **[2]**

It is now given that $w = 16$ and that the coefficient of friction between the block and the plane is 0.8. At first, the block is in equilibrium. As α is increased, equilibrium is broken by the block toppling, rather than sliding.

(b) Determine the range of possible values of h. **[4]**

For the remainder of the question you are given that $h = 24$, $m = 12$ and $\alpha = 35°$.
One end of a light inextensible string is attached to the top downslope edge of the block. The other end of the string is attached to a point further up the inclined plane, and the string passes above the block. The string makes an angle of 25° with the inclined plane, so that the block is held in equilibrium with its base horizontal and only its lower upslope edge in contact with the plane, as shown in the diagram below. Both the string and the centre of mass of the block lie in the same vertical plane as a line of greatest slope.



(c) Determine the tension in the string. **[4]**

(d) Determine the angle between the inclined plane and the total contact force that the inclined plane exerts on the block. **[6]**

Solution

(a) Let P be the lower downslope edge of the block and G its centre of mass. The block is uniform, so G

is $\frac{w}{2}$ cm along the plane from P and $\frac{h}{2}$ cm from the plane, measured perpendicular to it. If β is the angle between PG and the perpendicular to the plane through P, then

$$\tan\beta = \frac{\frac{w}{2}}{\frac{h}{2}} = \frac{w}{h}$$

The block is on the point of toppling about P when the weight acts through P, that is, when G is vertically above P. The perpendicular to the plane makes an angle α with the vertical, so PG is vertical when $\beta = \alpha$, which gives

$$\tan\alpha = \frac{w}{h}$$

as required.

(b) With $w = 16$, part (a) shows that the block is on the point of toppling when

$$\tan\alpha = \frac{16}{h}$$

When the block is on the point of sliding, resolving perpendicular to and along the plane gives $R = mg\cos\alpha$ and $F = mg\sin\alpha$, where R is the normal reaction and F the friction, with friction limiting, $F = 0.8R$:

$$mg\sin\alpha = 0.8\,mg\cos\alpha \quad \Longrightarrow \quad \tan\alpha = 0.8$$

As α increases, $\tan\alpha$ increases, so the block topples first if the toppling value of $\tan\alpha$ is reached before the sliding value:

$$\frac{16}{h} < 0.8$$

$$h > 20$$

(c) The string makes an angle of 25° with the plane, which is inclined at 35° to the horizontal, so the string is inclined at $35^\circ - 25^\circ = 10^\circ$ to the horizontal.

Let C be the lower upslope edge, the only point of contact. The base is horizontal, so G is 8 cm horizontally from C, on the downslope side. The string is attached 16 cm horizontally from C and 24 cm above it. Split the tension T into a horizontal component $T\cos 10^\circ$, acting upslope, and a vertical component $T\sin 10^\circ$, acting upwards; both turn the block about C in the opposite sense to the weight. The contact force at C has no moment about C.

Taking moments about C:

$$T\cos 10^\circ \times 24 + T\sin 10^\circ \times 16 = 12g \times 8$$

$$T = \frac{96g}{24\cos 10^\circ + 16\sin 10^\circ}$$

$$T \approx 35.6\,\text{N}$$

(d) Let the contact force at C have normal component N and frictional component F, taking F up the slope. The string is at 25° to the plane, below the upslope direction, and the weight $12g$ is at 35° to the normal.

Resolving along the plane:

$$F + T\cos 25^\circ = 12g\sin 35^\circ$$

so, using the unrounded value of T from part (c):

$$F = 12g\sin 35^\circ - T\cos 25^\circ \approx 35.17\,\text{N}$$

Resolving perpendicular to the plane:

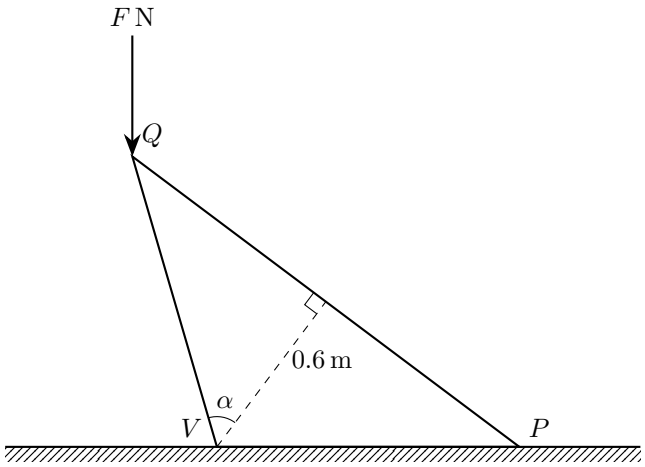
$$N = 12g\cos 35^\circ + T\sin 25^\circ \approx 111.39\,\text{N}$$

Since $F > 0$, the friction does act up the slope. The total contact force is the resultant of F along the plane and N perpendicular to it, so the angle θ it makes with the plane satisfies

$$\tan\theta = \frac{N}{F} \approx 3.167$$

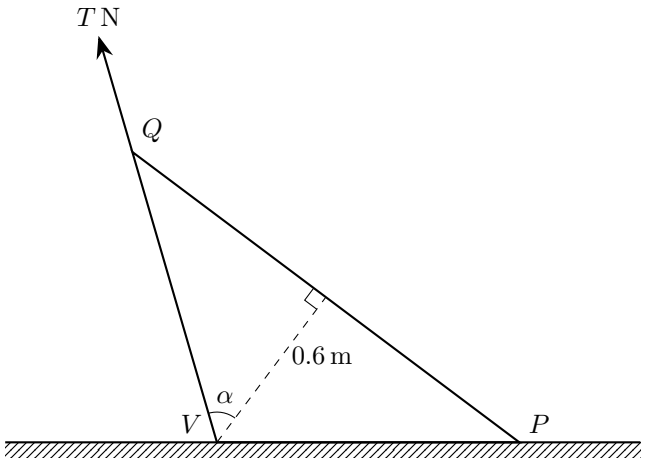
$$\theta \approx 72.5^\circ$$

Final answers with $g = 10$: (c) 36.3 N



12. A uniform solid cone has height 0.6 m, semi-vertical angle α, where $\tan\alpha = \frac{4}{3}$, and weight 280 N.
The cone lies with its curved surface on a horizontal plane. The point P on the circumference of its base is in contact with the plane, V is the vertex of the cone and PQ is a diameter of its base.
A vertical force of magnitude F N, acting downwards, is applied to the cone at Q, as shown in the first diagram above.

(a) Find the least value of F for which the cone topples about V. **[4]**



The force of magnitude F N is removed. A force of magnitude T N is now applied to the cone at Q, acting along VQ produced in the direction from V to Q, as shown in the second diagram above. The value of T is gradually increased from zero.
The coefficient of friction between the cone and the plane is 0.9.

(b) Determine whether the cone first slides or first topples about P, and find the value of T at which this happens. **[11]**

Solution

(a) Since $\tan\alpha = \frac{4}{3}$, $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$. The radius of the base is $0.6\tan\alpha = 0.8$ m, so the slant height is

$$VP = VQ = \sqrt{0.6^2 + 0.8^2} = 1\text{ m}$$

The angle between VP and VQ is 2α, where

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25} = -0.28$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25} = 0.96$$

Since $\cos 2\alpha < 0$, Q lies on the opposite side of V from P, at a horizontal distance of $1 \times 0.28 = 0.28\,\text{m}$ from V and a height of $1 \times 0.96 = 0.96\,\text{m}$ above the plane.

The centre of mass G of a uniform solid cone lies on its axis, a quarter of the height from the base, so $VG = \frac{3}{4} \times 0.6 = 0.45\,\text{m}$. The axis makes angle α with VP, so the horizontal distance of G from V, towards P, is

$$0.45\cos\alpha = 0.45 \times \frac{3}{5} = 0.27\,\text{m}$$

On the point of toppling about V, the cone touches the plane only at V, so the reaction from the plane acts at V, and the moment of F about V balances the moment of the weight. Taking moments about V:

$$0.28F = 280 \times 0.27 = 75.6$$

$$F = 270$$

(b) Let the normal reaction be $R\,\text{N}$ and the friction $S\,\text{N}$. The force T acts along VQ, at angle 2α to VP. Using part (a), $\cos 2\alpha = -0.28$ and $\sin 2\alpha = 0.96$, so its horizontal component is $0.28T$, directed away from P, and its vertical component is $0.96T$, upwards. Friction acts horizontally towards P.

