FURTHER MECHANICS / CIRCULAR MOTION & OSCILLATIONS

Simple Harmonic Motion

Worked solutions 10 questions · 139 marks

EXAM BOARDS Edexcel FM2 9FM0/4C · OCR MEI Mech Major Y421

Instructions

Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question gives a value. Where a question leaves g unspecified and a final answer depends on it, the solutions also give that answer for $g = 10\,\mathrm{m\,s^{-2}}$.

Questions

Question 1	*15 marks*	2
Question 2	*14 marks*	4
Question 3	*14 marks*	6
Question 4	*10 marks*	8
Question 5	*11 marks*	10
Question 6	*9 marks*	12
Question 7	*15 marks*	14
Question 8	*14 marks*	16
Question 9	*17 marks*	19
Question 10	*20 marks*	22

1. A particle P is moving along the x-axis. At time t seconds, the displacement of P from the origin O is x metres, where

$$x = 3\cos\left(\tfrac{1}{4}\pi t\right) + 3\sqrt{3}\sin\left(\tfrac{1}{4}\pi t\right)$$

(a) Prove that P is moving with simple harmonic motion. **[3]**

(b) Find the period of the motion. **[2]**

(c) Find the amplitude of the motion. **[2]**

(d) Find, in terms of π, the maximum speed of P. **[2]**

(e) Find the total time during each complete oscillation for which the speed of P is greater than $\frac{3\sqrt{3}}{4}\pi\,\mathrm{m\,s^{-1}}$. **[6]**

Solution

(a) Differentiating with respect to t:

$$\begin{aligned}
\dot{x} &= -\tfrac{3}{4}\pi\sin\left(\tfrac{1}{4}\pi t\right) + \tfrac{3\sqrt{3}}{4}\pi\cos\left(\tfrac{1}{4}\pi t\right) \\
\ddot{x} &= -\tfrac{3}{16}\pi^2\cos\left(\tfrac{1}{4}\pi t\right) - \tfrac{3\sqrt{3}}{16}\pi^2\sin\left(\tfrac{1}{4}\pi t\right) \\
&= -\tfrac{1}{16}\pi^2\left(3\cos\left(\tfrac{1}{4}\pi t\right) + 3\sqrt{3}\sin\left(\tfrac{1}{4}\pi t\right)\right) \\
&= -\tfrac{1}{16}\pi^2 x
\end{aligned}$$

This is of the form $\ddot{x} = -\omega^2 x$ with $\omega = \frac{\pi}{4}$, so P is moving with simple harmonic motion about O.

(b) The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{4}} = 8\,\mathrm{s}$$

(c) Write $3\cos\left(\frac{1}{4}\pi t\right) + 3\sqrt{3}\sin\left(\frac{1}{4}\pi t\right)$ as $R\cos\left(\frac{1}{4}\pi t - \alpha\right)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Expanding and comparing coefficients:

$$R\cos\alpha = 3, \qquad R\sin\alpha = 3\sqrt{3}$$

so

$$R = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = \sqrt{36} = 6$$

$$\tan\alpha = \sqrt{3} \quad \Longrightarrow \quad \alpha = \frac{\pi}{3}$$

Hence $x = 6\cos\left(\frac{1}{4}\pi t - \frac{1}{3}\pi\right)$, which oscillates between -6 and 6, so the amplitude is $6\,\mathrm{m}$.

(d) For simple harmonic motion with amplitude a, the maximum speed is $a\omega$, so

$$v_{\max} = 6 \times \frac{\pi}{4} = \frac{3\pi}{2}\,\mathrm{m\,s^{-1}}$$

(e) For simple harmonic motion with amplitude a, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

The speed is greater than $\frac{3\sqrt{3}}{4}\pi\,\mathrm{m\,s^{-1}}$ when $v^2 > \left(\frac{3\sqrt{3}}{4}\pi\right)^2$. Substituting $\omega = \frac{\pi}{4}$ and $a = 6$:

$$\frac{\pi^2}{16}(36 - x^2) > \left(\frac{3\sqrt{3}}{4}\pi\right)^2 = \frac{27\pi^2}{16}$$

$$36 - x^2 > 27$$

$$x^2 < 9$$

so when $-3 < x < 3$.

In each complete oscillation, P crosses the region $-3 < x < 3$ twice, once in each direction, and by symmetry each crossing takes twice the time taken to travel from O to $x = 3$. The required time is therefore four times the time from O to $x = 3$.

Measuring time τ seconds from an instant when P passes through O moving in the positive direction, the displacement is $x = 6\sin\left(\frac{1}{4}\pi\tau\right)$. Setting $x = 3$:

$$6\sin\left(\tfrac{1}{4}\pi\tau\right) = 3$$

$$\sin\left(\tfrac{1}{4}\pi\tau\right) = \frac{1}{2}$$

$$\tfrac{1}{4}\pi\tau = \frac{\pi}{6} \quad \Longrightarrow \quad \tau = \frac{2}{3}$$

The total time is

$$4 \times \frac{2}{3} = \frac{8}{3}\,\mathrm{s} \approx 2.67\,\mathrm{s}$$

2. A particle P of mass 0.5 kg moves in a straight line with simple harmonic motion about a centre O. The time taken for one complete oscillation is $\frac{\pi}{2}$ seconds.
When P is 0.3 m from O, its speed is $1.6\,\mathrm{m\,s^{-1}}$.

