

FURTHER MECHANICS / KINEMATICS & DYNAMICS

Motion Under a Variable Force

Worked solutions

14 questions · 146 marks

Instructions

Unless a question states otherwise, take $g = 9.8 \text{ m s}^{-2}$ wherever a numerical value of g is required. Solutions also give final answers for $g = 10 \text{ m s}^{-2}$ where these differ.

Questions

Question 1	5 marks	2
Question 2	8 marks	4
Question 3	10 marks	5
Question 4	13 marks	7
Question 5	14 marks	9
Question 6	14 marks	11
Question 7	10 marks	13
Question 8	10 marks	15
Question 9	11 marks	17
Question 10	8 marks	19
Question 11	11 marks	21
Question 12	8 marks	23
Question 13	12 marks	25
Question 14	12 marks	27

1. A particle P of mass 4 kg moves along a straight horizontal line. At time t s, its displacement in the positive direction from a fixed point O is x m and its velocity is v m s⁻¹.

The only horizontal force acting on P has magnitude

$$\frac{36}{(x+2)^3} \text{ N}$$

and acts in the direction of decreasing x . Initially, P is at O and is moving in the positive direction with speed $\frac{3}{2}$ m s⁻¹.

(a) Find an expression for v in terms of x . [3]

(b) Hence find an expression for x in terms of t . [2]

Solution

(a) Take the direction of increasing x as positive. Newton's second law gives

$$4a = -\frac{36}{(x+2)^3}$$

Using $a = v \frac{dv}{dx}$, then separating variables and integrating:

$$\begin{aligned} 4v \frac{dv}{dx} &= -\frac{36}{(x+2)^3} \\ \int 4v \, dv &= -36 \int (x+2)^{-3} \, dx \\ 2v^2 &= \frac{18}{(x+2)^2} + c \end{aligned}$$

Initially, $x = 0$ and $v = \frac{3}{2}$, so

$$2\left(\frac{3}{2}\right)^2 = \frac{18}{2^2} + c \implies c = 0$$

Thus $v^2 = \frac{9}{(x+2)^2}$. The velocity is initially positive and cannot become zero at any finite $x \geq 0$, so we take the positive root:

$$v = \frac{3}{x+2}$$

(b) Using part (a) and $v = \frac{dx}{dt}$, separate variables and integrate:

$$\begin{aligned} \frac{dx}{dt} &= \frac{3}{x+2} \\ \int (x+2) \, dx &= \int 3 \, dt \\ \frac{x^2}{2} + 2x &= 3t + d \end{aligned}$$

Since $x = 0$ when $t = 0$, $d = 0$. Completing the square gives

$$x^2 + 4x = 6t$$

$$(x + 2)^2 = 6t + 4$$

Taking the positive root since $x + 2 > 0$:

$$x = \sqrt{6t + 4} - 2, \quad t \geq 0$$

2. A lorry of mass 2000 kg travels along a straight horizontal road. Its engine delivers constant power of 80 kW. When the lorry is moving at speed $v \text{ m s}^{-1}$, the resistance to its motion has magnitude $200v \text{ N}$.

At time t seconds, the speed of the lorry is $v \text{ m s}^{-1}$. At $t = 0$, the speed of the lorry is 8 m s^{-1} .

- (a) Show that

$$10v \frac{dv}{dt} = 400 - v^2 \quad [3]$$

- (b) Verify that, while $8 \leq v < 20$,

$$t = 5 \ln \left(\frac{336}{400 - v^2} \right) \quad [5]$$

Solution

- (a) Take the direction of motion as positive. The driving force is

$$\frac{P}{v} = \frac{80\,000}{v} \text{ N}$$

Using acceleration $\frac{dv}{dt}$, Newton's second law gives

$$2000 \frac{dv}{dt} = \frac{80\,000}{v} - 200v$$

Multiplying by $\frac{v}{200}$:

$$10v \frac{dv}{dt} = 400 - v^2$$

- (b) Separating the variables in the equation from part (a):

$$\frac{10v}{400 - v^2} dv = dt$$

Since the derivative of $400 - v^2$ is $-2v$, integrating gives

$$\begin{aligned} \int 1 dt &= -5 \int \frac{-2v}{400 - v^2} dv \\ t &= -5 \ln |400 - v^2| + c \end{aligned}$$

For $8 \leq v < 20$, $400 - v^2 > 0$, so the modulus can be dropped. Using $v = 8$ when $t = 0$:

$$\begin{aligned} 0 &= -5 \ln(400 - 8^2) + c \\ c &= 5 \ln 336 \end{aligned}$$

Substituting and combining the logarithms:

$$\begin{aligned} t &= 5 \ln 336 - 5 \ln(400 - v^2) \\ &= 5 \ln \left(\frac{336}{400 - v^2} \right) \end{aligned}$$

3. A particle P of mass 0.5 kg moves along the positive x -axis under the action of a single force. At time t seconds, the particle is x metres from the origin O , where $x > 0$, and has speed $v \text{ m s}^{-1}$. The force acts in the direction of increasing x and has magnitude

$$\frac{k}{x^3} \text{ newtons}$$

where k is a positive constant.

When $x = 2$, the speed of P is 1 m s^{-1} , and when $x = 4$, its speed is 2 m s^{-1} .

- (a) Show that

$$v^2 = 5 - \frac{16}{x^2} \quad [6]$$

The time taken for the speed of P to increase from 1 m s^{-1} to 2 m s^{-1} is T seconds.

