

FURTHER MECHANICS / KINEMATICS & DYNAMICS

Projectile Motion

Worked solutions

9 questions · 93 marks

Instructions

Unless a question states otherwise, take $g = 9.8 \text{ m s}^{-2}$ wherever a numerical value of g is required. Solutions also give final answers for $g = 10 \text{ m s}^{-2}$ where these differ.

Questions

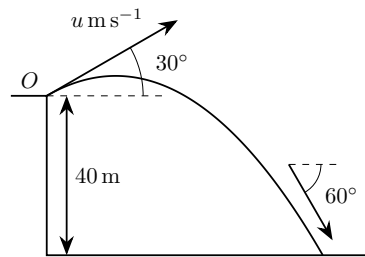
Question 1	8 marks	2
Question 2	6 marks	4
Question 3	9 marks	6
Question 4	9 marks	8
Question 5	14 marks	11
Question 6	12 marks	13
Question 7	11 marks	16
Question 8	7 marks	18
Question 9	17 marks	20

1. A particle P is projected from a point O at the edge of a cliff, 40 m above horizontal ground. Its initial speed is $u \text{ m s}^{-1}$ and its angle of projection is 30° above the horizontal, directed away from the cliff. The particle moves freely under gravity until it strikes the ground.

Immediately before impact, the direction of motion of P is perpendicular to its initial direction of motion.

Find the initial speed u and the time taken to reach the ground, giving both answers in exact form. [8]

Solution



Take horizontal motion away from the cliff and vertical motion upwards as positive. Use $g = 9.8 \text{ m s}^{-2}$, and let t be the time taken to reach the ground.

The initial horizontal and vertical velocity components are

$$v_{x0} = u \cos 30^\circ = \frac{\sqrt{3}u}{2}, \quad v_{y0} = u \sin 30^\circ = \frac{u}{2}$$

There is no horizontal acceleration, so the horizontal velocity remains $\frac{\sqrt{3}u}{2}$. Since this is positive, the impact direction perpendicular to the initial direction must be 60° below the horizontal. Hence the vertical velocity immediately before impact is

$$\begin{aligned} v_y &= -v_x \tan 60^\circ \\ &= -\frac{\sqrt{3}u}{2} \times \sqrt{3} \\ &= -\frac{3u}{2} \end{aligned}$$

The constant acceleration formula is

$$v^2 = v_0^2 + 2as$$

Applying this to the vertical motion, with initial velocity $\frac{u}{2}$, acceleration $-g$ and displacement -40 m :

$$\left(-\frac{3u}{2}\right)^2 = \left(\frac{u}{2}\right)^2 + 2(-g)(-40)$$

$$\frac{9u^2}{4} - \frac{u^2}{4} = 80g$$

$$2u^2 = 80g$$

$$u^2 = 40g = 392$$

The initial speed is positive, so

$$u = 14\sqrt{2} \text{ m s}^{-1}$$

The constant acceleration formula is

$$v = v_0 + at$$

Applying this to the vertical motion:

$$-\frac{3u}{2} = \frac{u}{2} - gt$$

$$gt = 2u$$

$$t = \frac{2u}{g} = \frac{2 \times 14\sqrt{2}}{9.8} = \frac{20\sqrt{2}}{7} \text{ s}$$

Final answers with $g = 10$: $u = 20 \text{ m s}^{-1}$; $t = 4 \text{ s}$

2. A particle P is projected from a point O , which is 30 m above horizontal ground, with speed $u \text{ m s}^{-1}$ at an angle θ above the horizontal, where $0 < \theta < 90^\circ$.

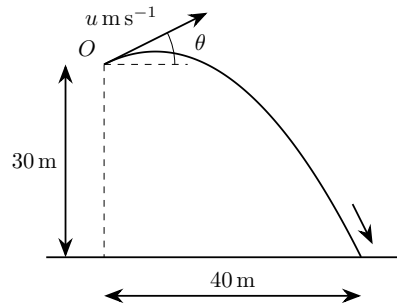
The particle moves freely under gravity and lands on the ground at a horizontal distance of 40 m from O .

The velocity of P immediately before it lands is perpendicular to its initial velocity.

Find u and θ , giving the angle to one decimal place.

[6]

Solution



Take horizontal motion away from O and vertical motion upwards as positive. Use $g = 9.8 \text{ m s}^{-2}$. Let the flight time be T s and the signed vertical velocity immediately before landing be $v_y \text{ m s}^{-1}$.

The horizontal velocity remains $u \cos \theta$. The constant acceleration formula is

$$s = v_0 t + \frac{1}{2} a t^2$$

Applying this horizontally, with $a = 0$, gives

$$40 = u \cos \theta T \implies T = \frac{40}{u \cos \theta}$$

The final velocity points forwards and downwards, perpendicular to the initial velocity, so its direction is $\theta - 90^\circ$. Hence

$$\frac{v_y}{u \cos \theta} = \tan(\theta - 90^\circ) = -\cot \theta \implies v_y = -u \cos \theta \cot \theta$$

The constant acceleration formula is

$$s = \frac{1}{2}(v_0 + v)t$$

Applying this vertically:

$$\begin{aligned} -30 &= \frac{1}{2}(u \sin \theta + v_y)T \\ &= \frac{1}{2}(u \sin \theta - u \cos \theta \cot \theta) \frac{40}{u \cos \theta} \\ &= 20(\tan \theta - \cot \theta) \end{aligned}$$