Resolving vertically:

$$R + 0.96T = 280 \quad \Longrightarrow \quad R = 280 - 0.96T$$

Resolving horizontally:

$$S = 0.28T$$

Sliding: the cone is on the point of sliding when $S = 0.9R$:

$$0.28T = 0.9(280 - 0.96T)$$

$$0.28T = 252 - 0.864T$$

$$1.144T = 252$$

$$T = \frac{31500}{143} \approx 220.3$$

Toppling about P: on the point of toppling about P, the cone touches the plane only at P, so R and S both act at P. The line of action of T passes through V, and angle $PVQ = 2\alpha$, so the perpendicular distance from P to this line is

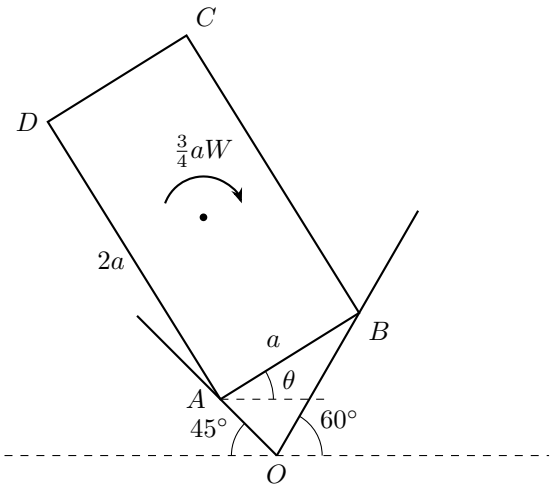
$$VP\sin 2\alpha = 1 \times 0.96 = 0.96\,\text{m}$$

The horizontal distance from P to the line of action of the weight is $1 - 0.27 = 0.73\,\text{m}$. Taking moments about P:

$$0.96T = 280 \times 0.73 = 204.4$$

$$T = \frac{2555}{12} \approx 212.9$$

Since $212.9 < 220.3$, the value of T for toppling is reached first as T increases. The cone first topples about P, when $T = \frac{2555}{12} \approx 213$.



13. The diagram above shows a uniform rectangular lamina $ABCD$, of weight W, in which $AB = a$ and $AD = 2a$. The lamina is in equilibrium in a vertical plane that also contains the point O. The vertex A rests on a smooth plane inclined at an angle of 45° to the horizontal, and the vertex B rests on a smooth plane inclined at an angle of 60° to the horizontal.
OA is a line of greatest slope of the plane inclined at 45° to the horizontal and OB is a line of greatest slope of the plane inclined at 60° to the horizontal.
The side AB is inclined at an acute angle θ to the horizontal, with B higher than A, and the lamina is kept in equilibrium in this position by a clockwise couple of magnitude $\frac{3}{4}aW$.

(a) By resolving horizontally and vertically, find, in terms of W, the magnitude of the normal contact force between the plane and the lamina at B, giving your answer in simplified surd form. **[6]**

(b) By taking moments about A, show that θ satisfies the equation

$$2\left(5-\sqrt{3}\right)\sin\theta - 2\left(2-\sqrt{3}\right)\cos\theta = 3$$

[5]

(c) Hence find θ, giving your answer in degrees correct to 1 decimal place. **[3]**

Solution

(a) Let R_A and R_B be the normal contact forces at A and B. The planes are smooth, so each force is perpendicular to its plane. The normal to the 45° plane is at 45° to the vertical, so R_A acts upwards and to the right at 45° to the vertical. The normal to the 60° plane is at 60° to the vertical, so R_B acts upwards and to the left at 60° to the vertical. A couple has no resultant force, so it does not appear when resolving.