- (b) Use algebraic integration to show that

$$T = \frac{6}{5} \quad [4]$$

Solution

- (a) Take the direction of increasing x as positive. Newton's second law gives

$$\frac{1}{2}a = +\frac{k}{x^3}$$

Using $a = v \frac{dv}{dx}$, and separating the variables:

$$\frac{1}{2}v \frac{dv}{dx} = \frac{k}{x^3}$$

$$2v \, dv = 4kx^{-3} \, dx$$

$$\int 2v \, dv = \int 4kx^{-3} \, dx$$

$$v^2 = -\frac{2k}{x^2} + c$$

Substituting the two given conditions:

$$1 = -\frac{k}{2} + c$$

$$4 = -\frac{k}{8} + c$$

Subtracting the first equation from the second:

$$3 = \frac{3k}{8}$$

$$k = 8$$

$$c = 1 + \frac{8}{2} = 5$$

Hence

$$v^2 = 5 - \frac{16}{x^2}$$

- (b) Since the particle moves in the direction of increasing x , use the positive square root from part (a).
As $x > 0$,

$$\frac{dx}{dt} = v = \sqrt{5 - \frac{16}{x^2}} = \frac{\sqrt{5x^2 - 16}}{x}$$

Separating the variables and integrating:

$$\begin{aligned} dt &= \frac{x}{\sqrt{5x^2 - 16}} dx \\ \int dt &= \frac{1}{10} \int 10x(5x^2 - 16)^{-1/2} dx \\ t &= \frac{1}{5} \sqrt{5x^2 - 16} + d \end{aligned}$$

Let t_1 and t_2 be the times when $x = 2$ and $x = 4$, respectively. Substituting these values:

$$\begin{aligned} t_1 &= \frac{1}{5} \sqrt{20 - 16} + d = \frac{2}{5} + d \\ t_2 &= \frac{1}{5} \sqrt{80 - 16} + d = \frac{8}{5} + d \\ T = t_2 - t_1 &= \frac{8}{5} - \frac{2}{5} = \frac{6}{5} \text{ s} \end{aligned}$$

4. A particle P of mass 3 kg moves along the positive x -axis. At time $t = 0$, it passes through the origin with speed 4 m s^{-1} .

While P is moving, the only force acting on it is a resistive force of magnitude

$$\frac{6}{v}\text{ N}$$

where $v\text{ m s}^{-1}$ is its speed. The force acts in the negative x -direction.

- (a) Find an expression for v in terms of the displacement x of P from the origin. [5]
- (b) Find the distance travelled by P while its speed decreases from 4 m s^{-1} to 2 m s^{-1} . [2]

A particle Q , also of mass 3 kg , starts from the origin with speed 4 m s^{-1} . The only force acting on Q is a constant resistive force of magnitude $\frac{3}{2}\text{ N}$.

- (c) By comparing the resistive forces acting on P and Q , explain why P must come to rest at a time $t < 8\text{ s}$ and at a displacement $x < 16\text{ m}$. [3]
- (d) Sketch the velocity–time graph for P from $t = 0$ until it first comes to rest. Your sketch should indicate the concavity of the graph, but no numerical values are required. [1]
- (e) Determine the maximum displacement of P from the origin. [2]

Solution

- (a) Take the positive direction to be along the positive x -axis. Newton's second law gives

$$3a = -\frac{6}{v}$$

Using $a = v\frac{dv}{dx}$:

$$3v\frac{dv}{dx} = -\frac{6}{v}$$

$$v^2\frac{dv}{dx} = -2$$

Separating the variables and integrating:

$$\int v^2 dv = -2 \int dx$$

$$\frac{v^3}{3} = -2x + c$$

When $x = 0$, $v = 4$, so $c = \frac{64}{3}$. Hence

$$v^3 = 64 - 6x$$

$$v = \sqrt[3]{64 - 6x}$$

(b) Substituting $v = 2$ into the result of part (a):

$$8 = 64 - 6x \implies x = \frac{28}{3} \text{ m}$$

Since P moves in the positive direction throughout, the distance travelled is $\frac{28}{3}$ m.

(c) For Q , Newton's second law in the same positive direction gives

$$3a_Q = -\frac{3}{2} \implies a_Q = -\frac{1}{2} \text{ m s}^{-2}$$

Using constant-acceleration formulae, its stopping time and displacement are

$$0 = 4 - \frac{1}{2}t \implies t = 8 \text{ s}$$

$$x_Q = 4(8) + \frac{1}{2}\left(-\frac{1}{2}\right)(8)^2 = 16 \text{ m}$$

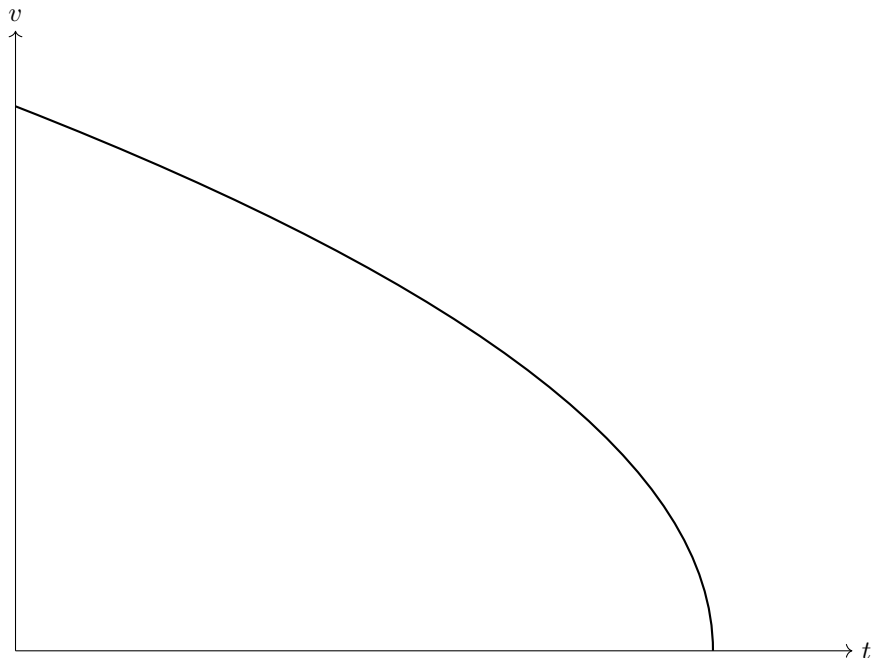
The initial resistive forces are equal. As P slows down, however,

$$0 < v < 4 \implies \frac{6}{v} > \frac{3}{2}$$

Thus, at every speed below 4 m s^{-1} , P has a greater deceleration than Q . Each further decrease in speed therefore takes less time and covers less distance for P , so it comes to rest with

$$t < 8 \text{ s}, \quad x < 16 \text{ m}$$

(d) The sketch should look like:



As the speed falls, the resistive force increases, making the acceleration more negative. The graph is therefore decreasing and concave down.

