Questions

1. **(a)** State the dimensions of force. **[1]**

(b) The braking distance of a vehicle can be modelled by the formula

$$d = m^a u^b R^c$$

where,

m is the mass of the vehicle in kg
u is the initial speed in $\mathrm{m\,s^{-1}}$
R is the constant braking force in N

Use dimensional analysis to find the values of a, b and c. **[4]**

Solution

(a) Force is mass × acceleration, so

$$[F] = M \times LT^{-2} = MLT^{-2}$$

So the dimensions of force are MLT^{-2}.

(b) Write the dimensions of each quantity:

$$[d] = L, \qquad [m] = M, \qquad [u] = LT^{-1}, \qquad [R] = MLT^{-2}$$

Given

$$d = m^a u^b R^c$$

the dimensions on the right-hand side are

$$\begin{aligned}[m^a u^b R^c] &= M^a (LT^{-1})^b (MLT^{-2})^c \\ &= M^a L^b T^{-b} \times M^c L^c T^{-2c} \\ &= M^{a+c} L^{b+c} T^{-b-2c}\end{aligned}$$

Since this must equal the dimensions of d,

$$M^{a+c} L^{b+c} T^{-b-2c} = L = M^0 L^1 T^0$$

Equating powers of M, L and T gives

$$a + c = 0, \qquad b + c = 1, \qquad -b - 2c = 0$$

From $-b - 2c = 0$,

$$b = -2c$$

Substitute into $b + c = 1$:

$$-2c + c = 1$$

$$-c = 1$$

$$c = -1$$

Then

$$b = -2(-1) = 2$$

and from $a + c = 0$,

$$a + (-1) = 0$$

so

$$a = 1$$

Hence

$$a = 1, \qquad b = 2, \qquad c = -1$$

2. A small satellite moves in a circular orbit of radius r about the centre of a much more massive planet of mass M. It is suggested that the period T of the orbit depends only on G, M and r, and that

$$T = kG^{\alpha} M^{\beta} r^{\gamma}$$

where k is a dimensionless constant.

It is given that

$$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$$

(c) Write down the dimensions of G. **[1]**

(d) By using dimensional analysis, determine the values of α, β and γ. **[3]**

Solution

(a) From

$$G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$$

the dimensions of G are

$$[G] = M^{-1}L^3T^{-2}$$

(b) Since k is dimensionless,

$$[T] = [G]^{\alpha}[M]^{\beta}[r]^{\gamma}$$

Now

$$[T] = T, \qquad [G] = M^{-1}L^3T^{-2}, \qquad [r] = L$$

so

$$\begin{aligned} T &= (M^{-1}L^3T^{-2})^{\alpha} M^{\beta} L^{\gamma} \\ &= M^{-\alpha+\beta} L^{3\alpha+\gamma} T^{-2\alpha} \end{aligned}$$

Equating powers of M, L and T with $M^0L^0T^1$:

$$\begin{aligned} -\alpha + \beta &= 0 \\ 3\alpha + \gamma &= 0 \\ -2\alpha &= 1 \end{aligned}$$

From $-2\alpha = 1$,

$$\alpha = -\frac{1}{2}$$

Then

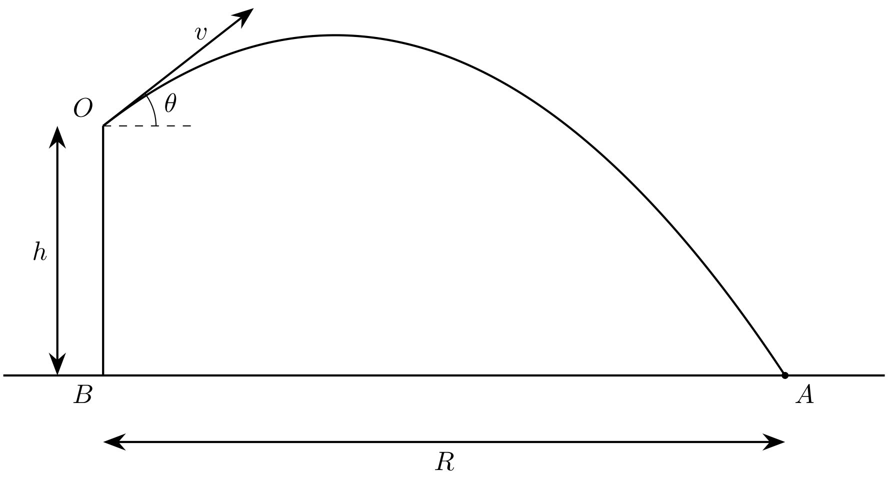
$$\beta = \alpha = -\frac{1}{2}$$

and

$$\gamma = -3\alpha = \frac{3}{2}$$

Hence

$$\alpha = -\frac{1}{2}, \qquad \beta = -\frac{1}{2}, \qquad \gamma = \frac{3}{2}$$