Resolving horizontally:

$$R_A \sin 45° = R_B \sin 60° \quad \Longrightarrow \quad \frac{\sqrt{2}}{2}R_A = \frac{\sqrt{3}}{2}R_B$$

Resolving vertically:

$$R_A \cos 45° + R_B \cos 60° = W \quad \Longrightarrow \quad \frac{\sqrt{2}}{2}R_A + \frac{1}{2}R_B = W$$

Substituting the horizontal equation into the vertical equation:

$$\frac{\sqrt{3}}{2}R_B + \frac{1}{2}R_B = W$$

$$R_B = \frac{2W}{\sqrt{3}+1}$$

$$= \frac{2W(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \Big(\sqrt{3}-1\Big)W$$

(b) The force R_A acts through A, so it has no moment about A.

The centre of the lamina is reached from A by moving $\frac{1}{2}a$ along AB and a along AD. Since AD is perpendicular to AB, it is inclined at θ to the vertical, leaning to the left. So the centre is a horizontal distance $\frac{1}{2}a\cos\theta - a\sin\theta$ to the right of A, and the weight has a clockwise moment of $Wa\big(\frac{1}{2}\cos\theta - \sin\theta\big)$ about A.

The point B is $a\cos\theta$ to the right of A and $a\sin\theta$ above it. The force at B has an upward vertical component $R_B\cos 60°$ and a horizontal component $R_B\sin 60°$ to the left, and both turn the lamina anticlockwise about A.

Taking moments about A, clockwise moments equal anticlockwise moments:

$$Wa\left(\frac{1}{2}\cos\theta - \sin\theta\right) + \frac{3}{4}aW = R_B\cos 60° \times a\cos\theta + R_B\sin 60° \times a\sin\theta$$

Dividing by a, substituting $R_B = \big(\sqrt{3}-1\big)W$ from part (a), then multiplying through by $\frac{4}{W}$:

$$W\left(\frac{1}{2}\cos\theta - \sin\theta\right) + \frac{3}{4}W = \Big(\sqrt{3}-1\Big)W\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$

$$2\cos\theta - 4\sin\theta + 3 = 2\Big(\sqrt{3}-1\Big)\cos\theta + 2\Big(3-\sqrt{3}\Big)\sin\theta$$

$$3 = \Big(6 - 2\sqrt{3} + 4\Big)\sin\theta + \Big(2\sqrt{3} - 2 - 2\Big)\cos\theta$$

$$2\Big(5-\sqrt{3}\Big)\sin\theta - 2\Big(2-\sqrt{3}\Big)\cos\theta = 3$$

as required.

(c) Write the left-hand side in the form $R\sin(\theta-\alpha)$, where $R>0$ and $0<\alpha<90°$. Expanding gives

$$R\sin(\theta-\alpha) = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$$

Comparing coefficients:

$$R\cos\alpha = 2\Big(5-\sqrt{3}\Big), \qquad R\sin\alpha = 2\Big(2-\sqrt{3}\Big)$$

so

$$R = \sqrt{4\Big(5-\sqrt{3}\Big)^2 + 4\Big(2-\sqrt{3}\Big)^2} = \sqrt{4\Big(28-10\sqrt{3}\Big) + 4\Big(7-4\sqrt{3}\Big)} = \sqrt{140-56\sqrt{3}} \approx 6.558$$

$$\tan\alpha = \frac{2-\sqrt{3}}{5-\sqrt{3}} \quad\Longrightarrow\quad \alpha \approx 4.687°$$

The equation from part (b) becomes

$$\sin(\theta - \alpha) = \frac{3}{\sqrt{140 - 56\sqrt{3}}} \approx 0.4575$$

Since θ is acute, $\theta - \alpha$ lies between $-\alpha$ and $90^\circ - \alpha$, so the only solution is

$$\theta - \alpha \approx 27.22^\circ \quad \Longrightarrow \quad \theta \approx 31.9^\circ \qquad (1 \text{ d.p.})$$