Multiplying by $\tan \theta$ and rearranging:

$$\begin{aligned} 2 \tan^2 \theta + 3 \tan \theta - 2 &= 0 \\ (2 \tan \theta - 1)(\tan \theta + 2) &= 0 \end{aligned}$$

Since $0 < \theta < 90^\circ$, we reject $\tan \theta = -2$, giving

$$\tan \theta = \frac{1}{2} \implies \theta = \arctan\left(\frac{1}{2}\right) \approx 26.6^\circ \quad (1 \text{ d.p.})$$

The constant acceleration formula is

$$v = v_0 + at$$

Applying this vertically:

$$v_y = u \sin \theta - gT$$

$$\begin{aligned} gT &= u \sin \theta + u \cos \theta \cot \theta \\ &= \frac{u(\sin^2 \theta + \cos^2 \theta)}{\sin \theta} = \frac{u}{\sin \theta} \end{aligned}$$

Substituting $T = \frac{40}{u \cos \theta}$:

$$\begin{aligned} \frac{40g}{u \cos \theta} &= \frac{u}{\sin \theta} \\ u^2 &= 40g \tan \theta = 20g \\ u &= \sqrt{20 \times 9.8} = 14 \text{ m s}^{-1} \end{aligned}$$

Final answers with $g = 10$: $u = 14.1 \text{ m s}^{-1}$

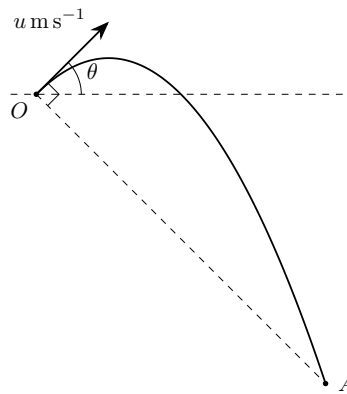
3. A particle P is projected from a point O at time $t = 0$ with speed $u \text{ m s}^{-1}$ at an angle θ above the horizontal, where $0 < \theta < 90^\circ$.
The particle moves freely under gravity, with $g = 10 \text{ m s}^{-2}$.
After 6 seconds, its speed is $\sqrt{5}u \text{ m s}^{-1}$.

(a) Show that

$$u^2 + 30u \sin \theta - 900 = 0 \quad [3]$$

- (b) At time $t = 6 \text{ s}$, the particle is at a point A below the horizontal level of O .
The displacement vector $\overrightarrow{OA}$ is perpendicular to the initial velocity of P .
Find the exact value of u , the value of θ , and the exact distance OA . [6]

Solution



- (a) Take horizontal motion to the right and vertical motion upwards as positive. The acceleration components are 0 and -10 m s^{-2} .

The constant acceleration formula is

$$v = v_0 + at$$

Applying this, the velocity components after 6 seconds are

$$v_x = u \cos \theta, \quad v_y = u \sin \theta - 10(6) = u \sin \theta - 60$$

The square of the speed is $v_x^2 + v_y^2$, so

$$\begin{aligned} (u \cos \theta)^2 + (u \sin \theta - 60)^2 &= (\sqrt{5}u)^2 \\ u^2(\cos^2 \theta + \sin^2 \theta) - 120u \sin \theta + 3600 &= 5u^2 \\ 4u^2 + 120u \sin \theta - 3600 &= 0 \\ u^2 + 30u \sin \theta - 900 &= 0 \end{aligned}$$

- (b) The constant acceleration formula is

$$s = v_0 t + \frac{1}{2}at^2$$

Applying this, the horizontal and vertical displacements after 6 seconds are

$$x = (u \cos \theta)(6) = 6u \cos \theta$$

$$y = (u \sin \theta)(6) + \frac{1}{2}(-10)(6^2) = 6u \sin \theta - 180$$

Perpendicular vectors have zero scalar product. Hence the displacement and initial velocity satisfy

$$\begin{pmatrix} 6u \cos \theta \\ 6u \sin \theta - 180 \end{pmatrix} \cdot \begin{pmatrix} u \cos \theta \\ u \sin \theta \end{pmatrix} = 0$$

$$6u^2 \cos^2 \theta + 6u^2 \sin^2 \theta - 180u \sin \theta = 0$$

$$6u^2 - 180u \sin \theta = 0$$

Since $u > 0$, dividing by $6u$ gives

$$u = 30 \sin \theta$$

Substituting $30u \sin \theta = u^2$ into the result from part (a):

$$u^2 + u^2 - 900 = 0$$

$$u^2 = 450$$

$$u = 15\sqrt{2} \text{ m s}^{-1}$$

Using $u = 30 \sin \theta$ and $0 < \theta < 90^\circ$:

$$\sin \theta = \frac{15\sqrt{2}}{30} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

The displacement is therefore

$$\overrightarrow{OA} = \begin{pmatrix} 6(15\sqrt{2}) \cos 45^\circ \\ 6(15\sqrt{2}) \sin 45^\circ - 180 \end{pmatrix} \text{ m} = \begin{pmatrix} 90 \\ -90 \end{pmatrix} \text{ m}$$

Its magnitude gives the distance:

$$OA = \sqrt{90^2 + (-90)^2} = 90\sqrt{2} \text{ m}$$

4. At time $t = 0$, a particle P is projected from a point O on horizontal ground with speed 26 m s^{-1} at an angle α above the horizontal, where $\sin \alpha = \frac{12}{13}$.
A point A on the ground lies 40 m from O in the direction of the horizontal motion of P .