3. A particle is projected with speed v from a point O at the top of a cliff of height h above horizontal ground. The angle of projection is θ above the horizontal. The particle lands at the point A on the ground, and B is the foot of the cliff, so that $BA = R$, as shown in the diagram.

You are given that

$$R = \frac{v^\alpha \cos\theta}{g^\beta}\left(v^\gamma \sin\theta + \sqrt{v^\delta \sin^2\theta + 2gh}\right)$$

Use dimensional analysis to find α, β, γ and δ. **[4]**

Solution

Using dimensions,

$$[R] = L, \qquad [v] = LT^{-1}, \qquad [g] = LT^{-2}, \qquad [h] = L$$

and $\sin\theta$ and $\cos\theta$ are dimensionless.
First look at the expression inside the square root. Since terms being added must have the same dimensions,

$$[v^\delta \sin^2\theta] = [2gh]$$

Now

$$[2gh] = [g][h] = (LT^{-2})(L) = L^2T^{-2}$$

So

$$(LT^{-1})^\delta = L^2T^{-2}$$

Comparing powers of L and T,

$$\delta = 2$$

Hence the whole square-root term has dimensions

$$\sqrt{L^2T^{-2}} = LT^{-1}$$

Therefore $v^\gamma \sin\theta$ must also have dimensions LT^{-1}, so

$$(LT^{-1})^\gamma = LT^{-1}$$

which gives

$$\gamma = 1$$

Now the bracket

$$v^\gamma \sin\theta + \sqrt{v^\delta \sin^2\theta + 2gh}$$

has dimensions LT^{-1}. So

$$\begin{aligned}[R] &= \left[\frac{v^\alpha \cos\theta}{g^\beta}\right]\left[v^\gamma \sin\theta + \sqrt{v^\delta \sin^2\theta + 2gh}\right] \\ L &= \frac{(LT^{-1})^\alpha}{(LT^{-2})^\beta}(LT^{-1}) \\ &= L^{\alpha-\beta+1}T^{-\alpha+2\beta-1}\end{aligned}$$

Comparing powers with L^1T^0,

$$\alpha - \beta + 1 = 1, \qquad -\alpha + 2\beta - 1 = 0$$

From the first equation,

$$\alpha = \beta$$

Substitute into the second:

$$-\beta + 2\beta - 1 = 0$$

so

$$\beta = 1$$

and therefore

$$\alpha = 1$$

So the values are

$$\alpha = 1, \qquad \beta = 1, \qquad \gamma = 1, \qquad \delta = 2$$

4. The increase in pressure p pascals between the surface and the bottom of a liquid of depth h metres and density $\rho\,\mathrm{kg\,m^{-3}}$ is modelled by

$$p = k\rho^{\alpha}h^{\beta}g$$

where $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity and k is a dimensionless constant.

(a) State the dimensions of p. **[1]**

(b) Use dimensional analysis to find the value of α and the value of β. **[3]**

Solution

(a) Pressure is force per unit area.
Since [force] $= MLT^{-2}$ and [area] $= L^2$,

$$[p] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

So the dimensions of p are $ML^{-1}T^{-2}$.

(b) Using

$$p = k\rho^{\alpha}h^{\beta}g$$

and noting that k is dimensionless,

$$[p] = [\rho]^{\alpha}[h]^{\beta}[g]$$

Now

$$[\rho] = ML^{-3}, \qquad [h] = L, \qquad [g] = LT^{-2}$$

So the dimensions of the right-hand side are

$$\begin{aligned}(ML^{-3})^{\alpha}(L)^{\beta}(LT^{-2}) &= M^{\alpha}L^{-3\alpha}L^{\beta}LT^{-2}\\ &= M^{\alpha}L^{-3\alpha+\beta+1}T^{-2}\end{aligned}$$

But from part (a),

$$[p] = ML^{-1}T^{-2}$$

Equating powers of M, L and T:

$$\alpha = 1$$

and

$$-3\alpha + \beta + 1 = -1$$

Substituting $\alpha = 1$,

$$\begin{aligned}-3(1) + \beta + 1 &= -1\\ \beta - 2 &= -1\\ \beta &= 1\end{aligned}$$

Hence

$$\alpha = 1, \qquad \beta = 1$$

5. The force, F, exerted on a flat plate by a water jet is thought to depend on the density of the water, ρ, the cross-sectional area of the jet, A, and the speed of the water, v. You are given that ρ is measured in kg m^{-3}.