(a) Show that the amplitude of the motion is 0.5 m. **[4]**

(b) Find the length of time within each complete oscillation for which the speed of P is greater than $1\,\mathrm{m\,s^{-1}}$, giving your answer as an exact multiple of π. **[6]**

(c) Find the magnitude of the resultant force acting on P at an instant when the kinetic energy of P is 0.36 J. **[4]**

Solution

(a) Let the amplitude be a m and the angular frequency $\omega\,\mathrm{rad\,s^{-1}}$. The period of the motion is

$$T = \frac{2\pi}{\omega}$$

so

$$\frac{2\pi}{\omega} = \frac{\pi}{2} \quad \Longrightarrow \quad \omega = 4$$

For simple harmonic motion with amplitude a, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

Substituting $x = 0.3$ and $v = 1.6$:

$$1.6^2 = 4^2\left(a^2 - 0.3^2\right)$$

$$2.56 = 16\left(a^2 - 0.09\right)$$

$$a^2 - 0.09 = 0.16$$

$$a^2 = 0.25$$

$$a = 0.5$$

So the amplitude of the motion is 0.5 m, as required.

(b) The speed is greater than $1\,\mathrm{m\,s^{-1}}$ when $v^2 > 1$, that is, when

$$16\left(0.25 - x^2\right) > 1$$

$$0.25 - x^2 > \frac{1}{16}$$

$$x^2 < \frac{3}{16}$$

So the speed exceeds $1\,\mathrm{m\,s^{-1}}$ exactly when P is within $\frac{\sqrt{3}}{4}$ m of O.

Measure t from an instant when P passes through O moving in the positive direction. Then

$$x = a\sin\omega t = 0.5\sin 4t$$

P first reaches $x = \frac{\sqrt{3}}{4}$ when

$$0.5 \sin 4t = \frac{\sqrt{3}}{4}$$

$$\sin 4t = \frac{\sqrt{3}}{2}$$

$$4t = \frac{\pi}{3}$$

$$t = \frac{\pi}{12}$$

So P takes $\frac{\pi}{12}$ s to travel from O to a point $\frac{\sqrt{3}}{4}$ m from O. The motion is symmetric, so in each of the four quarter-oscillations (from O out to an end point, and from an end point back to O, on each side of O) P spends $\frac{\pi}{12}$ s within $\frac{\sqrt{3}}{4}$ m of O. The total time within each complete oscillation is

$$4 \times \frac{\pi}{12} = \frac{\pi}{3} \text{ s}$$

(c) The kinetic energy is $\frac{1}{2}mv^2$, so

$$\frac{1}{2} \times 0.5 \times v^2 = 0.36$$

$$v^2 = 1.44$$

Using $v^2 = \omega^2\left(a^2 - x^2\right)$:

$$1.44 = 16\left(0.25 - x^2\right)$$

$$0.25 - x^2 = 0.09$$

$$x^2 = 0.16$$

$$x = \pm 0.4$$

So P is 0.4 m from O. In simple harmonic motion the acceleration is $\ddot{x} = -\omega^2 x$, so its magnitude here is

$$\omega^2 \times 0.4 = 16 \times 0.4 = 6.4 \,\text{m}\,\text{s}^{-2}$$

Using Newton's second law, the magnitude of the resultant force is

$$F = 0.5 \times 6.4 = 3.2 \,\text{N}$$

3. A particle P moves in a straight line with simple harmonic motion about a fixed point O.
When P is 0.5 m from O, its speed is $3.6\,\mathrm{m\,s^{-1}}$, and when P is 1.2 m from O, its speed is $1.5\,\mathrm{m\,s^{-1}}$.
The amplitude of the motion is a metres.

(a) Show that $a = 1.3$. **[5]**

(b) Find the exact period of the motion and the greatest magnitude of the acceleration of P. **[3]**

During one complete oscillation, the magnitude of the acceleration of P is at least $6.3\,\mathrm{m\,s^{-2}}$ for a total of T seconds.

(c) Find the value of T. **[6]**

Solution

(a) For simple harmonic motion with amplitude a, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

Substituting the two given positions:

$$3.6^2 = \omega^2\left(a^2 - 0.5^2\right) \qquad \Longrightarrow \qquad 12.96 = \omega^2\left(a^2 - 0.25\right)$$

$$1.5^2 = \omega^2\left(a^2 - 1.2^2\right) \qquad \Longrightarrow \qquad 2.25 = \omega^2\left(a^2 - 1.44\right)$$

Subtracting the second equation from the first:

$$10.71 = \omega^2(1.44 - 0.25)$$

$$10.71 = 1.19\,\omega^2$$

$$\omega^2 = 9$$

so $\omega = 3$. Substituting into the first equation:

$$12.96 = 9\left(a^2 - 0.25\right)$$

$$a^2 - 0.25 = 1.44$$

$$a^2 = 1.69$$

Since $a > 0$, $a = 1.3$, as required.