(e) The displacement increases while P is moving, so its maximum occurs as P comes to rest. Taking $v \rightarrow 0$ in part (a):

$$0 = 64 - 6x_{\max} \implies x_{\max} = \frac{32}{3} \text{ m}$$

5. A particle B of mass 2 kg moves along the positive x -axis. When B passes through the origin, its speed is $u\text{ m s}^{-1}$.

The only horizontal force acting on B is a resistance whose magnitude is directly proportional to the cube of the speed of B . When the speed of B is 3 m s^{-1} , the magnitude of the resistance is 54 N .

- (a) Show that, while B is moving in the positive x -direction, its motion satisfies

$$\frac{1}{v^2} \frac{dv}{dx} = -1 \quad [3]$$

- (b) (i) Solve the differential equation in part (a) to obtain v in terms of x and u . [4]

- (ii) By considering the behaviour of v as $x \rightarrow \infty$, state one respect in which this model of the motion is unrealistic. [1]

The particle subsequently reaches a point A with coordinate X , where its speed is $V\text{ m s}^{-1}$. The work done by the resistance as B moves from the origin to A is $W\text{ J}$.

- (c) (i) Using $W = \int F dx$, where F is the signed force in the x -direction, find an expression for W in terms of X and u . [3]

- (ii) Explain the physical significance of the sign of W . [1]

- (iii) Rewrite your answer to part (c)(i) in terms of V and u , and hence show how W is related to the change in the kinetic energy of B . [2]

Solution

- (a) Take the positive direction to be the positive x -direction. Let the magnitude of the resistance be kv^3 . The given force and speed give

$$54 = k(3^3) \quad \Rightarrow \quad k = 2$$

Newton's second law in the positive x -direction is

$$2a = -kv^3 = -2v^3$$

Using $a = v \frac{dv}{dx}$:

$$2v \frac{dv}{dx} = -2v^3$$

While the particle is moving in the positive direction, $v > 0$, so dividing by $2v^3$ gives

$$\frac{1}{v^2} \frac{dv}{dx} = -1$$

- (b) (i) Separating the variables and integrating:

$$\int v^{-2} dv = - \int 1 dx$$

$$-\frac{1}{v} = -x + c$$

At the origin, $x = 0$ and $v = u$, so $c = -\frac{1}{u}$. Hence

$$\begin{aligned} -\frac{1}{v} &= -x - \frac{1}{u} \\ \frac{1}{v} &= x + \frac{1}{u} = \frac{1+ux}{u} \\ v &= \frac{u}{1+ux} \end{aligned}$$

(ii) As $x \rightarrow \infty$, $v \rightarrow 0$, but $v > 0$ for every finite x . Thus the model unrealistically predicts that the particle travels an unlimited distance without coming to rest.

(c) (i) The signed force is $F = -2v^3$. Using the speed from part (b)(i):

$$\begin{aligned} W &= \int_0^X F \, dx \\ &= -2u^3 \int_0^X (1+ux)^{-3} \, dx \\ &= -2u^3 \left[\frac{(1+ux)^{-2}}{-2u} \right]_0^X \\ &= u^2 [(1+uX)^{-2} - 1] \end{aligned}$$

(ii) $W < 0$: the resistance acts opposite to the displacement, so it does negative work and reduces the particle's kinetic energy.

(iii) From part (b)(i), $V = \frac{u}{1+uX}$, so the expression in part (c)(i) becomes

$$W = \frac{u^2}{(1+uX)^2} - u^2 = V^2 - u^2$$

Since the mass is 2 kg, the change in kinetic energy is

$$\Delta \text{KE} = \frac{1}{2}(2)V^2 - \frac{1}{2}(2)u^2 = V^2 - u^2 = W$$

6. A small sphere P of mass 5 kg is released into a liquid and moves vertically downwards. Take the downward direction as positive. In a model of the motion, the forces acting on P are:

- its weight;
- a constant upthrust of magnitude 14 N ;
- a resistance to motion whose magnitude is proportional to the speed of P .

When the speed of P is 4 ms^{-1} , the resistance has magnitude 10 N . At time $t = 0$, the sphere is at a point O and is moving downwards with speed $u\text{ ms}^{-1}$, where $u > 0$. At time t seconds, its downward velocity is $v\text{ ms}^{-1}$. Take $g = 9.8\text{ ms}^{-2}$.

(a) Show that, according to the model,

$$\frac{dv}{dt} = 7 - \frac{1}{2}v \quad [2]$$

(b) (i) Solve the differential equation in part (a) to show that

$$v = 14 + (u - 14)e^{-t/2} \quad [5]$$

(ii) Describe briefly how the speed of P varies over time in each of the following cases:

- $0 < u < 14$;
- $u = 14$;
- $u > 14$.

In each case, state the limiting speed of P as $t \rightarrow \infty$.

[3]

(c) Given that $u = 9$, find the exact downward displacement of P from O when $t = 6$.

[4]

Solution

(a) Write the resistance as kv . Since $4k = 10$, $k = \frac{5}{2}$.

Taking downwards as positive, Newton's second law gives:

$$5\frac{dv}{dt} = 5(9.8) - 14 - \frac{5}{2}v$$

$$\frac{dv}{dt} = 7 - \frac{1}{2}v$$

(b) (i) For $u \neq 14$, separate the variables in the equation from part (a):

$$\frac{dv}{dt} = -\frac{1}{2}(v - 14)$$

$$\int \frac{1}{v - 14} dv = -\frac{1}{2} \int 1 dt$$

$$\ln |v - 14| = -\frac{t}{2} + c$$

Using $v = u$ when $t = 0$ gives $c = \ln |u - 14|$. Hence:

$$\ln |v - 14| - \ln |u - 14| = -\frac{t}{2}$$

$$|v - 14| = |u - 14|e^{-t/2}$$

The right-hand side is non-zero for finite t , so $v - 14$ cannot change sign. It therefore has the same sign as its initial value $u - 14$, giving:

$$v - 14 = (u - 14)e^{-t/2}$$

If $u = 14$, the differential equation gives the constant solution $v = 14$. Thus, for every $u > 0$,

$$v = 14 + (u - 14)e^{-t/2}$$

(ii) Since $e^{-t/2}$ decreases from 1 towards 0:

- If $0 < u < 14$, the speed increases from u towards 14 m s^{-1} .
- If $u = 14$, the speed remains constant at 14 m s^{-1} .
- If $u > 14$, the speed decreases from u towards 14 m s^{-1} .