At time $t = T \text{ s}$, where $T > 0$, a second particle Q is projected vertically upwards from A with speed 22 m s^{-1} .

The particles collide while Q is travelling upwards.

Before the collision, both particles move freely under gravity.

Neglect air resistance throughout.

(a) Find the time at which the particles collide and their height above the ground at that instant. [2]

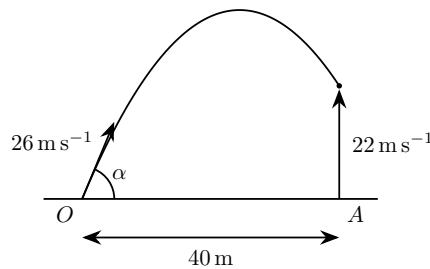
(b) Find the exact value of T . [3]

(c) Immediately after the collision, the speed of P is $v \text{ m s}^{-1}$ and its direction of motion is above the horizontal.

Its horizontal velocity component is unchanged by the collision, and it subsequently moves freely under gravity.

Find its greatest height above the ground in terms of v . Hence find the least possible value of v for P to reach a height of 40 m above the ground. [4]

Solution



Take horizontal motion from O towards A and vertical motion upwards as positive. Use $g = 9.8 \text{ m s}^{-2}$.

(a) Since α is above the horizontal, $\cos \alpha = \frac{5}{13}$. The initial velocity components of P are

$$v_{x0} = 26 \times \frac{5}{13} = 10 \text{ m s}^{-1}, \quad v_{y0} = 26 \times \frac{12}{13} = 24 \text{ m s}^{-1}$$

The collision occurs directly above A . The constant acceleration formula is

$$s = v_0 t + \frac{1}{2} a t^2$$

Applying this horizontally, where the acceleration is zero, and then vertically:

$$40 = 10t \implies t = 4 \text{ s}$$

$$\text{Height} = 24(4) - \frac{1}{2}(9.8)(4^2) = 17.6 \text{ m}$$

- (b) At the collision, Q has been moving for $(4 - T)$ s. Applying the displacement formula from part (a) to Q :

$$17.6 = 22(4 - T) - 4.9(4 - T)^2$$

$$17.6 = 9.6 + 17.2T - 4.9T^2$$

$$49T^2 - 172T + 80 = 0$$

$$T = \frac{86 \pm 2\sqrt{869}}{49}$$

The constant acceleration formula is

$$v_y = v_{y0} + at$$

Applying this to Q , the condition that Q is travelling upwards gives

$$22 - 9.8(4 - T) > 0$$

$$T > \frac{86}{49}$$

Therefore the required root is

$$T = \frac{86 + 2\sqrt{869}}{49} \text{ s}$$

- (c) Let $w > 0$ be the upward velocity component of P immediately after the collision. Its horizontal component remains 10 m s^{-1} , so

$$w^2 + 10^2 = v^2 \implies w^2 = v^2 - 100$$

Let the greatest height above the ground be H m. At this height the vertical velocity is zero. The constant acceleration formula is

$$v_y^2 = v_{y0}^2 + 2as$$

Applying this to the vertical motion after the collision, with $v_{y0} = w$:

$$0 = w^2 - 2(9.8)(H - 17.6)$$

$$H = 17.6 + \frac{v^2 - 100}{19.6}$$

Thus the greatest height is

$$\left(\frac{88}{5} + \frac{5(v^2 - 100)}{98} \right) \text{ m}$$

Hence, for P to reach 40 m:

$$17.6 + \frac{v^2 - 100}{19.6} \geq 40$$

$$v^2 \geq 100 + 19.6(40 - 17.6) = \frac{13476}{25}$$

Since v is positive, the least possible speed is

$$v_{\min} = \frac{2\sqrt{3369}}{5} \text{ m s}^{-1}$$

Final answers with $g = 10$: (a) height 16 m; (b) $T = \frac{9+\sqrt{41}}{5}$ s; (c) greatest height $\left(11 + \frac{v^2}{20}\right)$ m,
 $v_{\min} = 2\sqrt{145} \text{ m s}^{-1}$

5. A particle P is projected from a point A which is vertically above a fixed point O on horizontal ground, with $OA = h$.

Its initial velocity is $u(4\mathbf{i} + 3\mathbf{j})$, where $u > 0$ and $h > 0$.

The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertically upwards respectively.

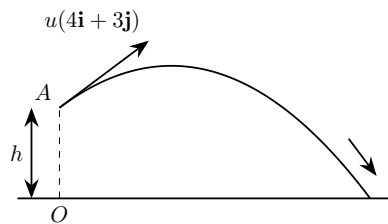
The particle moves freely under gravity, with constant acceleration $-g\mathbf{j}$.

At time $t = 0$, the particle leaves A .