(b) The period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{3}\,\mathrm{s}$$

The acceleration is $\ddot{x} = -\omega^2 x$, so its magnitude is greatest at the extreme positions, where $|x| = a$:

$$\omega^2 a = 9 \times 1.3 = 11.7\,\mathrm{m\,s^{-2}}$$

(c) The magnitude of the acceleration is $9|x|$, so it is at least $6.3\,\mathrm{m\,s^{-2}}$ when

$$9|x| \geq 6.3 \qquad \Longrightarrow \qquad |x| \geq 0.7$$

Measure time from an instant when P is at an extreme position on the positive side of O, so that

$$x = 1.3\cos 3t$$

Moving from this extreme position towards O, P has $x \geq 0.7$ until the time t_1 given by

$$1.3\cos 3t_1 = 0.7$$

$$\cos 3t_1 = \frac{7}{13}$$

$$t_1 = \frac{1}{3}\arccos\left(\frac{7}{13}\right)$$

The motion is symmetrical about each extreme position and about O, so in one complete oscillation P spends a time t_1 with $|x| \geq 0.7$ on each side of each extreme position. This is four intervals of length t_1:

$$T = 4t_1 = \frac{4}{3}\arccos\left(\frac{7}{13}\right) \approx 1.34$$

4. The depth of water at the entrance to a harbour varies with the tides, and the water level moves with simple harmonic motion.
On a particular day, high tide occurs at 05:40, when the depth of water is 11.2 m, and the next low tide occurs at 11:55, when the depth of water is 3.6 m.

(a) Between 05:40 and 11:55 there are two moments at which the water level is falling at a rate of $1.5\,\text{m}\,\text{h}^{-1}$. Find the depth of water at each of these moments. **[6]**

A ship can only use the harbour when the depth of water at the entrance is at least 5.5 m.

(b) Find the length of time between 05:40 and the following high tide during which the ship cannot use the harbour. **[4]**

Solution

The centre of the motion is the mean of the high and low tide depths, and the amplitude is half their difference:

$$\text{centre depth} = \frac{11.2 + 3.6}{2} = 7.4\,\text{m}$$

$$a = \frac{11.2 - 3.6}{2} = 3.8\,\text{m}$$

From high tide at 05:40 to low tide at 11:55 is 6 hours 15 minutes, which is half a period, so the period is $T = 12.5$ hours. Using $T = \frac{2\pi}{\omega}$:

$$\omega = \frac{2\pi}{12.5} = \frac{4\pi}{25}\,\text{rad}\,\text{h}^{-1}$$

Let x m be the height of the water level above its centre level of 7.4 m.

(a) For simple harmonic motion with amplitude a, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

Between high tide and low tide the water level is falling throughout, so the two moments are where the speed is $1.5\,\text{m}\,\text{h}^{-1}$. Substituting $v = 1.5$, $a = 3.8$ and $\omega = \frac{4\pi}{25}$:

$$1.5^2 = \left(\frac{4\pi}{25}\right)^2\left(3.8^2 - x^2\right)$$

$$x^2 = 14.44 - \frac{2.25 \times 625}{16\pi^2}$$

$$x^2 \approx 5.535$$

$$x \approx \pm 2.353$$

The depths of water are therefore

$$7.4 + 2.353 \approx 9.75\,\text{m} \qquad \text{and} \qquad 7.4 - 2.353 \approx 5.05\,\text{m}$$

(b) Let t be the time in hours after 05:40. At $t = 0$ the water is at high tide, so $x = a$ when $t = 0$, and

$$x = 3.8\cos\left(\frac{4\pi t}{25}\right)$$

The following high tide is at $t = 12.5$. The ship cannot use the harbour when the depth is less than 5.5 m, that is when $x < 5.5 - 7.4 = -1.9$. The boundary values satisfy

$$3.8\cos\left(\frac{4\pi t}{25}\right) = -1.9$$

$$\cos\left(\frac{4\pi t}{25}\right) = -\frac{1}{2}$$

For $0 \le t \le 12.5$, the angle $\frac{4\pi t}{25}$ runs from 0 to 2π, so

$$\frac{4\pi t}{25} = \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi t}{25} = \frac{4\pi}{3} \quad \Longrightarrow \quad t = \frac{25}{6} \quad \text{or} \quad t = \frac{25}{3}$$

Between these times $\cos\left(\frac{4\pi t}{25}\right) < -\frac{1}{2}$, so the depth is below 5.5 m (from 09:50 to 14:00). The length of time is

$$\frac{25}{3} - \frac{25}{6} = \frac{25}{6} \text{ hours}$$

This is 4 hours 10 minutes.

5. A particle P moves along the x-axis with simple harmonic motion. At time t seconds, its displacement from the origin O is x metres, where

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 9(2-x)$$

and P is instantaneously at rest when $x = 5$.