In all three cases, the limiting speed is 14 m s^{-1} .

(c) Let x be the downward displacement from O , so $x = 0$ when $t = 0$. Using $u = 9$ in part (b)(i):

$$\frac{dx}{dt} = v = 14 - 5e^{-t/2}$$

Integrating with respect to t :

$$x = 14t + 10e^{-t/2} + A$$

$$0 = 10 + A$$

$$A = -10$$

Therefore, at $t = 6$, the downward displacement is:

$$x = 14(6) + 10e^{-3} - 10 = (74 + 10e^{-3}) \text{ m}$$

7. A vehicle of mass 900 kg moves along a straight horizontal road. Its engine provides a constant driving force of 576 N. When the vehicle has speed $v \text{ m s}^{-1}$, the resistance to its motion has magnitude $(v^2 + 14v) \text{ N}$.

The time taken for the speed of the vehicle to increase from 3 m s^{-1} to 13 m s^{-1} is T seconds.

- (a) Show that

$$T = 18 \ln \left(\frac{27}{7} \right) \quad [7]$$

When $t = 0$, the speed of the vehicle is 8 m s^{-1} .

- (b) Show that

$$\frac{v + 32}{18 - v} = pe^{qt}$$

where the rational numbers p and q are to be found. [2]

- (c) Hence explain why $v < 18$ for all $t \geq 0$. [1]

Solution

- (a) Take the direction of motion as positive. Using $a = \frac{dv}{dt}$, Newton's second law gives

$$\begin{aligned} 900 \frac{dv}{dt} &= 576 - (v^2 + 14v) \\ &= (18 - v)(v + 32) \end{aligned}$$

Separating the variables:

$$\frac{900}{(18 - v)(v + 32)} dv = dt$$

For the partial fractions, write

$$\frac{1}{(18 - v)(v + 32)} = \frac{A}{18 - v} + \frac{B}{v + 32}$$

Multiplying by $(18 - v)(v + 32)$ and comparing coefficients gives

$$\begin{aligned} 1 &= A(v + 32) + B(18 - v) \\ &= (A - B)v + 32A + 18B \\ A - B &= 0, \quad 32A + 18B = 1 \\ A &= B = \frac{1}{50} \end{aligned}$$

Hence, integrating both sides:

$$\begin{aligned} 18 \int \left(\frac{1}{18 - v} + \frac{1}{v + 32} \right) dv &= \int 1 dt \\ 18(-\ln|18 - v| + \ln|v + 32|) &= t + c \end{aligned}$$

For $3 \leq v \leq 13$, both $18 - v$ and $v + 32$ are positive, so

$$18 \ln \left(\frac{v + 32}{18 - v} \right) = t + c$$

Let t_3 and t_{13} be the times at which the speeds are 3 and 13 m s^{-1} , respectively. Substituting these speeds:

$$t_3 + c = 18 \ln \left(\frac{35}{15} \right) = 18 \ln \left(\frac{7}{3} \right)$$

$$t_{13} + c = 18 \ln \left(\frac{45}{5} \right) = 18 \ln 9$$

Subtracting eliminates c :

$$\begin{aligned} T = t_{13} - t_3 &= 18 \left(\ln 9 - \ln \frac{7}{3} \right) \\ &= 18 \ln \left(\frac{27}{7} \right) \end{aligned}$$

(b) Using the integrated relation from part (a), the condition $v = 8$ when $t = 0$ gives

$$c = 18 \ln \left(\frac{8 + 32}{18 - 8} \right) = 18 \ln 4$$

Thus, rearranging and exponentiating:

$$\begin{aligned} \ln \left(\frac{v + 32}{18 - v} \right) &= \frac{t}{18} + \ln 4 \\ \frac{v + 32}{18 - v} &= 4e^{t/18} \end{aligned}$$

This is the required form pe^{qt} , with

$$p = 4, \quad q = \frac{1}{18}$$

(c) From part (b), $v + 32 = 4e^{t/18}(18 - v)$. Since $v + 32 = 50 - (18 - v)$,

$$18 - v = \frac{50}{1 + 4e^{t/18}} > 0$$

The denominator is positive for every $t \geq 0$, so $v < 18$ for all $t \geq 0$.

8. A particle P of mass 2 kg moves along the positive x -axis in the direction of increasing x . At time t seconds, its displacement from the origin is x metres and its speed is $v\text{ m s}^{-1}$.

While P is moving, the resultant force acting on it is directed towards the origin and has magnitude $kv(v+2)\text{ N}$, where k is a positive constant and v denotes the numerical value of the speed in metres per second.

When $x = 1$, the speed of P is 7 m s^{-1} , and when $x = 3$, its speed is 1 m s^{-1} .

- (a) Find k and show that

$$v = 9 \cdot 3^{(1-x)/2} - 2 \quad [6]$$

The time taken for the speed of P to decrease from 7 m s^{-1} to 1 m s^{-1} is T seconds.