Its position vector relative to O at time t is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

- (a) Starting from an expression for $\ddot{\mathbf{r}}$, use integration to obtain $\mathbf{r}$ in terms of t , u , h and g . [4]
- (b) Eliminate t to obtain the Cartesian equation of the path of P . [3]
- (c) Hence find the maximum height of P above the ground, in terms of u , h and g . [3]
- (d) When P first strikes the ground, its velocity is perpendicular to its initial velocity. Determine h in terms of u and g . [4]

Solution



- (a) The acceleration is

$$\ddot{\mathbf{r}} = -g\mathbf{j}, \quad \ddot{x} = 0, \quad \ddot{y} = -g$$

Integrating each component with respect to t :

$$\dot{x} = c_1, \quad \dot{y} = -gt + c_2$$

The initial velocity is $4u\mathbf{i} + 3u\mathbf{j}$, so $c_1 = 4u$ and $c_2 = 3u$. Integrating again:

$$x = 4ut + c_3, \quad y = 3ut - \frac{1}{2}gt^2 + c_4$$

At $t = 0$, the particle is at A , so $x = 0$ and $y = h$. Hence $c_3 = 0$ and $c_4 = h$, giving

$$\mathbf{r} = 4ut\mathbf{i} + \left(h + 3ut - \frac{1}{2}gt^2\right)\mathbf{j}$$

- (b) From the horizontal component in part (a):

$$x = 4ut \implies t = \frac{x}{4u}$$

Substituting into the vertical component:

$$\begin{aligned} y &= h + 3u\left(\frac{x}{4u}\right) - \frac{1}{2}g\left(\frac{x}{4u}\right)^2 \\ &= h + \frac{3x}{4} - \frac{gx^2}{32u^2} \end{aligned}$$

(c) Completing the square in the equation from part (b):

$$\begin{aligned}y &= h - \frac{g}{32u^2} \left(x^2 - \frac{24u^2}{g}x \right) \\&= h - \frac{g}{32u^2} \left[\left(x - \frac{12u^2}{g} \right)^2 - \frac{144u^4}{g^2} \right] \\&= h + \frac{9u^2}{2g} - \frac{g}{32u^2} \left(x - \frac{12u^2}{g} \right)^2\end{aligned}$$

Since the squared term is non-negative and $g > 0$, the maximum occurs at $x = \frac{12u^2}{g} > 0$. The maximum height above the ground is therefore

$$h + \frac{9u^2}{2g}$$

(d) The horizontal velocity remains $4u$. Let the vertical velocity when the particle strikes the ground be v_y . Perpendicular vectors have zero scalar product, so the two velocities satisfy

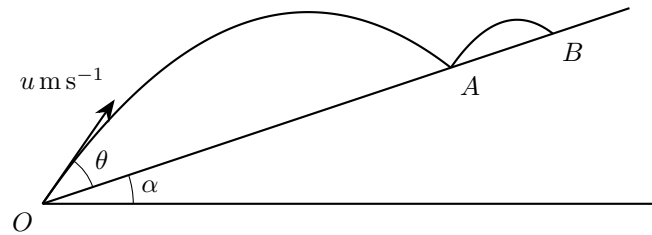
$$\begin{aligned}(4u)(4u) + (3u)v_y &= 0 \\v_y &= -\frac{16u}{3}\end{aligned}$$

The constant acceleration formula is

$$v^2 = v_0^2 + 2as$$

For the vertical motion, $v = v_y$, $v_0 = 3u$, $a = -g$ and the displacement to the ground is $s = -h$. Substituting:

$$\begin{aligned}\left(-\frac{16u}{3} \right)^2 &= (3u)^2 + 2(-g)(-h) \\ \frac{256u^2}{9} &= 9u^2 + 2gh \\ 2gh &= \frac{175u^2}{9} \\ h &= \frac{175u^2}{18g}\end{aligned}$$



6. A small ball P is projected from a point O at the foot of a smooth plane inclined at an angle α to the horizontal, where $\tan \alpha = \frac{1}{3}$. Its initial speed is $u \text{ m s}^{-1}$, and its initial velocity makes an angle θ above the plane, where $\tan \theta = \frac{3}{4}$.

The ball first strikes the plane at A , where $OA = 15\sqrt{10} \text{ m}$.

Its motion takes place in the vertical plane containing a line of greatest slope of the inclined plane, and air resistance is negligible.

- (a) Find u and the components of the ball's velocity parallel and perpendicular to the plane immediately before impact at A . State the direction of each component. [8]

The ball rebounds from the plane at A and next strikes it at B , further up the plane, where

$$AB = \frac{15\sqrt{10}}{4} \text{ m}$$

- (b) Find the coefficient of restitution between the ball and the plane, explaining why any other algebraic solution must be rejected. [4]

Solution

- (a) Take horizontal motion to the right and vertical motion upwards as positive. Use $g = 9.8 \text{ m s}^{-2}$.

From the given tangents:

$$\sin \alpha = \frac{1}{\sqrt{10}}, \quad \cos \alpha = \frac{3}{\sqrt{10}}, \quad \sin \theta = \frac{3}{5}, \quad \cos \theta = \frac{4}{5}$$

The launch angle above the horizontal is $\alpha + \theta$, so the initial horizontal and vertical velocity components are

$$v_{x0} = u \cos(\alpha + \theta) = u \left(\frac{3}{\sqrt{10}} \frac{4}{5} - \frac{1}{\sqrt{10}} \frac{3}{5} \right) = \frac{9u}{5\sqrt{10}}$$

$$v_{y0} = u \sin(\alpha + \theta) = u \left(\frac{1}{\sqrt{10}} \frac{4}{5} + \frac{3}{\sqrt{10}} \frac{3}{5} \right) = \frac{13u}{5\sqrt{10}}$$