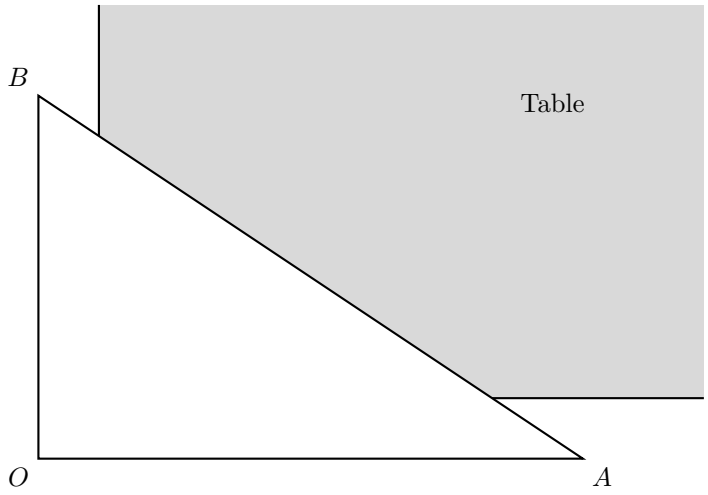
14. A uniform triangular lamina OAB of mass M has vertices at the origin $O(0,0)$, $A(18a,0)$ and $B(0,12a)$, where a is a positive constant. A small object P of mass m is attached to the lamina at the point (x,y). The centre of mass of the system consisting of the lamina and P is at the point $(\bar{x},\bar{y})$. P is modelled as a particle.

(a) Show that

$$\bar{x} = \frac{6Ma + mx}{M+m}$$ **[1]**

(b) Find a corresponding expression for $\bar{y}$. **[1]**

The lamina, with P no longer attached, is placed on a horizontal rectangular table, with OA and OB parallel to the edges of the table, and partly overhanging two edges of the table, as shown in the diagram below. The corner of the table is at the point $(2a,2a)$.



When P is placed on the lamina at O, the lamina topples over one of the edges of the table.

(c) Show that $m > M$. **[4]**

The lamina is now put back on the table in the same position as before. P is placed at the point $(12a,4a)$ on the smooth upper surface of the lamina, and is projected towards O. At a subsequent instant during the motion, P is at the point $(12ak,4ak)$, where $0 < k < 1$.

(d) Given that $m > 2M$, and assuming that the lamina has not yet toppled, find, in terms of M and m, the value of k for which the centre of mass of the system lies on the line of the table edge parallel to OB. **[3]**

(e) Given now that $m > M$, show that the lamina topples before P reaches O, and determine which table edge it topples over, justifying your answer. **[5]**

Solution

(a) The centre of mass of a uniform triangular lamina is $\frac{2}{3}$ of the way along a median from the vertex. The median from $B(0,12a)$ meets OA at its midpoint $(9a,0)$, so the centre of mass of the lamina is

$$\left(0 + \frac{2}{3}(9a-0),\ 12a + \frac{2}{3}(0-12a)\right) = (6a,4a)$$

For a body made of pieces, the centre of mass is given by

$$\left(\sum m_i\right)\bar{x} = \sum m_i x_i$$

Taking moments in the x-direction gives

$$(M+m)\bar{x} = 6Ma + mx \quad \Longrightarrow \quad \bar{x} = \frac{6Ma+mx}{M+m}$$

as required.

(b) In the same way, in the y-direction:

$$(M+m)\bar{y} = 4Ma + my \quad \Longrightarrow \quad \bar{y} = \frac{4Ma+my}{M+m}$$

(c) The part of the lamina in contact with the table is where $x \geq 2a$ and $y \geq 2a$. The centre of mass of the system always lies within the lamina, so the lamina stays on the table provided $\bar{x} \geq 2a$ and $\bar{y} \geq 2a$. It topples over the edge parallel to OB, the line $x = 2a$, if $\bar{x} < 2a$, and over the edge parallel to OA, the line $y = 2a$, if $\bar{y} < 2a$.