(a) Show that v, the velocity of P in $\mathrm{m\,s^{-1}}$ at time t, satisfies

$$v^2 = 9(5-x)(x+1) \qquad \textbf{[5]}$$

(b) When $t = 0$, P is at the point where $x = \frac{1}{2}$ and is moving in the positive x-direction. Hence, or otherwise, show that

$$x = 2 + 3\sin\left(3t - \frac{\pi}{6}\right) \qquad \textbf{[6]}$$

Solution

(a) Writing the acceleration as $v\frac{\mathrm{d}v}{\mathrm{d}x}$:

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = 9(2-x) = 18 - 9x$$

Separating the variables and integrating:

$$\int v\,\mathrm{d}v = \int (18-9x)\,\mathrm{d}x$$

$$\tfrac{1}{2}v^2 = 18x - \tfrac{9}{2}x^2 + c$$

When $x = 5$, $v = 0$, so

$$0 = 90 - \tfrac{225}{2} + c \quad \Longrightarrow \quad c = \tfrac{45}{2}$$

Hence

$$\begin{aligned}\tfrac{1}{2}v^2 &= 18x - \tfrac{9}{2}x^2 + \tfrac{45}{2}\\ v^2 &= 36x - 9x^2 + 45\\ &= 9\left(5 + 4x - x^2\right)\\ &= 9(5-x)(x+1)\end{aligned}$$

as required.

(b) From part (a), completing the square:

$$(5-x)(x+1) = 5 + 4x - x^2 = 9 - (x-2)^2$$

so $v^2 = 9\left(9 - (x-2)^2\right)$. At $t = 0$, P is moving in the positive x-direction, so $v > 0$ and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3\sqrt{9 - (x-2)^2}$$

Separating the variables:

$$\int \frac{1}{\sqrt{9-(x-2)^2}}\,\mathrm{d}x = \int 3\,\mathrm{d}t$$

The formula booklet gives

$$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + K \qquad (|x| < a)$$

Using this with $a = 3$ and x replaced by $x - 2$:

$$\arcsin\left(\frac{x-2}{3}\right) = 3t + K$$

When $t = 0$, $x = \frac{1}{2}$, so

$$K = \arcsin\left(\frac{\frac{1}{2}-2}{3}\right) = \arcsin\left(-\tfrac{1}{2}\right) = -\frac{\pi}{6}$$

Therefore

$$\arcsin\left(\frac{x-2}{3}\right) = 3t - \frac{\pi}{6}$$

$$\frac{x-2}{3} = \sin\left(3t - \frac{\pi}{6}\right)$$

$$x = 2 + 3\sin\left(3t - \frac{\pi}{6}\right)$$

as required.

6. A box of mass m kg rests on a horizontal platform. The platform moves vertically with simple harmonic motion of angular frequency $\omega\,\text{rad}\,\text{s}^{-1}$ and amplitude a m about a mean position O. At time t seconds, the upward displacement of the platform from O is x m.
The box remains in contact with the platform throughout the motion, and the greatest magnitude of the force exerted by the platform on the box is $\frac{7}{3}$ times the least magnitude of this force.

(a) Show that $\omega^2 = \frac{2g}{5a}$. **[3]**

The angular frequency is unchanged, but the amplitude of the motion is increased to $3a$ m.

(b) Show that the box leaves the platform, and find, in terms of a, the height above O at which this first happens. **[3]**

(c) Find, in terms of a and g, the speed of the box at the instant it leaves the platform. **[3]**

Solution

(a) The box stays in contact with the platform, so it moves with the platform. For simple harmonic motion with angular frequency ω, the acceleration is

$$\ddot{x} = -\omega^2 x$$

Let the force exerted by the platform on the box be R N. Take the direction of increasing x (upwards) as positive.

Using Newton's second law, with acceleration $-\omega^2 x$:

$$R - mg = -m\omega^2 x$$

$$R = m(g - \omega^2 x)$$

Since $-a \leq x \leq a$, R is greatest at the lowest point, $x = -a$, and least at the highest point, $x = a$:

$$R_{\max} = m(g + \omega^2 a), \qquad R_{\min} = m(g - \omega^2 a)$$

The greatest force is $\frac{7}{3}$ times the least, so

$$3m(g + \omega^2 a) = 7m(g - \omega^2 a)$$

$$3g + 3a\omega^2 = 7g - 7a\omega^2$$

$$10a\omega^2 = 4g$$

$$\omega^2 = \frac{2g}{5a}$$

as required.

(b) While the box is in contact with the platform, the result of part (a) still holds:

$$R = m(g - \omega^2 x) = m\left(g - \frac{2gx}{5a}\right)$$

The box leaves the platform when $R = 0$, which gives

$$g - \frac{2gx}{5a} = 0$$

$$x = \frac{5a}{2}$$

The platform now moves between $x = -3a$ and $x = 3a$, and $\frac{5a}{2} < 3a$, so the platform reaches $x = \frac{5a}{2}$ on its way up. For $x > \frac{5a}{2}$ the formula would give $R < 0$, which is impossible, since the platform can only push on the box. Hence the box leaves the platform, and this first happens at a height of $\frac{5a}{2}$ m above O.

(c) Up to this instant the box moves with the platform, whose amplitude is $3a$. For simple harmonic motion with amplitude A, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(A^2 - x^2\right)$$

Substituting $\omega^2 = \frac{2g}{5a}$, $A = 3a$ and $x = \frac{5a}{2}$:

$$v^2 = \frac{2g}{5a}\left(9a^2 - \frac{25a^2}{4}\right)$$

$$= \frac{2g}{5a} \times \frac{11a^2}{4}$$

$$= \frac{11ga}{10}$$

$$v = \sqrt{\frac{11ga}{10}}$$

The speed of the box as it leaves the platform is $\sqrt{\frac{11ga}{10}}\,\mathrm{m\,s^{-1}}$.