The horizontal and vertical displacements from O to A are

$$x_A = OA \cos \alpha = 15\sqrt{10} \frac{3}{\sqrt{10}} = 45 \text{ m}$$

$$y_A = OA \sin \alpha = 15\sqrt{10} \frac{1}{\sqrt{10}} = 15 \text{ m}$$

The constant acceleration formula is

$$s = v_0 t + \frac{1}{2} a t^2$$

Let the flight time from O to A be t_A . Applying this formula horizontally and vertically gives

$$45 = \frac{9u}{5\sqrt{10}}t_A \implies ut_A = 25\sqrt{10}$$

$$15 = \frac{13u}{5\sqrt{10}}t_A - \frac{1}{2}gt_A^2$$

$$15 = 65 - \frac{1}{2}gt_A^2$$

$$t_A = \frac{10}{\sqrt{g}} \text{ s}$$

Hence the initial speed is

$$u = \frac{25\sqrt{10}}{10/\sqrt{g}} = \frac{5}{2}\sqrt{10g} \text{ m s}^{-1}$$

The constant acceleration formula is

$$v = v_0 + at$$

Immediately before impact, the horizontal and vertical components are therefore

$$v_x = \frac{9u}{5\sqrt{10}} = \frac{9}{2}\sqrt{g} \text{ m s}^{-1}$$

$$v_y = \frac{13u}{5\sqrt{10}} - gt_A = \frac{13}{2}\sqrt{g} - 10\sqrt{g} = -\frac{7}{2}\sqrt{g} \text{ m s}^{-1}$$

Resolving parallel to the plane, positive up the plane, and perpendicular to it, positive away from the plane:

$$v_{\parallel} = v_x \cos \alpha + v_y \sin \alpha = \frac{27\sqrt{g} - 7\sqrt{g}}{2\sqrt{10}} = \sqrt{10g} \text{ m s}^{-1}$$

$$v_{\perp} = -v_x \sin \alpha + v_y \cos \alpha = \frac{-9\sqrt{g} - 21\sqrt{g}}{2\sqrt{10}} = -\frac{3}{2}\sqrt{10g} \text{ m s}^{-1}$$

Using $g = 9.8 \text{ m s}^{-2}$, the required speed and component magnitudes are

$$u = \frac{35\sqrt{2}}{2} \text{ m s}^{-1} \approx 24.7 \text{ m s}^{-1}$$

$$\text{Parallel component} = 7\sqrt{2} \text{ m s}^{-1} \approx 9.90 \text{ m s}^{-1} \quad \text{up the plane}$$

$$\text{Perpendicular component} = \frac{21\sqrt{2}}{2} \text{ m s}^{-1} \approx 14.8 \text{ m s}^{-1} \quad \text{towards the plane}$$

- (b) Let the coefficient of restitution be e . Since the plane is smooth, the parallel velocity component is unchanged by impact. The perpendicular component reverses direction and its magnitude is multiplied by e . Thus, immediately after impact, the components are

$$v_{\parallel} = \sqrt{10g} \text{ m s}^{-1}, \quad v_{\perp} = \frac{3e}{2}\sqrt{10g} \text{ m s}^{-1}$$

Take up the plane and away from the plane as positive. The corresponding acceleration components are $-g \sin \alpha = -\frac{g}{\sqrt{10}}$ and $-g \cos \alpha = -\frac{3g}{\sqrt{10}}$.

Let the flight time from A to B be t_B . Using the displacement formula from part (a), the perpendicular displacement is zero, so

$$\begin{aligned} 0 &= \frac{3e}{2}\sqrt{10g}t_B - \frac{3g}{2\sqrt{10}}t_B^2 \\ 0 &= \frac{3}{2}t_B\left(e\sqrt{10g} - \frac{g}{\sqrt{10}}t_B\right) \\ t_B &= \frac{10e}{\sqrt{g}} \text{ s} \end{aligned}$$

Applying the same displacement formula parallel to the plane:

$$\begin{aligned} AB &= \sqrt{10g}t_B - \frac{g}{2\sqrt{10}}t_B^2 \\ &= \sqrt{10g}\frac{10e}{\sqrt{g}} - \frac{g}{2\sqrt{10}}\frac{100e^2}{g} \\ &= 5\sqrt{10}(2e - e^2) \text{ m} \end{aligned}$$

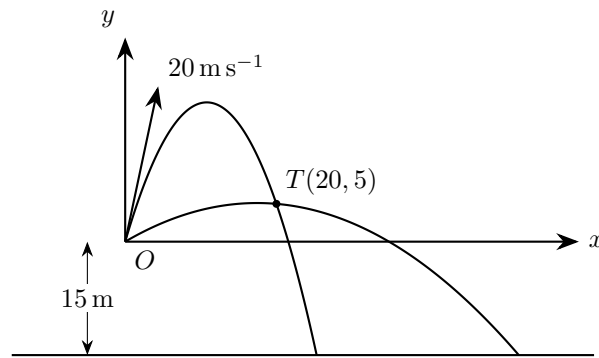
Substituting the given distance:

$$\begin{aligned} \frac{15\sqrt{10}}{4} &= 5\sqrt{10}(2e - e^2) \\ 4e^2 - 8e + 3 &= 0 \\ (2e - 1)(2e - 3) &= 0 \\ e &= \frac{1}{2} \quad \text{or} \quad e = \frac{3}{2} \end{aligned}$$

A coefficient of restitution must satisfy $0 \leq e \leq 1$, so $\frac{3}{2}$ is rejected. Therefore

$$e = \frac{1}{2}$$

Final answers with $g = 10$: (a) $u = 25 \text{ m s}^{-1}$, parallel component 10 m s^{-1} up the plane, perpendicular component 15 m s^{-1} towards the plane



7. A launcher projects balls at speed 20 m s^{-1} from a point O , which is 15 m above horizontal ground. Horizontal and vertically upwards displacements from O are denoted by $x \text{ m}$ and $y \text{ m}$ respectively. A small target T has coordinates $(20, 5)$, as shown in the diagram above.