(a) Find the dimensions of density ρ. **[2]**

A model of the force suggests the relationship

$$F = \frac{1}{2}\rho^{\alpha} A^{\beta} v^{\gamma}$$

(b) Explain whether or not it is possible to use dimensional analysis to verify that the constant $\frac{1}{2}$ is correct. **[1]**

(c) Use dimensional analysis to find the values of α, β and γ. **[4]**

Solution

(a) Density is mass per unit volume.

$$[\rho] = \frac{M}{L^3} = ML^{-3}$$

So the dimensions of density are ML^{-3}.

(b) No, it is not possible.

Dimensional analysis only checks that both sides of an equation have the same dimensions. The number $\frac{1}{2}$ is a dimensionless constant, so dimensional analysis cannot show whether $\frac{1}{2}$ is the correct numerical value.

(c) We use

$$F = \frac{1}{2}\rho^{\alpha} A^{\beta} v^{\gamma}$$

The factor $\frac{1}{2}$ is dimensionless, so it does not affect the dimensions.
Now write the dimensions of each quantity:

$$[F] = MLT^{-2}, \qquad [\rho] = ML^{-3}, \qquad [A] = L^2, \qquad [v] = LT^{-1}$$

So

$$MLT^{-2} = (ML^{-3})^{\alpha}(L^2)^{\beta}(LT^{-1})^{\gamma}$$

Combining powers gives

$$MLT^{-2} = M^{\alpha}L^{-3\alpha+2\beta+\gamma}T^{-\gamma}$$

Equating powers of M, L and T:

$$\begin{aligned} \text{For } M: &\quad \alpha = 1 \\ \text{For } T: &\quad -\gamma = -2 \Rightarrow \gamma = 2 \\ \text{For } L: &\quad -3\alpha + 2\beta + \gamma = 1 \end{aligned}$$

Substitute $\alpha = 1$ and $\gamma = 2$ into the L-equation:

$$\begin{aligned} -3(1) + 2\beta + 2 &= 1 \\ -1 + 2\beta &= 1 \\ 2\beta &= 2 \\ \beta &= 1 \end{aligned}$$

Hence

$$\alpha = 1, \qquad \beta = 1, \qquad \gamma = 2$$

So the model is

$$F = \frac{1}{2}\rho A v^2$$

6. A motor pulls a crate of mass m along a rough horizontal floor. The equation below is suggested to model the pulling force exerted by the motor, F, while the crate moves in a straight line.

$$F = k_1 mv\frac{\mathrm{d}v}{\mathrm{d}x} + k_2\mu mg + k_3\frac{Q}{x}$$

in which,

- v is the speed of the crate when it has travelled distance x
- μ is the coefficient of friction between the crate and the floor
- g is the acceleration due to gravity
- Q is the total thermal energy transferred to the surroundings after the crate has moved distance x
- k_1, k_2 and k_3 are dimensionless constants

Determine whether the equation is dimensionally consistent. **[6]**

Solution

For the equation to be dimensionally consistent, every term must have the dimensions of force.

$$[F] = MLT^{-2}$$

Also, k_1, k_2 and k_3 are dimensionless, and μ is dimensionless.
Now consider each term on the right-hand side.

$$[v] = LT^{-1}, \qquad [x] = L$$

so

$$\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right] = \frac{[v]}{[x]} = \frac{LT^{-1}}{L} = T^{-1}$$

Hence for the first term,

$$\begin{aligned}\left[k_1 mv\frac{\mathrm{d}v}{\mathrm{d}x}\right] &= [m][v]\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right] \\ &= M\cdot LT^{-1}\cdot T^{-1} \\ &= MLT^{-2}\end{aligned}$$

For the second term,

$$\begin{aligned}[k_2\mu mg] &= [\mu][m][g] \\ &= 1\cdot M\cdot LT^{-2} \\ &= MLT^{-2}\end{aligned}$$

For the third term, Q is thermal energy transferred, so it has the dimensions of energy:

$$[Q] = ML^2T^{-2}$$

Therefore

$$\begin{aligned}\left[k_3\frac{Q}{x}\right] &= \frac{[Q]}{[x]} \\ &= \frac{ML^2T^{-2}}{L} \\ &= MLT^{-2}\end{aligned}$$

Each term on the right-hand side has dimensions MLT^{-2}, the same as force.
Therefore the equation is dimensionally consistent.