7. Throughout this question, use $g = 10\,\mathrm{m\,s^{-2}}$.
A light elastic string has natural length 0.9 m and modulus of elasticity 36 N.
A particle P of mass 0.4 kg is attached to one end of the string. The other end of the string is attached to a fixed point A. Particle P hangs freely in equilibrium with P vertically below A.
The particle is then pulled vertically down to a point B and released from rest.

(a) Show that, while the string is taut, P moves with simple harmonic motion with period $\frac{\pi}{5}$ seconds. **[6]**

When the string first becomes slack, the speed of P is $\sqrt{3}\,\mathrm{m\,s^{-1}}$.

(b) Find the distance AB. **[3]**

After the string becomes slack, P continues to move vertically upwards until it first comes to instantaneous rest at the point C.

(c) Find the distance BC and the exact time taken for P to move from B to C. **[6]**

Solution

(a) Hooke's law is

$$T = \frac{\lambda e}{l}$$

where e is the extension and l the natural length. Here $\lambda = 36$ and $l = 0.9$, so $T = 40e$.

In equilibrium, let the extension be e_0. Equating the tension to the weight gives

$$40e_0 = 0.4 \times 10 = 4 \quad \Longrightarrow \quad e_0 = 0.1\,\mathrm{m}$$

Let x be the displacement of P below the equilibrium position. While the string is taut, its extension is $0.1 + x$, so the tension is $40(0.1 + x)$.

Take the downward direction as positive. Using Newton's second law, with acceleration $\ddot{x}$:

$$0.4\ddot{x} = 0.4 \times 10 - 40(0.1 + x)$$

$$0.4\ddot{x} = 4 - 4 - 40x$$

$$\ddot{x} = -100x$$

This is of the form $\ddot{x} = -\omega^2 x$ with $\omega = 10$, so while the string is taut P moves with simple harmonic motion about the equilibrium position. The period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}\,\mathrm{s}$$

as required.

(b) The string becomes slack when its extension is zero, which is when $x = -0.1$. Let the amplitude be a. Since P is released from rest at B, B is at $x = a$.

For simple harmonic motion with amplitude a, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

Substituting $v = \sqrt{3}$, $\omega = 10$ and $x = -0.1$:

$$3 = 100\left(a^2 - 0.01\right)$$

$$a^2 = 0.04$$

$$a = 0.2$$

So B is 0.2 m below the equilibrium position, which is $0.9 + 0.1 = 1$ m below A:

$$AB = 0.9 + 0.1 + 0.2 = 1.2\,\text{m}$$

(c) Once the string is slack, the only force on P is its weight, so P moves freely under gravity with constant acceleration. Take the upward direction as positive, so the acceleration is $\alpha = -10$, the initial velocity $u = \sqrt{3}$ and the final velocity $v = 0$.

The constant acceleration formula used is

$$v^2 = u^2 + 2\alpha s$$

Substituting:

$$0 = 3 - 20s \quad \Longrightarrow \quad s = 0.15$$

So C is 0.15 m above the point where the string becomes slack. That point is $0.2 + 0.1 = 0.3$ m above B, so

$$BC = 0.3 + 0.15 = 0.45\,\text{m}$$

For the time from B to the point where the string becomes slack, measure t from the release at B. Since P starts from rest at $x = a$, the motion is

$$x = 0.2\cos 10t$$

The string becomes slack when $x = -0.1$:

$$0.2\cos 10t = -0.1$$

$$\cos 10t = -\frac{1}{2}$$

$$10t = \frac{2\pi}{3}$$

$$t = \frac{\pi}{15}$$

taking the first positive solution.

Let the time spent moving freely under gravity be t_2. The constant acceleration formula used is

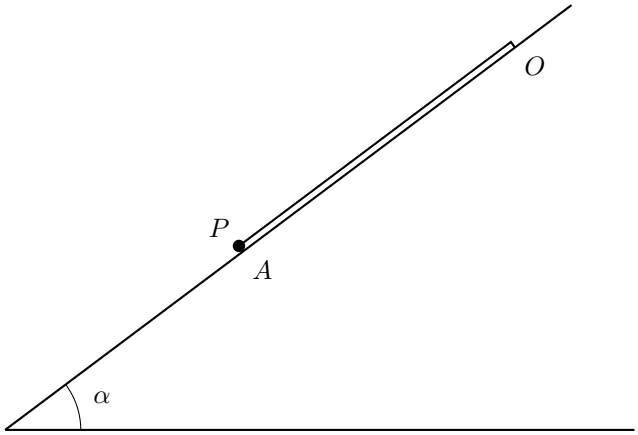
$$v = u + \alpha t$$

Substituting $t = t_2$:

$$0 = \sqrt{3} - 10t_2 \quad \Longrightarrow \quad t_2 = \frac{\sqrt{3}}{10}$$

The total time from B to C is

$$\frac{\pi}{15} + \frac{\sqrt{3}}{10} = \frac{2\pi + 3\sqrt{3}}{30}\,\text{s}$$



8. A particle P of mass m is attached to one end of a light elastic string of natural length $5a$ and modulus of elasticity $3mg$. The other end of the string is fixed to a point O on a smooth plane, which is inclined at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. The string lies along a line of greatest slope of the plane and P rests in equilibrium on the plane at a point A below O, as shown in the diagram above.