At time $t = 0$, a ball B is projected at an angle θ above the positive horizontal direction, where $0 \leq \theta < 90^\circ$. Model the ball as a particle moving freely under gravity until it first reaches the ground.

- (a) By first writing x and y in terms of t , find the two possible values of θ for which the ball passes through T . Give the angles to the nearest 0.1° . [5]
- (b) For a fixed horizontal displacement $x > 0$, use a quadratic in $\tan \theta$ to determine the greatest height y that the ball can reach. Hence state the equation of the upper boundary of the reachable region to the right of the y -axis. [3]
- (c) Hence find the greatest possible horizontal distance from O to the point where the ball first reaches the ground, giving your answer exactly. Find also the launch angle that achieves this distance, to the nearest 0.1° . [3]

Solution

Use $g = 9.8 \text{ m s}^{-2}$.

- (a) The initial horizontal and vertical velocities are $20 \cos \theta$ and $20 \sin \theta$. The constant acceleration formula is

$$s = v_0 t + \frac{1}{2} a t^2$$

Applying this in each direction:

$$x = 20t \cos \theta$$

$$y = 20t \sin \theta - 4.9t^2$$

From the horizontal equation, $t = \frac{x}{20 \cos \theta}$. Substituting into the vertical equation and using $\sec^2 \theta = 1 + \tan^2 \theta$:

$$\begin{aligned} y &= x \tan \theta - \frac{49x^2}{4000 \cos^2 \theta} \\ &= x \tan \theta - \frac{49x^2}{4000} (1 + \tan^2 \theta) \end{aligned}$$

At T , $x = 20$ and $y = 5$, so

$$5 = 20 \tan \theta - \frac{49}{10}(1 + \tan^2 \theta)$$

$$49 \tan^2 \theta - 200 \tan \theta + 99 = 0$$

$$\tan \theta = \frac{100 \pm \sqrt{5149}}{49}$$

Both roots are positive and give angles in the permitted interval:

$$\theta = 30.0^\circ \quad \text{or} \quad 74.1^\circ \quad (1 \text{ d.p.})$$

(b) Rearranging the trajectory equation from part (a) gives a quadratic in $\tan \theta$:

$$\frac{49x^2}{4000} \tan^2 \theta - x \tan \theta + y + \frac{49x^2}{4000} = 0$$

For a real launch angle, the discriminant must be non-negative:

$$\begin{aligned} x^2 - 4 \left(\frac{49x^2}{4000} \right) \left(y + \frac{49x^2}{4000} \right) &\geq 0 \\ y + \frac{49x^2}{4000} &\leq \frac{1000}{49} \\ y &\leq \frac{1000}{49} - \frac{49x^2}{4000} \end{aligned}$$

Equality gives the greatest height. The repeated root is $\tan \theta = \frac{2000}{49x} > 0$, so the corresponding angle is permitted. Since the motion ends at $y = -15$, this boundary is reachable only where

$$\frac{1000}{49} - \frac{49x^2}{4000} \geq -15 \implies x^2 \leq \frac{6\,940\,000}{2401}$$

Thus the upper boundary of the reachable region is

$$y = \frac{1000}{49} - \frac{49x^2}{4000}, \quad 0 < x \leq \frac{100\sqrt{694}}{49}$$

(c) Using part (b), the greatest horizontal distance occurs where the upper boundary meets the ground. Let this distance be R m. Setting $y = -15$:

$$\begin{aligned} -15 &= \frac{1000}{49} - \frac{49R^2}{4000} \\ R^2 &= \frac{4000}{49} \left(15 + \frac{1000}{49} \right) \\ R &= \frac{100\sqrt{694}}{49} \text{ m} \end{aligned}$$

The repeated-root condition from part (b) gives the launch angle:

$$\begin{aligned} \tan \theta &= \frac{2000}{49R} = \frac{20}{\sqrt{694}} \\ \theta &\approx 37.2^\circ \quad (1 \text{ d.p.}) \end{aligned}$$

Final answers with $g = 10$: (a) 30.4° , 73.7° ; (b) $y = 20 - \frac{x^2}{80}$, $0 < x \leq 20\sqrt{7}$; (c) $20\sqrt{7}$ m, launch angle 37.1°

8. A plane is inclined at an unknown angle α above the horizontal, where $0 < \alpha < 90^\circ$. Points O and A lie on a line of greatest slope of the plane, with A uphill from O and $OA = 18$ m.

A particle is projected from O with speed $u \text{ m s}^{-1}$ at an angle θ above the horizontal, where $\tan \theta = 3$ and $0 < \theta < 90^\circ$.

Its initial horizontal velocity is directed towards A .