7. The speed of a particle falling vertically through a resistive medium is modelled by the formula

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

where,

v is the speed of the particle
m is the mass of the particle
g is the acceleration due to gravity
t is the time since the particle was released
k is a constant

Use dimensional analysis to determine the dimensions of k. **[4]**

Solution

For

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

the factor $1 - e^{-kt/m}$ must be dimensionless.
In particular, the exponent of an exponential must have no dimensions, so

$$\left[\frac{kt}{m}\right] = 1$$

Using

$$[t] = T, \qquad [m] = M$$

we get

$$\frac{[k][t]}{[m]} = 1$$

so

$$\frac{[k]T}{M} = 1$$

Hence

$$[k] = MT^{-1}$$

So the dimensions of k are

$$MT^{-1}$$

As a check,

$$\left[\frac{mg}{k}\right] = \frac{M \cdot LT^{-2}}{MT^{-1}} = LT^{-1}$$

which is the dimension of speed, as required.
Therefore k has dimensions MT^{-1}, with SI unit $\mathrm{kg\,s^{-1}}$.

8. A comet of mass m measured in kg follows an elliptical orbit around a planet. The semi-major axis of the orbit is a measured in metres and the orbital period is T measured in seconds. The eccentricity e of the orbit is dimensionless.

A constant μ associated with the planet is defined by

$$T^2 = \frac{4\pi^2 a^3}{\mu}$$

and the angular momentum of the comet is given by

$$H = m\sqrt{\mu a(1-e^2)}$$

By using dimensional analysis, find the dimensions of:

(a) μ **[3]**

(b) H **[3]**

Solution

(a) From

$$T^2 = \frac{4\pi^2 a^3}{\mu}$$

and since $4\pi^2$ is dimensionless,

$$[T]^2 = \frac{[a]^3}{[\mu]}$$

Using $[T] = T$ and $[a] = L$,

$$T^2 = \frac{L^3}{[\mu]}$$

so

$$[\mu] = \frac{L^3}{T^2} = L^3T^{-2}$$

Hence the dimensions of μ are L^3T^{-2}.

(b) The angular momentum is

$$H = m\sqrt{\mu a(1-e^2)}$$

Here $[m] = M$, and since e is dimensionless, $(1-e^2)$ is also dimensionless.

Therefore

$$[H] = [m]\sqrt{[\mu][a]}$$

Using $[\mu] = L^3T^{-2}$ from part (a) and $[a] = L$,

$$\begin{aligned}[H] &= M\sqrt{(L^3T^{-2})(L)} \\ &= M\sqrt{L^4T^{-2}} \\ &= ML^2T^{-1}\end{aligned}$$

Hence the dimensions of H are ML^2T^{-1}.

9. A small ball of mass m kg is projected vertically upwards with speed $u\,\mathrm{m\,s^{-1}}$. During the motion, when its speed is $v\,\mathrm{m\,s^{-1}}$, it experiences an air resistance of magnitude kv^2 N, where k is a constant.

(a) Determine the dimensions of k. **[3]**

The greatest height reached by the ball is H m. A formula approximating H is

$$H = \frac{u^2}{2g} - \frac{ku^4}{4mg^2} + \frac{1}{6}k^2 m^\alpha u^\beta g^\gamma$$

where $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(b) Use dimensional analysis to find α, β and γ. **[4]**

Solution

(a) The air resistance has magnitude kv^2, and this is a force.

$$[k][v]^2 = [F]$$

Now

$$[F] = MLT^{-2}, \qquad [v] = LT^{-1}$$

So

$$\begin{aligned}[k] &= \frac{[F]}{[v]^2} \\ &= \frac{MLT^{-2}}{(LT^{-1})^2} \\ &= \frac{MLT^{-2}}{L^2T^{-2}} \\ &= ML^{-1}\end{aligned}$$

So the dimensions of k are ML^{-1}.
Hence the SI unit of k is $\mathrm{kg\,m^{-1}}$.