- (b) Show that

$$T = \frac{\ln(7/3)}{\ln 3} \quad [4]$$

Solution

- (a) Take the positive direction to be away from the origin, in the direction of increasing x . The resultant force acts in the negative direction, so Newton's second law gives

$$2a = -kv(v+2)$$

Using $a = v \frac{dv}{dx}$:

$$2v \frac{dv}{dx} = -kv(v+2)$$

Since $v > 0$ while the particle is moving, divide by v and separate the variables:

$$\begin{aligned} \frac{1}{v+2} dv &= -\frac{k}{2} dx \\ \int \frac{1}{v+2} dv &= -\frac{k}{2} \int dx \\ \ln|v+2| &= -\frac{kx}{2} + c \end{aligned}$$

Here $v+2 > 0$, so the modulus can be dropped. Substituting the two given conditions:

$$\begin{aligned} \ln 9 &= -\frac{k}{2} + c \\ \ln 3 &= -\frac{3k}{2} + c \end{aligned}$$

Subtracting the first equation from the second:

$$\begin{aligned} \ln 3 - \ln 9 &= -k \\ k &= \ln 3 \end{aligned}$$

Subtracting the relation at $x = 1$, $v = 7$ from the general relation now gives

$$\ln(v+2) - \ln 9 = -\frac{k}{2}(x-1)$$

$$\ln\left(\frac{v+2}{9}\right) = \frac{1-x}{2} \ln 3$$

$$\frac{v+2}{9} = 3^{(1-x)/2}$$

$$v = 9 \cdot 3^{(1-x)/2} - 2$$

(b) Using $a = \frac{dv}{dt}$, Newton's second law in the same positive direction gives

$$2\frac{dv}{dt} = -kv(v+2)$$

Separating the variables, and using partial fractions:

$$\frac{2}{v(v+2)} dv = -k dt$$

$$\frac{2}{v(v+2)} = \frac{1}{v} - \frac{1}{v+2}$$

$$\int \left(\frac{1}{v} - \frac{1}{v+2} \right) dv = -k \int dt$$

$$\ln|v| - \ln|v+2| = -kt + d$$

Both v and $v+2$ are positive, so this becomes

$$\ln\left(\frac{v}{v+2}\right) = -kt + d$$

Let t_0 be the time when $v = 7$. Then $v = 1$ at time $t_0 + T$, giving

$$\ln \frac{7}{9} = -kt_0 + d$$

$$\ln \frac{1}{3} = -k(t_0 + T) + d$$

Subtracting eliminates d and t_0 . Using $k = \ln 3$ from part (a):

$$\ln \frac{1}{3} - \ln \frac{7}{9} = -kT$$

$$\ln \frac{3}{7} = -T \ln 3$$

$$T = \frac{\ln(7/3)}{\ln 3}$$

9. A particle P of mass 4 kg is released from rest and moves vertically downwards through a fluid. While P is moving with speed $v\text{ m s}^{-1}$, it experiences a resistance of magnitude $6v\text{ N}$ acting vertically upwards. After P has descended a distance $x\text{ m}$, its speed is $v\text{ m s}^{-1}$. Take $g = 9.8\text{ m s}^{-2}$.

(a) Show that

$$\left(\frac{10v}{98 - 15v}\right) \frac{dv}{dx} = 1 \quad [2]$$

(b) Verify that x and v satisfy

$$45x = 196 \ln\left(\frac{98}{98 - 15v}\right) - 30v \quad [4]$$

(c) Determine the work done against the resistance from the instant P is released until its acceleration has magnitude 2.45 m s^{-2} . Give your answer to 3 significant figures. [5]

Solution

(a) Take vertically downwards as positive. Newton's second law gives

$$4a = 4(9.8) - 6v$$

Using $a = v \frac{dv}{dx}$:

$$4v \frac{dv}{dx} = 39.2 - 6v$$

$$10v \frac{dv}{dx} = 98 - 15v$$

$$\left(\frac{10v}{98 - 15v}\right) \frac{dv}{dx} = 1$$

(b) Separating variables in the equation from part (a):

$$\int \frac{10v}{98 - 15v} dv = \int 1 dx$$

To integrate, write

$$\frac{10v}{98 - 15v} = -\frac{2}{3} + \frac{196}{3(98 - 15v)}$$

Hence

$$x = -\frac{2v}{3} - \frac{196}{45} \ln|98 - 15v| + c$$

Initially, $x = 0$ and $v = 0$, so

$$0 = -\frac{196}{45} \ln 98 + c \implies c = \frac{196}{45} \ln 98$$

The terminal speed is $\frac{98}{15} \text{ m s}^{-1}$. Since the particle accelerates towards this speed from rest, $98 - 15v > 0$, so the modulus may be removed. Substituting for c and combining the logarithms:

$$\begin{aligned} 45x &= -30v - 196 \ln(98 - 15v) + 196 \ln 98 \\ &= 196 \ln\left(\frac{98}{98 - 15v}\right) - 30v \end{aligned}$$

(c) The acceleration is downwards throughout the motion. From Newton's second law,

$$a = 9.8 - 1.5v$$

When its magnitude is 2.45 m s^{-2} ,

$$2.45 = 9.8 - 1.5v \quad \implies \quad v = 4.9 \text{ m s}^{-1}$$

Using part (b), the distance descended is

$$\begin{aligned} 45x &= 196 \ln\left(\frac{98}{98 - 15(4.9)}\right) - 30(4.9) \\ x &= \frac{196 \ln 4 - 147}{45} \text{ m} \end{aligned}$$

Let W be the positive work done against resistance. The work done by resistance is $-W$. The work-energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

Taking gravitational potential energy to be zero at the final position:

$$\begin{aligned} 0 + 4gx - W &= \frac{1}{2}(4)v^2 + 0 \\ W &= 39.2x - 2v^2 \\ &= 39.2\left(\frac{196 \ln 4 - 147}{45}\right) - 2(4.9)^2 \\ &\approx 60.6 \text{ J} \quad (3 \text{ significant figures}) \end{aligned}$$

10. A particle P , of mass m , moves on a rough straight track inclined at an angle α to the horizontal. A fixed point O lies on the track, and x denotes the distance of P from O , measured up the track.

In addition to its weight, P is acted on by a force directed down the track and having magnitude

$$\frac{kmg}{x+a}$$

where a and k are positive constants with dimensions of length. The coefficient of friction between P and the track is μ .

The particle is projected up the track and first comes to rest at the point where $x = 2a$.