With P at O, $x = y = 0$, so parts (a) and (b) give

$$\bar{x} = \frac{6Ma}{M+m}, \qquad \bar{y} = \frac{4Ma}{M+m}$$

Since $M + m > 0$, each inequality below may be multiplied through by $M + m$. If the lamina topples over the edge parallel to OB, then $\bar{x} < 2a$, which gives

$$\begin{aligned} \frac{6Ma}{M+m} &< 2a \\ 6M &< 2M + 2m \\ m &> 2M \end{aligned}$$

so $m > M$. If the lamina topples over the edge parallel to OA, then $\bar{y} < 2a$, which gives

$$\begin{aligned} \frac{4Ma}{M+m} &< 2a \\ 4M &< 2M + 2m \\ m &> M \end{aligned}$$

In either case $m > M$, as required.

(d) Using part (a) with $x = 12ak$:

$$\bar{x} = \frac{6Ma + 12mak}{M+m}$$

The table edge parallel to OB is the line $x = 2a$, so

$$\begin{aligned} \frac{6Ma+12mak}{M+m} &= 2a \\ 6M + 12mk &= 2M + 2m \\ 12mk &= 2m - 4M \\ k &= \frac{m-2M}{6m} \end{aligned}$$

(e) Using parts (a) and (b) with $x = 12ak$ and $y = 4ak$:

$$\bar{x} = \frac{6Ma + 12mak}{M + m}, \qquad \bar{y} = \frac{4Ma + 4mak}{M + m}$$

When $k = 1$, $\bar{y} = 4a > 2a$ and $\bar{x} = \frac{(6M+12m)a}{M+m} > 2a$, so the centre of mass starts over the table. As P moves towards O, k decreases from 1 towards 0, and both $\bar{x}$ and $\bar{y}$ decrease.

The centre of mass lies on the edge parallel to OA, $\bar{y} = 2a$, when

$$\begin{aligned} \frac{4Ma + 4mak}{M + m} &= 2a \\ 4M + 4mk &= 2M + 2m \\ 4mk &= 2m - 2M \\ k &= \frac{m - M}{2m} \end{aligned}$$

Since $m > M > 0$, $0 < \frac{m-M}{2m} < \frac{1}{2}$. So for $0 < k < \frac{m-M}{2m}$ the centre of mass has $\bar{y} < 2a$, and the lamina topples before P reaches O.

The lamina topples over the edge that the centre of mass crosses first, that is, at the larger value of k.

If $M < m \leq 2M$, then $6M \geq 2M + 2m$, so for $k > 0$

$$\bar{x} > \frac{6Ma}{M + m} \geq 2a$$

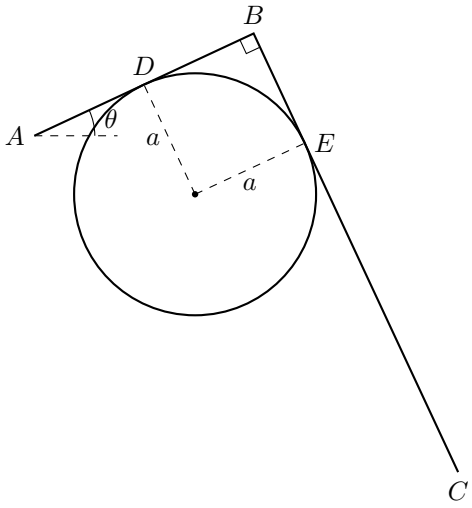
and the centre of mass never crosses the edge parallel to OB before P reaches O.

If $m > 2M$, then by part (d) $\bar{x} = 2a$ when $k = \frac{m-2M}{6m}$. Comparing the two values of k:

$$\frac{m - M}{2m} - \frac{m - 2M}{6m} = \frac{3m - 3M - m + 2M}{6m} = \frac{2m - M}{6m} > 0$$

since $m > M$. So $\bar{y}$ reaches $2a$ while $\bar{x}$ is still greater than $2a$.

In both cases, the lamina topples over the table edge parallel to OA, the line $y = 2a$, when $k = \frac{m-M}{2m}$.