At time $t = 0$, P is projected from A with speed $2\sqrt{\frac{3ag}{5}}$ up the line of greatest slope towards O. At time t, the displacement of P from A is x, where the positive direction of x is up the plane.

(a) Show that, while the string is taut, x satisfies the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{3gx}{5a} = 0 \qquad \textbf{[6]}$$

(b) Find, in terms of a and g, the time at which the string first becomes slack. **[5]**

(c) Find, in terms of a, the distance from A of the highest point on the plane reached by P. **[3]**

Solution

(a) Hooke's law is

$$T = \frac{\lambda e}{l}$$

where e is the extension and l the natural length. Here $\lambda = 3mg$ and $l = 5a$.

Let the extension at A be e_0. Resolving parallel to the plane for the equilibrium at A:

$$\frac{3mge_0}{5a} = mg\sin\alpha$$

$$\frac{3mge_0}{5a} = \frac{3mg}{5}$$

$$e_0 = a$$

When the displacement of P from A is x up the plane, the extension is $a - x$, so while the string is taut the tension, acting up the plane, is

$$T = \frac{3mg(a-x)}{5a}$$

The component of the weight down the plane is $mg\sin\alpha = \frac{3mg}{5}$, and the normal reaction has no component along the plane.

Using Newton's second law, with acceleration $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{3mg(a-x)}{5a} - \frac{3mg}{5}$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{3mg}{5} - \frac{3mgx}{5a} - \frac{3mg}{5}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{3gx}{5a}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{3gx}{5a} = 0$$

as required.

(b) By part (a), while the string is taut P moves with simple harmonic motion about A, with

$$\omega^2 = \frac{3g}{5a} \quad \Longrightarrow \quad \omega = \sqrt{\frac{3g}{5a}}$$

At $t = 0$, P is at the centre of the motion, so its speed $2\sqrt{\frac{3ag}{5}}$ is the maximum speed. Let the amplitude be R. The maximum speed is $R\omega$, so

$$R\sqrt{\frac{3g}{5a}} = 2\sqrt{\frac{3ag}{5}}$$

$$R = 2\sqrt{\frac{3ag}{5}} \times \sqrt{\frac{5a}{3g}}$$

$$R = 2a$$

Since $x = 0$ when $t = 0$ and P starts moving in the positive direction, the displacement is

$$x = 2a\sin(\omega t)$$

The extension is $a - x$, so the string first becomes slack when $x = a$. Since $a < 2a$, P does reach this point:

$$2a\sin(\omega t) = a$$

$$\sin(\omega t) = \frac{1}{2}$$

The first time this happens is when $\omega t = \frac{\pi}{6}$, so

$$t = \frac{\pi}{6}\sqrt{\frac{5a}{3g}}$$

(c) For simple harmonic motion with amplitude R, the speed v at displacement x from the centre satisfies

$$v^2 = \omega^2\left(R^2 - x^2\right)$$

The string becomes slack when $x = a$. Substituting $\omega^2 = \frac{3g}{5a}$, $R = 2a$ and $x = a$:

$$v^2 = \frac{3g}{5a}\left(4a^2 - a^2\right) = \frac{9ag}{5}$$

After this the string is slack, and P moves a further distance d up the plane before coming to rest.

The work–energy principle is

$$KE_0 + GPE_0 + EPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1 + EPE_1$$

where state 0 is the point where the string becomes slack and state 1 is the highest point. Take the initial position as the zero level for gravitational potential energy. The string is slack from state 0 to state 1, so $EPE_0 = EPE_1 = 0$. The normal reaction is perpendicular to the motion, so no other force does work, and $\sum W_{\text{other}} = 0$. This gives

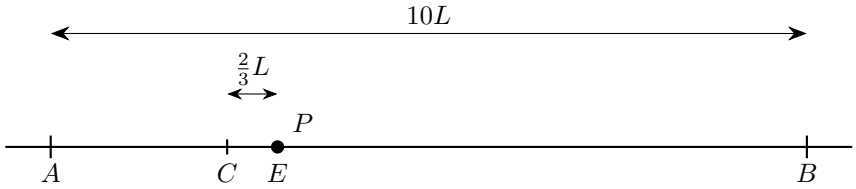
$$\frac{1}{2}m \times \frac{9ag}{5} = mgd\sin\alpha$$

$$\frac{9mag}{10} = \frac{3mgd}{5}$$

$$d = \frac{3a}{2}$$

The distance from A of the highest point is

$$a + \frac{3a}{2} = \frac{5a}{2}$$



9. A particle P of mass m lies on a smooth horizontal table.
One end of a light elastic string, of natural length $2L$ and modulus of elasticity $3mg$, is attached to P, and the other end is attached to a fixed point A on the table.
One end of a second light elastic string, of natural length $4L$ and modulus of elasticity $2mg$, is also attached to P, and the other end is attached to a fixed point B on the table, where $AB = 10L$.
The particle P rests in equilibrium at the point E, where AEB is a straight line, as shown in the diagram above.

(a) Show that $AE = 3L$. **[2]**

The particle P is now held at the point C on AE, where $CE = \frac{2}{3}L$, and released from rest.