- (a) Show that, while P is moving up the track with speed v ,

$$\frac{d}{dx}(v^2) + \frac{2gk}{x+a} + 2g(\sin \alpha + \mu \cos \alpha) = 0 \quad [3]$$

- (b) Verify that

$$v^2 = 2gk \ln \left(\frac{3a}{x+a} \right) + 2g(\sin \alpha + \mu \cos \alpha)(2a - x) \quad [3]$$

- (c) Find, in terms of a , k and α , the range of values of μ for which P remains at rest when $x = 2a$. [2]

Solution

- (a) Take up the track as the positive direction, and let the normal reaction be R . Resolving perpendicular to the track gives

$$R = mg \cos \alpha$$

While P moves up the track, friction acts down the track with magnitude μR . Using $v \frac{dv}{dx}$ for the acceleration, Newton's second law along the track gives

$$\begin{aligned} mv \frac{dv}{dx} &= -mg \sin \alpha - \mu R - \frac{kmg}{x+a} \\ 2v \frac{dv}{dx} &= -2g(\sin \alpha + \mu \cos \alpha) - \frac{2gk}{x+a} \end{aligned}$$

Since $\frac{d}{dx}(v^2) = 2v \frac{dv}{dx}$, this becomes

$$\frac{d}{dx}(v^2) + \frac{2gk}{x+a} + 2g(\sin \alpha + \mu \cos \alpha) = 0$$

- (b) Integrating the equation in part (a) with respect to x :

$$v^2 = -2gk \ln |x+a| - 2g(\sin \alpha + \mu \cos \alpha)x + c$$

Since x is a distance and $a > 0$, $x+a > 0$, so the modulus can be dropped. Substituting $v = 0$ when $x = 2a$ gives

$$\begin{aligned} 0 &= -2gk \ln(3a) - 4ga(\sin \alpha + \mu \cos \alpha) + c \\ c &= 2gk \ln(3a) + 4ga(\sin \alpha + \mu \cos \alpha) \end{aligned}$$

Hence

$$\begin{aligned}v^2 &= 2gk(\ln(3a) - \ln(x+a)) + 2g(\sin \alpha + \mu \cos \alpha)(2a-x) \\&= 2gk \ln\left(\frac{3a}{x+a}\right) + 2g(\sin \alpha + \mu \cos \alpha)(2a-x)\end{aligned}$$

- (c) At $x = 2a$, let the frictional force acting up the track be F . For equilibrium, Newton's second law gives

$$F - mg \sin \alpha - \frac{kmg}{3a} = 0$$

The required friction must not exceed its limiting value $\mu R = \mu mg \cos \alpha$, so

$$\begin{aligned}mg \sin \alpha + \frac{kmg}{3a} &\leq \mu mg \cos \alpha \\ \mu &\geq \tan \alpha + \frac{k}{3a \cos \alpha}\end{aligned}$$

11. A small instrument package of mass m kg enters a deep liquid moving vertically downwards at 7 m s^{-1} . While the package is moving downwards, it experiences a resistive force of magnitude $\frac{m}{20}v^2 \text{ N}$, where $v \text{ m s}^{-1}$ is its speed.

At time t s after entering the liquid, the package has descended a distance x m.

- (a) Find an expression for v in terms of t . [6]
- (b) Find an expression for x in terms of t . [3]
- (c) Find the exact distance descended by the package when its speed first reaches $\frac{21}{2} \text{ m s}^{-1}$. [2]

Solution

- (a) Take vertically downwards as positive. Newton's second law gives

$$m \frac{dv}{dt} = mg - \frac{m}{20}v^2$$

$$20 \frac{dv}{dt} = 196 - v^2$$

The terminal speed is 14 m s^{-1} . Since the initial speed is 7 m s^{-1} , the speed increases towards 14 m s^{-1} and remains below it.

Separating variables and using partial fractions:

$$\int \frac{1}{196 - v^2} dv = \frac{1}{20} \int 1 dt$$

$$\frac{1}{28} \int \left(\frac{1}{14 - v} + \frac{1}{14 + v} \right) dv = \frac{t}{20} + c$$

$$\frac{1}{28} (\ln |14 + v| - \ln |14 - v|) = \frac{t}{20} + c$$

Both $14 + v$ and $14 - v$ are positive, so the modulus signs can be dropped. Using $v = 7$ when $t = 0$:

$$c = \frac{1}{28} \ln \frac{21}{7} = \frac{\ln 3}{28}$$

Hence

$$\ln \frac{14 + v}{14 - v} = \frac{7t}{5} + \ln 3$$

$$\frac{14 + v}{14 - v} = 3e^{7t/5}$$

$$14 + v = 42e^{7t/5} - 3ve^{7t/5}$$

$$v(1 + 3e^{7t/5}) = 14(3e^{7t/5} - 1)$$

$$v = \frac{14(3e^{7t/5} - 1)}{3e^{7t/5} + 1}$$

(b) Using $v = \frac{dx}{dt}$, rewrite the result from part (a) to recognise a logarithmic integral:

$$\begin{aligned}\frac{dx}{dt} &= -14 + \frac{84e^{7t/5}}{3e^{7t/5} + 1} \\ &= -14 + 20 \frac{\frac{21}{5}e^{7t/5}}{3e^{7t/5} + 1} \\ x &= -14t + 20 \ln(3e^{7t/5} + 1) + d\end{aligned}$$

Since $x = 0$ when $t = 0$, $d = -20 \ln 4$. Therefore

$$x = 20 \ln\left(\frac{3e^{7t/5} + 1}{4}\right) - 14t$$

(c) Using $a = v \frac{dv}{dx}$, the equation of motion becomes

$$v \frac{dv}{dx} = \frac{196 - v^2}{20}$$

Separating variables and integrating, with $196 - v^2 > 0$:

$$\begin{aligned}\int 1 \, dx &= \int \frac{20v}{196 - v^2} \, dv \\ x &= -10 \ln |196 - v^2| + h\end{aligned}$$

At entry, $x = 0$ and $v = 7$, so $h = 10 \ln 147$. Thus, when $v = \frac{21}{2}$,

$$x = 10 \ln\left(\frac{147}{196 - (\frac{21}{2})^2}\right) = 10 \ln\left(\frac{12}{7}\right) \text{ m}$$

Final answers with $g = 10$: (a) $v = 10\sqrt{2} \frac{(10\sqrt{2}+7)e^{\sqrt{2}t} - (10\sqrt{2}-7)}{(10\sqrt{2}+7)e^{\sqrt{2}t} + (10\sqrt{2}-7)}$;

(b) $x = 20 \ln\left(\frac{(10\sqrt{2}+7)e^{\sqrt{2}t} + 10\sqrt{2}-7}{20\sqrt{2}}\right) - 10\sqrt{2}t$; (c) $10 \ln\left(\frac{604}{359}\right) \text{ m}$

12. A package of mass m kg is released from rest and falls vertically. When its downward speed is v ms⁻¹, it experiences a resistive force of magnitude mkv^2 N, where k is a positive constant. Take the downward direction as positive.