(b) Since H is a height,

$$[H] = L$$

Therefore each term in

$$H = \frac{u^2}{2g} - \frac{ku^4}{4mg^2} + \frac{1}{6}k^2 m^\alpha u^\beta g^\gamma$$

must have dimensions of length.
So the third term must satisfy

$$[k^2 m^\alpha u^\beta g^\gamma] = L$$

Using

$$[k] = ML^{-1}, \qquad [m] = M, \qquad [u] = LT^{-1}, \qquad [g] = LT^{-2}$$

we get

$$\begin{aligned}[k^2 m^\alpha u^\beta g^\gamma] &= (ML^{-1})^2(M)^\alpha(LT^{-1})^\beta(LT^{-2})^\gamma \\ &= M^{2+\alpha}L^{-2+\beta+\gamma}T^{-\beta-2\gamma}\end{aligned}$$

This must equal $M^0L^1T^0$, so equating powers gives

$$\begin{aligned}2 + \alpha &= 0 \\ -2 + \beta + \gamma &= 1 \\ -\beta - 2\gamma &= 0\end{aligned}$$

From the first equation,

$$\alpha = -2$$

From the third equation,

$$\beta = -2\gamma$$

Substitute into the second equation:

$$\begin{aligned} -2 + (-2\gamma) + \gamma &= 1 \\ -2 - \gamma &= 1 \\ -\gamma &= 3 \\ \gamma &= -3 \end{aligned}$$

Then

$$\beta = -2(-3) = 6$$

Hence

$$\alpha = -2, \qquad \beta = 6, \qquad \gamma = -3$$

10. A small probe is launched vertically upwards from the surface of a spherical moon. The radius of the moon is R metres, and the probe is at a distance r metres from the centre of the moon. The gravitational field strength g at distance r is modelled by

$$g = \frac{\Gamma}{r^2}$$

where Γ is a constant associated with the moon.

The probe has initial speed u at the surface and speed v when it reaches distance r. These speeds are related by

$$v^2 = u^2 - 2\Gamma\left(\frac{1}{R} - \frac{1}{r}\right)$$

The work done against gravity in moving a probe of mass m from the surface to distance r is

$$W = m\Gamma\left(\frac{1}{R} - \frac{1}{r}\right)$$

(a) Find the dimensions of Γ. **[2]**

(b) Use dimensional analysis to show that the equation for v^2 is dimensionally consistent. **[2]**

(c) Find the dimensions and SI units of W. **[2]**

Solution

(a) From

$$g = \frac{\Gamma}{r^2}$$

and since

$$[g] = LT^{-2}, \qquad [r] = L$$

we have

$$[\Gamma] = [g][r^2] = LT^{-2} \times L^2 = L^3T^{-2}$$

So the dimensions of Γ are

$$[\Gamma] = L^3T^{-2}$$

(b) In

$$v^2 = u^2 - 2\Gamma\left(\frac{1}{R} - \frac{1}{r}\right)$$

the dimensions of v^2 and u^2 are

$$[v^2] = [u^2] = (LT^{-1})^2 = L^2T^{-2}$$

Also,

$$\left[\frac{1}{R}\right] = \left[\frac{1}{r}\right] = L^{-1}$$

so

$$\left[\left(\frac{1}{R} - \frac{1}{r}\right)\right] = L^{-1}$$

Since 2 is dimensionless,

$$\left[2\Gamma\left(\frac{1}{R} - \frac{1}{r}\right)\right] = [\Gamma]L^{-1} = L^3T^{-2} \times L^{-1} = L^2T^{-2}$$

Therefore every term in the equation has dimensions L^2T^{-2}, so the equation is dimensionally consistent.

(c) Using

$$W = m\Gamma\left(\frac{1}{R} - \frac{1}{r}\right)$$

with

$$[m] = M, \qquad [\Gamma] = L^3T^{-2}, \qquad \left[\left(\frac{1}{R} - \frac{1}{r}\right)\right] = L^{-1}$$

gives

$$[W] = M \times L^3T^{-2} \times L^{-1} = ML^2T^{-2}$$

So the dimensions of W are

$$[W] = ML^2T^{-2}$$

Its SI units are

$$\mathrm{kg\,m^2\,s^{-2}}$$

which is the joule, J.