(b) Show that P moves with simple harmonic motion with centre E and period $\pi\sqrt{\frac{2L}{g}}$. **[6]**

(c) Find, in terms of L and g, the speed of P at the instant when $AP = \frac{5}{2}L$. **[3]**

(d) Find, in terms of L and g, the exact length of time in each complete oscillation for which the tension in the string AP is greater than $2mg$. **[6]**

Solution

(a) Hooke's law is

$$T = \frac{\lambda x}{l}$$

where x is the extension and l the natural length.

At E, the string AP has extension $AE - 2L$, and the string BP has length $10L - AE$, so its extension is $6L - AE$. The tensions are

$$T_{AP} = \frac{3mg(AE - 2L)}{2L}$$

$$T_{BP} = \frac{2mg(6L - AE)}{4L} = \frac{mg(6L - AE)}{2L}$$

In equilibrium the two tensions are equal, so

$$3(AE - 2L) = 6L - AE$$

$$4AE = 12L$$

$$AE = 3L$$

as required.

(b) Let y be the displacement of P from E, measured in the direction from A to B. Then $AP = 3L + y$,

with extension $L+y$, and $BP = 7L - y$, with extension $3L - y$. While both strings are taut, the tensions are

$$T_{AP} = \frac{3mg(L+y)}{2L}$$

$$T_{BP} = \frac{2mg(3L-y)}{4L} = \frac{mg(3L-y)}{2L}$$

Take the direction from A to B as positive. The tension in BP acts towards B and the tension in AP acts towards A.

Using Newton's second law, with acceleration $\ddot{y}$:

$$\begin{aligned} m\ddot{y} &= \frac{mg(3L-y)}{2L} - \frac{3mg(L+y)}{2L} \\ &= \frac{mg}{2L}(3L - y - 3L - 3y) \\ &= -\frac{2mg}{L}\,y \end{aligned}$$

so

$$\ddot{y} = -\frac{2g}{L}\,y$$

This is of the form $\ddot{y} = -\omega^2 y$ with $\omega^2 = \frac{2g}{L}$, so P moves with simple harmonic motion with centre E, where $y = 0$.

P is released from rest at C, where $y = -\frac{2}{3}L$, so the amplitude is $\frac{2}{3}L$ and $-\frac{2}{3}L \leq y \leq \frac{2}{3}L$ throughout the motion. The extension of AP is then at least $L - \frac{2}{3}L = \frac{1}{3}L > 0$, and the extension of BP is at least $3L - \frac{2}{3}L = \frac{7}{3}L > 0$, so both strings stay taut and the equation of motion holds throughout.

The period is

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{2g}} = \pi\sqrt{\frac{4L}{2g}} = \pi\sqrt{\frac{2L}{g}}$$

as required.

(c) When $AP = \frac{5}{2}L$, the displacement is $y = \frac{5}{2}L - 3L = -\frac{1}{2}L$. The amplitude is $a = \frac{2}{3}L$. For simple harmonic motion with amplitude a, the speed v at displacement y from the centre satisfies

$$v^2 = \omega^2\left(a^2 - y^2\right)$$

Substituting $\omega^2 = \frac{2g}{L}$:

$$\begin{aligned} v^2 &= \frac{2g}{L}\left(\frac{4L^2}{9} - \frac{L^2}{4}\right) \\ &= \frac{2g}{L} \times \frac{7L^2}{36} \\ &= \frac{7gL}{18} \end{aligned}$$

so the speed of P is

$$v = \sqrt{\frac{7gL}{18}} = \frac{\sqrt{14gL}}{6}$$

(d) The tension in AP is greater than $2mg$ when

$$\frac{3mg(L+y)}{2L} > 2mg$$

$$3(L+y) > 4L$$

$$y > \frac{1}{3}L$$

Measure the time t from the moment of release. P starts from rest at $y = -\frac{2}{3}L$, so

$$y = -\frac{2}{3}L\cos\omega t$$

The condition $y > \frac{1}{3}L$ becomes

$$-\frac{2}{3}L\cos\omega t > \frac{1}{3}L$$

$$\cos\omega t < -\frac{1}{2}$$

During one complete oscillation, $0 \leq \omega t \leq 2\pi$, this holds for

$$\frac{2\pi}{3} < \omega t < \frac{4\pi}{3}$$

an interval of $\frac{2\pi}{3}$ in ωt. With $\omega = \sqrt{\frac{2g}{L}}$, the length of time is

$$\frac{2\pi}{3} \times \sqrt{\frac{L}{2g}} = \frac{\pi}{3}\sqrt{\frac{4L}{2g}} = \frac{\pi}{3}\sqrt{\frac{2L}{g}}$$

10. A particle P of mass m is attached to one end of a light elastic string of natural length l and modulus of elasticity mg. The other end of the string is attached to a fixed point O.
Particle P is projected vertically downwards from O with speed $\sqrt{gl}$.

(a) Show that the greatest length of the string in the subsequent motion is $4l$. **[5]**

The string first becomes taut at time $t = 0$. At time $t \geq 0$, the length of the string is $l + x$, where x is the extension in the string.