(a) Show that, at time t s after release,

$$v = \sqrt{\frac{g}{k}} \left(\frac{e^{2\sqrt{gk}t} - 1}{e^{2\sqrt{gk}t} + 1} \right) \quad [4]$$

It is given that $k = 0.0125 \text{ m}^{-1}$.

(b) Find the distance fallen by the package during the first 2 seconds after its release. [4]

Solution

(a) Taking vertically downwards as positive, Newton's second law gives

$$\begin{aligned} m \frac{dv}{dt} &= mg - mkv^2 \\ \frac{dv}{dt} &= g - kv^2 \end{aligned}$$

Let $V = \sqrt{g/k}$ be the terminal speed. The package accelerates from rest towards this speed, so $0 \leq v < V$.

Separating variables and using partial fractions:

$$\begin{aligned} \int \frac{1}{V^2 - v^2} dv &= \int k dt \\ \frac{1}{2V} \int \left(\frac{1}{V+v} + \frac{1}{V-v} \right) dv &= \int k dt \\ \frac{1}{2V} \ln \left| \frac{V+v}{V-v} \right| &= kt + c \end{aligned}$$

Since $V+v > 0$ and $V-v > 0$, the modulus can be removed. At release, $t = 0$ and $v = 0$, giving $c = 0$. Hence

$$\begin{aligned} \ln \left(\frac{V+v}{V-v} \right) &= 2kVt \\ V+v &= e^{2kVt}(V-v) \\ v(1 + e^{2kVt}) &= V(e^{2kVt} - 1) \\ v &= \sqrt{\frac{g}{k}} \left(\frac{e^{2\sqrt{gk}t} - 1}{e^{2\sqrt{gk}t} + 1} \right) \end{aligned}$$

(b) Let s be the distance fallen after t seconds. Using $g = 9.8$ and $k = 0.0125$ in part (a):

$$\begin{aligned} \frac{ds}{dt} = v &= 28 \left(\frac{e^{0.7t} - 1}{e^{0.7t} + 1} \right) \\ &= \frac{56e^{0.7t}}{e^{0.7t} + 1} - 28 \end{aligned}$$

Integrating by recognising the derivative of the denominator:

$$\begin{aligned}s &= \int \left(\frac{56e^{0.7t}}{e^{0.7t} + 1} - 28 \right) dt \\ &= 80 \ln|e^{0.7t} + 1| - 28t + d\end{aligned}$$

Since $e^{0.7t} + 1 > 0$, the modulus can be removed. Using $s = 0$ when $t = 0$ gives $d = -80 \ln 2$, so

$$s = 80 \ln \left(\frac{e^{0.7t} + 1}{2} \right) - 28t$$

The distance fallen in the first 2 seconds is therefore

$$s(2) = \left(80 \ln \left(\frac{e^{1.4} + 1}{2} \right) - 56 \right) \text{ m} \approx 18.2 \text{ m}$$

Final answers with $g = 10$: (b) $\left(80 \ln \left(\frac{e^{\sqrt{2}} + 1}{2} \right) - 40\sqrt{2} \right) \text{ m} \approx 18.5 \text{ m}$

13. A particle P of mass 16 kg moves in a straight line on a smooth horizontal plane.

At time $t = 0$, the particle passes through a fixed point O with velocity $\frac{5\sqrt{3}}{4} \text{ m s}^{-1}$.

From this instant, a horizontal force opposing the motion acts on P . The magnitude of the force is proportional to $\sqrt{25 - 4v^2}$, where $v \text{ m s}^{-1}$ is the velocity of P .

When $v = 2$, the magnitude of the force is 6 N. The force ceases to act when the velocity has decreased to $\frac{5}{4} \text{ m s}^{-1}$.

- (a) Show that, while the force is acting, the motion of P is modelled by the differential equation

$$\frac{1}{\sqrt{25 - 4v^2}} \frac{dv}{dt} = -\frac{1}{8} \quad [3]$$

- (b) Hence find v in terms of t , showing that your expression is valid for

$$0 \leq t \leq \frac{2\pi}{3} \quad [5]$$

- (c) Determine the total distance travelled by P during the interval $0 \leq t \leq 8$. Give your answer in an exact form. [4]

Solution

- (a) Take the initial direction of motion as positive. Let the magnitude of the opposing force be $F = k\sqrt{25 - 4v^2}$, where $k > 0$.

Using the given force when $v = 2$:

$$6 = k\sqrt{25 - 4(2)^2}$$

$$6 = 3k$$

$$k = 2$$

Using $a = \frac{dv}{dt}$, Newton's second law in the positive direction gives

$$16 \frac{dv}{dt} = -2\sqrt{25 - 4v^2}$$

$$\frac{1}{\sqrt{25 - 4v^2}} \frac{dv}{dt} = -\frac{1}{8}$$

- (b) Separating the variables and factoring out the numerical scale before integrating:

$$\frac{dv}{\sqrt{25 - 4v^2}} = -\frac{1}{8} dt$$

$$\frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2 - v^2}} dv = -\frac{1}{8} \int dt$$

$$\frac{1}{2} \arcsin\left(\frac{2v}{5}\right) = -\frac{t}{8} + c$$

Substituting $t = 0$ and $v = \frac{5\sqrt{3}}{4}$:

$$c = \frac{1}{2} \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Hence

$$\arcsin\left(\frac{2v}{5}\right) = \frac{\pi}{3} - \frac{t}{4}$$

The force ceases when $v = \frac{5}{4}$, so

$$\begin{aligned}\arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{3} - \frac{t}{4} \\ \frac{\pi}{6} &= \frac{\pi}{3} - \frac{t}{4} \\ t &= \frac{2\pi}{3}\end{aligned}$$

Taking sine gives the required velocity while the force acts:

$$v = \frac{5}{2} \sin\left(\frac{\pi}{3} - \frac{t}{4}\right), \quad 0 \leq t \leq \frac{2\pi}{3}$$