15. A rough cylinder of radius a is fixed with its axis horizontal. A rigid frame consists of two uniform rods AB and BC, rigidly joined at B with AB perpendicular to BC. The rod AB has length $2a$ and weight W, and the rod BC has length $4a$ and weight $2W$.
The frame rests on the cylinder in a vertical plane perpendicular to the axis of the cylinder, with AB at an angle θ to the horizontal, as shown in the diagram above. The rods touch the cylinder at D and E, where D is the midpoint of AB and E is the point of BC with $BE = a$.
The coefficient of friction between each rod and the cylinder is μ, and the frame is about to slip in a clockwise direction. The normal reaction between AB and the cylinder is R and the normal reaction between BC and the cylinder is N.

(a) Find the ratio $R : N$ in terms of μ. **[7]**

(b) Given that $\mu = \frac{2}{3}$, find the value of $\tan\theta$ and find R in terms of W, giving your answer in exact form. **[4]**

Solution

(a) Let O be the centre of the cross-section of the cylinder. The radii OD and OE are perpendicular to AB and BC respectively, AB is perpendicular to BC, and $DB = BE = a$, so $ODBE$ is a square of side a. In particular, OE is parallel to AB and OD is parallel to BC.

The weight W of AB acts at its midpoint D. The weight $2W$ of BC acts at its midpoint G, where $BG = 2a$, so $EG = a$.

The frame is about to slip clockwise, so friction is limiting at both contacts and acts to give an anticlockwise moment about O: a force μR at D along AB towards A, and a force μN at E along BC towards B. The normal reaction R acts along OD produced and N along OE produced, so N is parallel to AB and R is perpendicular to it.

Resolving parallel to AB:

$$N - \mu R = 3W\sin\theta$$

Resolving perpendicular to AB:

$$R + \mu N = 3W\cos\theta$$

Since OD makes an angle θ with the vertical, D is a horizontal distance $a\sin\theta$ to the left of O. Moving from O to E is a distance a parallel to AB, and from E to G a distance a along BC, so G is a horizontal distance $a\cos\theta + a\sin\theta$ to the right of O.

Taking moments about O, where the normal reactions pass through O and each friction force acts along a tangent, at perpendicular distance a from O:

$$\mu Ra + \mu Na + Wa\sin\theta = 2Wa(\cos\theta + \sin\theta)$$

so

$$\mu(R+N) = W(\sin\theta + 2\cos\theta)$$

Multiplying by 3 and using the two resolving equations to eliminate W and θ:

$$\begin{aligned} 3\mu(R+N) &= 3W\sin\theta + 2(3W\cos\theta) \\ &= (N - \mu R) + 2(R + \mu N) \end{aligned}$$

Rearranging:

$$\begin{aligned} 3\mu R + 3\mu N &= N - \mu R + 2R + 2\mu N \\ 4\mu R - 2R &= N - \mu N \\ R(4\mu - 2) &= N(1-\mu) \end{aligned}$$

so

$$R : N = (1-\mu) : (4\mu - 2)$$

(b) With $\mu = \frac{2}{3}$, part (a) gives

$$R : N = \tfrac{1}{3} : \tfrac{2}{3} = 1 : 2 \quad \Longrightarrow \quad N = 2R$$

Substituting $N = 2R$ and $\mu = \frac{2}{3}$ into the resolving equations from part (a):

$$\begin{aligned} 2R - \tfrac{2}{3}R = 3W\sin\theta \quad &\Longrightarrow \quad \tfrac{4}{3}R = 3W\sin\theta \\ R + \tfrac{4}{3}R = 3W\cos\theta \quad &\Longrightarrow \quad \tfrac{7}{3}R = 3W\cos\theta \end{aligned}$$

Dividing the first equation by the second gives

$$\tan\theta = \frac{4}{7}$$

The hypotenuse of the corresponding right-angled triangle is $\sqrt{4^2+7^2} = \sqrt{65}$, so $\sin\theta = \frac{4}{\sqrt{65}}$. Then

$$R = \frac{9W\sin\theta}{4} = \frac{9W}{4} \times \frac{4}{\sqrt{65}} = \frac{9\sqrt{65}}{65}W$$