(b) Show that, while the string is taut, x satisfies the differential equation

$$\ddot{x} + \omega^2 x = g, \qquad \text{where } \omega^2 = \frac{g}{l} \quad \textbf{[3]}$$

(c) By using the substitution $x = y + \frac{g}{\omega^2}$, solve the differential equation and hence find, in terms of l and g, the exact length of time for which the string is taut before it first becomes slack. **[12]**

Solution

(a) Until the string becomes taut, P falls freely a distance l with acceleration g. The constant acceleration formula used is

$$v^2 = u^2 + 2as$$

Substituting $u = \sqrt{gl}$, $a = g$ and $s = l$, where v is the speed when the string becomes taut:

$$v^2 = gl + 2gl = 3gl$$

Let the greatest extension of the string be d. At the lowest point P is instantaneously at rest. The work–energy principle is

$$KE_0 + GPE_0 + EPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1 + EPE_1$$

where state 0 is when the string first becomes taut and state 1 is the lowest point. The elastic potential energy is

$$EPE = \frac{kx^2}{2l}$$

where k is the modulus of elasticity, x is the extension and l the natural length; here $k = mg$. Take the final position as the zero level for gravitational potential energy. No other force does work, so $\sum W_{\text{other}} = 0$. Substituting:

$$\tfrac{1}{2}m(3gl) + mgd + 0 + 0 = 0 + 0 + \frac{mgd^2}{2l}$$

$$3l^2 + 2ld = d^2$$

$$d^2 - 2ld - 3l^2 = 0$$

$$(d - 3l)(d + l) = 0$$

Since $d > 0$, $d = 3l$, so the greatest length of the string is $l + 3l = 4l$, as required.

(b) Hooke's law is

$$T = \frac{kx}{l}$$

where k is the modulus of elasticity, x is the extension and l the natural length. Here $k = mg$, so while the string is taut the tension is $\frac{mgx}{l}$, acting upwards on P, and the weight mg acts downwards. Take the direction of increasing x (downwards) as positive. Using Newton's second law, with acceleration $\ddot{x}$:

$$m\ddot{x} = mg - \frac{mgx}{l}$$

$$\ddot{x} + \frac{g}{l}x = g$$

so $\ddot{x} + \omega^2 x = g$, where $\omega^2 = \frac{g}{l}$, as required.

(c) Since $\frac{g}{\omega^2} = g \times \frac{l}{g} = l$, the substitution is $x = y + l$, so $\ddot{x} = \ddot{y}$. Substituting into the differential equation:

$$\ddot{y} + \omega^2(y + l) = g$$

$$\ddot{y} + \omega^2 y + g = g$$

$$\ddot{y} + \omega^2 y = 0$$

The auxiliary equation is

$$\lambda^2 + \omega^2 = 0 \quad \Longrightarrow \quad \lambda = \pm i\omega$$

so

$$y = A\cos\omega t + B\sin\omega t$$

$$x = l + A\cos\omega t + B\sin\omega t$$

Differentiating:

$$\dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$$

When $t = 0$, the string first becomes taut, so $x = 0$, and from part (a) the speed is $\sqrt{3gl}$ downwards, so $\dot{x} = \sqrt{3gl}$. Using $x = 0$ when $t = 0$:

$$0 = l + A \quad \Longrightarrow \quad A = -l$$

Using $\dot{x} = \sqrt{3gl}$ when $t = 0$:

$$B\omega = \sqrt{3gl} \quad \Longrightarrow \quad B = \sqrt{3gl} \times \sqrt{\frac{l}{g}} = \sqrt{3}\,l$$

Hence

$$x = l - l\cos\omega t + \sqrt{3}\,l\sin\omega t$$

Write $\sqrt{3}\,l\sin\omega t - l\cos\omega t$ as $R\sin(\omega t - \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Expanding:

$$R\sin(\omega t - \alpha) = R\cos\alpha\sin\omega t - R\sin\alpha\cos\omega t$$

Comparing coefficients:

$$R\cos\alpha = \sqrt{3}\,l, \qquad R\sin\alpha = l$$

so

$$R = \sqrt{3l^2 + l^2} = 2l, \qquad \tan\alpha = \frac{1}{\sqrt{3}} \quad \Longrightarrow \quad \alpha = \frac{\pi}{6}$$

Therefore the solution, valid while the string is taut, is

$$x = l + 2l\sin\Big(\omega t - \frac{\pi}{6}\Big), \qquad \text{where } \omega = \sqrt{\frac{g}{l}}$$

The string becomes slack when $x = 0$:

$$l + 2l\sin\Big(\omega t - \frac{\pi}{6}\Big) = 0$$

$$\sin\Big(\omega t - \frac{\pi}{6}\Big) = -\frac{1}{2}$$

At $t = 0$, $\omega t - \frac{\pi}{6} = -\frac{\pi}{6}$, and $\omega t - \frac{\pi}{6}$ increases with t. For $-\frac{\pi}{6} < \omega t - \frac{\pi}{6} < \frac{7\pi}{6}$, $\sin\big(\omega t - \frac{\pi}{6}\big) > -\frac{1}{2}$, so $x > 0$ and the string stays taut. The next solution is therefore

$$\omega t - \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\omega t = \frac{4\pi}{3}$$

$$t = \frac{4\pi}{3}\sqrt{\frac{l}{g}}$$