- (c) Let x be the displacement from O in the positive direction. Using $v = \frac{dx}{dt}$ and integrating while the force acts:

$$\begin{aligned}\frac{dx}{dt} &= \frac{5}{2} \sin\left(\frac{\pi}{3} - \frac{t}{4}\right) \\ x &= 10 \cos\left(\frac{\pi}{3} - \frac{t}{4}\right) + d\end{aligned}$$

Since $x = 0$ at $t = 0$,

$$0 = 10 \cos \frac{\pi}{3} + d \implies d = -5$$

The velocity remains positive, so the distance travelled while the force acts is

$$x\left(\frac{2\pi}{3}\right) = 10 \cos \frac{\pi}{6} - 5 = 5\sqrt{3} - 5$$

After the force ceases, there is no horizontal force, so the particle continues at the constant velocity $\frac{5}{4} \text{ m s}^{-1}$. Therefore,

$$\begin{aligned}\text{Total distance} &= (5\sqrt{3} - 5) + \frac{5}{4} \left(8 - \frac{2\pi}{3}\right) \\ &= \left(5\sqrt{3} + 5 - \frac{5\pi}{6}\right) \text{ m}\end{aligned}$$

14. A vehicle of mass m kg moves along a horizontal straight road. Its engine works at a constant rate of $2kmg$ W, where k is a positive constant with units of speed. When the vehicle has speed v m s⁻¹, it experiences a resistance of magnitude $\frac{mgv}{k}$ N.

Initially, the vehicle is at rest at $x = 0$, where x m is the distance travelled.

- (a) Show that

$$v^2 \frac{dv}{dx} = \frac{g}{k} (2k^2 - v^2) \quad [3]$$

- (b) Verify that

$$gx = -kv + \frac{k^2}{\sqrt{2}} \ln \left(\frac{\sqrt{2}k + v}{\sqrt{2}k - v} \right) \quad [5]$$

- (c) By using the work-energy principle, show that the time taken for the vehicle to accelerate from rest to a speed V , where $0 < V < \sqrt{2}k$, is

$$\frac{k}{2g} \ln \left(\frac{2k^2}{2k^2 - V^2} \right) \quad [4]$$

Solution

- (a) Take the direction of motion as positive. The driving force is power divided by speed:

$$F = \frac{P}{v} = \frac{2kmg}{v}$$

Newton's second law in this direction gives

$$\begin{aligned} ma &= \frac{2kmg}{v} - \frac{mgv}{k} \\ a &= \frac{g}{kv} (2k^2 - v^2) \end{aligned}$$

Using $a = v \frac{dv}{dx}$:

$$\begin{aligned} v \frac{dv}{dx} &= \frac{g}{kv} (2k^2 - v^2) \\ v^2 \frac{dv}{dx} &= \frac{g}{k} (2k^2 - v^2) \end{aligned}$$

- (b) Separating the variables in part (a):

$$g \, dx = k \frac{v^2}{2k^2 - v^2} \, dv$$

Use algebraic division followed by partial fractions:

$$\begin{aligned} \frac{v^2}{2k^2 - v^2} &= -1 + \frac{2k^2}{2k^2 - v^2} \\ &= -1 + \frac{k}{\sqrt{2}} \left(\frac{1}{\sqrt{2}k - v} + \frac{1}{\sqrt{2}k + v} \right) \end{aligned}$$

Integrating gives

$$\begin{aligned} gx &= k \int \left[-1 + \frac{k}{\sqrt{2}} \left(\frac{1}{\sqrt{2}k - v} + \frac{1}{\sqrt{2}k + v} \right) \right] dv \\ &= -kv + \frac{k^2}{\sqrt{2}} \left(\ln |\sqrt{2}k + v| - \ln |\sqrt{2}k - v| \right) + c \end{aligned}$$

The resultant force is zero at the terminal speed $\sqrt{2}k$. During acceleration from rest, $0 \leq v < \sqrt{2}k$, so both logarithm arguments are positive and the moduli can be removed.

When $x = 0$, $v = 0$, giving

$$0 = \frac{k^2}{\sqrt{2}} \left(\ln(\sqrt{2}k) - \ln(\sqrt{2}k) \right) + c \implies c = 0$$

Combining the logarithms:

$$gx = -kv + \frac{k^2}{\sqrt{2}} \ln \left(\frac{\sqrt{2}k + v}{\sqrt{2}k - v} \right)$$

- (c) Let $W(v)$ be the work done by resistance from rest until the speed is v . This work is negative because resistance opposes the motion. From part (a),

$$dx = \frac{kv^2}{g(2k^2 - v^2)} dv$$

Hence

$$\begin{aligned} dW &= -\frac{mgv}{k} dx \\ \frac{dW}{dv} &= -\frac{mv^3}{2k^2 - v^2} = mv - \frac{2mk^2v}{2k^2 - v^2} \end{aligned}$$

Integrating by recognising the derivative of $2k^2 - v^2$:

$$W(v) = \frac{1}{2}mv^2 + mk^2 \ln |2k^2 - v^2| + d$$

Using $W(0) = 0$ gives $d = -mk^2 \ln(2k^2)$. Since $0 < V < \sqrt{2}k$, we have $2k^2 - V^2 > 0$, so

$$W(V) = \frac{1}{2}mV^2 - mk^2 \ln \left(\frac{2k^2}{2k^2 - V^2} \right)$$

The work-energy principle, with work positive when a force acts in the direction of motion, is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

Let the required time be t . The engine does work $Pt = 2kmg t$, and the gravitational potential energy is unchanged. Taking it to be zero:

$$\begin{aligned} 0 + 0 + 2kmg t + W(V) &= \frac{1}{2}mV^2 + 0 \\ 2kmg t + \frac{1}{2}mV^2 - mk^2 \ln \left(\frac{2k^2}{2k^2 - V^2} \right) &= \frac{1}{2}mV^2 \\ 2kmg t &= mk^2 \ln \left(\frac{2k^2}{2k^2 - V^2} \right) \\ t &= \frac{k}{2g} \ln \left(\frac{2k^2}{2k^2 - V^2} \right) \end{aligned}$$