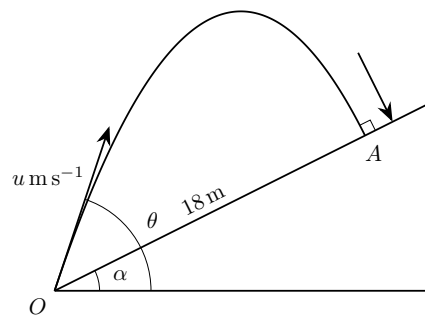
The particle moves freely under gravity in the vertical plane containing O and A .

The particle strikes the inclined plane at A , and its velocity immediately before impact is perpendicular to the inclined plane.

Find all possible pairs of values of α and u , giving each value to 3 significant figures.

[7]

Solution



Take up the plane and away from the plane as positive. The initial velocity components are $u \cos(\theta - \alpha)$ and $u \sin(\theta - \alpha)$, and the corresponding accelerations are $-g \sin \alpha$ and $-g \cos \alpha$.

Let $t > 0$ be the time taken to reach A . For each component, the constant acceleration formulae are

$$s = v_0 t + \frac{1}{2} a t^2, \quad v = v_0 + a t$$

Here v_0 denotes the initial velocity component. At impact, the velocity component along the plane is zero, so

$$0 = u \cos(\theta - \alpha) - g t \sin \alpha$$

$$u \cos(\theta - \alpha) = g t \sin \alpha$$

The displacement perpendicular to the plane is zero:

$$0 = u t \sin(\theta - \alpha) - \frac{1}{2} g t^2 \cos \alpha$$

$$u \sin(\theta - \alpha) = \frac{1}{2} g t \cos \alpha$$

Dividing these two component relations and using $\tan \theta = 3$ gives

$$\tan(\theta - \alpha) = \frac{\cos \alpha}{2 \sin \alpha}$$

$$\frac{3 - \tan \alpha}{1 + 3 \tan \alpha} = \frac{1}{2 \tan \alpha}$$

$$2 \tan \alpha (3 - \tan \alpha) = 1 + 3 \tan \alpha$$

$$2 \tan^2 \alpha - 3 \tan \alpha + 1 = 0$$

$$(2 \tan \alpha - 1)(\tan \alpha - 1) = 0$$

Since $0 < \alpha < 90^\circ$, the possible angles are

$$\alpha = \arctan\left(\frac{1}{2}\right) \quad \text{or} \quad \alpha = 45^\circ$$

Both satisfy $\alpha < \theta$, so the particle initially moves away from the plane.

Now apply the displacement formula along the plane, using $u \cos(\theta - \alpha) = gt \sin \alpha$:

$$\begin{aligned} 18 &= ut \cos(\theta - \alpha) - \frac{1}{2}gt^2 \sin \alpha \\ &= gt^2 \sin \alpha - \frac{1}{2}gt^2 \sin \alpha \\ &= \frac{1}{2}gt^2 \sin \alpha \\ t^2 &= \frac{36}{g \sin \alpha} \end{aligned}$$

Squaring the relation for the initial component along the plane gives

$$\begin{aligned} u^2 \cos^2(\theta - \alpha) &= g^2 t^2 \sin^2 \alpha \\ &= 36g \sin \alpha \\ u^2 &= \frac{36g \sin \alpha}{\cos^2(\theta - \alpha)} \end{aligned}$$

As $\tan \theta = 3$ and θ is acute, $\cos \theta = \frac{1}{\sqrt{10}}$ and $\sin \theta = \frac{3}{\sqrt{10}}$, so

$$\cos(\theta - \alpha) = \cos \theta \cos \alpha + \sin \theta \sin \alpha = \frac{\cos \alpha + 3 \sin \alpha}{\sqrt{10}}$$

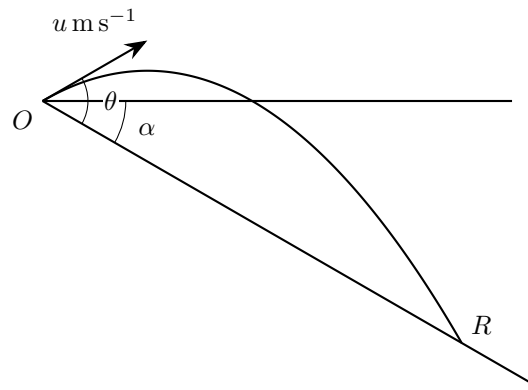
Substituting the two angles:

$$\begin{aligned} \alpha = \arctan\left(\frac{1}{2}\right) : \quad u^2 &= \frac{36g(1/\sqrt{5})}{(1/\sqrt{2})^2} = \frac{72g}{\sqrt{5}} \\ \alpha = 45^\circ : \quad u^2 &= \frac{36g(1/\sqrt{2})}{(2/\sqrt{5})^2} = \frac{45g}{\sqrt{2}} \end{aligned}$$

Using $g = 9.8 \text{ m s}^{-2}$, the possible pairs are

$$(\alpha, u) = (26.6^\circ, 17.8 \text{ m s}^{-1}) \quad \text{or} \quad (45.0^\circ, 17.7 \text{ m s}^{-1}) \quad (3 \text{ s.f.})$$

Final answers with $g = 10$: $(\alpha, u) = (26.6^\circ, 17.9 \text{ m s}^{-1})$; $(\alpha, u) = (45.0^\circ, 17.8 \text{ m s}^{-1})$



9. A small ball P is projected from a point O on a plane which slopes downwards at an angle α to the horizontal, where $0 < \alpha < \frac{\pi}{2}$.

Its initial speed is $u \text{ m s}^{-1}$, and its initial velocity makes an angle θ above the downhill direction of the plane.

The ball moves in a vertical plane containing a line of greatest slope and next meets the plane at R , where $OR = d \text{ m}$.

Assume that air resistance is negligible and that $0 < \theta < \frac{\pi}{2} + \alpha$.

- (a) Derive an expression for the time of flight in terms of u , θ , α and g .
Hence show that the speed required to reach R satisfies

$$u^2 = \frac{gd \cos^2 \alpha}{2 \sin \theta \cos(\theta - \alpha)} \quad [8]$$

- (b) The point R is fixed. Using a trigonometric identity, determine the minimum possible value of u , and the value of θ for which this minimum occurs. Give your answers in terms of d , g and α . [4]

- (c) Now take $\alpha = 30^\circ$, $d = 20$ and $g = 10 \text{ m s}^{-2}$. The ball is projected with speed $5\sqrt{6} \text{ m s}^{-1}$. Find the two possible values of θ and the corresponding times of flight, giving the times exactly. [5]

Solution

- (a) Take horizontal motion to the right and vertical motion upwards as positive. The initial velocity makes an angle $\theta - \alpha$ with the horizontal, so its components are

$$v_{x0} = u \cos(\theta - \alpha), \quad v_{y0} = u \sin(\theta - \alpha)$$

Let t be the time of flight. The constant acceleration formula is

$$s = v_0 t + \frac{1}{2} a t^2$$

With horizontal acceleration zero and vertical acceleration $-g$, the displacements from O are

$$x = u \cos(\theta - \alpha) t, \quad y = u \sin(\theta - \alpha) t - \frac{1}{2} g t^2$$

At R , $x = d \cos \alpha$ and $y = -d \sin \alpha$, so $y = -x \tan \alpha$. Substituting the displacement expressions:

$$\begin{aligned} -u \cos(\theta - \alpha)t \tan \alpha &= u \sin(\theta - \alpha)t - \frac{1}{2}gt^2 \\ \frac{1}{2}gt^2 &= ut[\sin(\theta - \alpha) + \cos(\theta - \alpha) \tan \alpha] \\ &= \frac{ut}{\cos \alpha}[\sin(\theta - \alpha) \cos \alpha + \cos(\theta - \alpha) \sin \alpha] \\ &= \frac{ut \sin \theta}{\cos \alpha} \end{aligned}$$

Since the next meeting with the plane occurs at $t > 0$, dividing by t gives

$$t = \frac{2u \sin \theta}{g \cos \alpha} \text{ s}$$

Hence, using the horizontal displacement to R :

$$\begin{aligned} d \cos \alpha &= u \cos(\theta - \alpha) \left(\frac{2u \sin \theta}{g \cos \alpha} \right) \\ gd \cos^2 \alpha &= 2u^2 \sin \theta \cos(\theta - \alpha) \\ u^2 &= \frac{gd \cos^2 \alpha}{2 \sin \theta \cos(\theta - \alpha)} \end{aligned}$$

- (b) The numerator in part (a) is fixed and positive, so u is minimised by maximising the denominator. Using the identity $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$:

$$2 \sin \theta \cos(\theta - \alpha) = \sin(2\theta - \alpha) + \sin \alpha \leq 1 + \sin \alpha$$

Equality occurs when $2\theta - \alpha = \frac{\pi}{2}$, giving $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$, which lies in the permitted interval. Therefore

$$\begin{aligned} u_{\min}^2 &= \frac{gd \cos^2 \alpha}{1 + \sin \alpha} \\ &= \frac{gd(1 - \sin \alpha)(1 + \sin \alpha)}{1 + \sin \alpha} \\ &= gd(1 - \sin \alpha) \end{aligned}$$

The minimum speed and its projection angle are

$$u_{\min} = \sqrt{gd(1 - \sin \alpha)} \text{ m s}^{-1}, \quad \theta = \frac{\pi}{4} + \frac{\alpha}{2}$$

- (c) Substituting the given values into the result of part (a):

$$\begin{aligned} (5\sqrt{6})^2 &= \frac{10 \times 20 \cos^2 30^\circ}{2 \sin \theta \cos(\theta - 30^\circ)} \\ 150 &= \frac{150}{2 \sin \theta \cos(\theta - 30^\circ)} \\ 2 \sin \theta \cos(\theta - 30^\circ) &= 1 \\ \sin(2\theta - 30^\circ) + \sin 30^\circ &= 1 \\ \sin(2\theta - 30^\circ) &= \frac{1}{2} \end{aligned}$$

The permitted interval is $0 < \theta < 120^\circ$, so $-30^\circ < 2\theta - 30^\circ < 210^\circ$. Thus

$$2\theta - 30^\circ = 30^\circ \quad \text{or} \quad 150^\circ$$

$$\theta = 30^\circ \quad \text{or} \quad 90^\circ$$

Using the time of flight from part (a):

$$t = \frac{2(5\sqrt{6}) \sin \theta}{10 \cos 30^\circ} = 2\sqrt{2} \sin \theta \text{ s}$$

The two angles and corresponding times are

$$\theta = 30^\circ : \quad t = \sqrt{2} \text{ s}, \quad \theta = 90^\circ : \quad t = 2\sqrt{2} \text{ s}$$