FURTHER MECHANICS / CIRCULAR MOTION & OSCILLATIONS

Motion in a Vertical Circle

Worked solutions 15 questions · 206 marks

EXAM BOARDS Edexcel FM2 9FM0/4C · AQA 7367/3M · OCR A Y543 · OCR MEI Mech Major Y421 CIE 9231 Paper 3

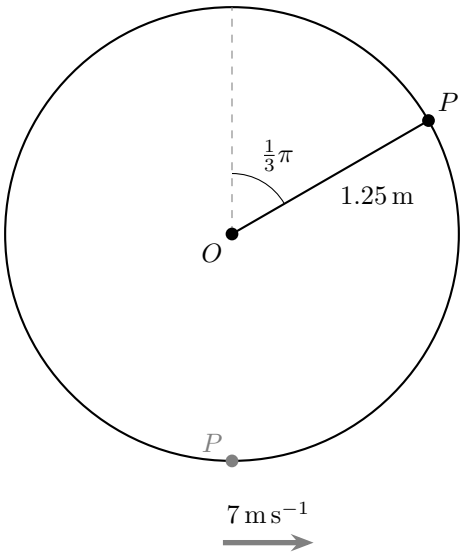
Instructions

Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question gives a value. Where a question leaves g unspecified and a final answer depends on it, the solutions also give that answer for $g = 10\,\mathrm{m\,s^{-2}}$.

CIE (9231 Paper 3): papers take $g = 10\,\mathrm{m\,s^{-2}}$; use the $g = 10$ final answers where the solutions give them.

Questions

Question 1	*10 marks*	2
Question 2	*12 marks*	4
Question 3	*12 marks*	7
Question 4	*9 marks*	9
Question 5	*11 marks*	11
Question 6	*13 marks*	13
Question 7	*16 marks*	15
Question 8	*15 marks*	18
Question 9	*13 marks*	20
Question 10	*14 marks*	23
Question 11	*21 marks*	25
Question 12	*17 marks*	28
Question 13	*11 marks*	30
Question 14	*13 marks*	32
Question 15	*19 marks*	35



1. One end of a light inextensible string of length 1.25 m is attached to a particle P of mass m kg. The other end of the string is attached to a fixed point O.
Initially P hangs in equilibrium vertically below O. It is then projected horizontally with a speed of $7\,\text{m s}^{-1}$ so that it moves in a vertical circle with centre O, as shown in the diagram above.
At a certain instant, P is above the level of O and first reaches the point where the string makes an angle of $\frac{1}{3}\pi$ radians with the upward vertical through O.

(a) Find the speed of P at this instant. **[3]**

(b) Find the magnitude and direction of the radial acceleration of P at this instant. **[3]**

(c) Find the magnitude of the resultant acceleration of P at this instant. **[4]**

Solution

(a) Let the speed of P at this instant be $v\,\text{m s}^{-1}$. Let state 0 be the lowest point and state 1 the position where the string makes an angle of $\frac{1}{3}\pi$ with the upward vertical. The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

Take the initial position as the zero level for gravitational potential energy. The height of P above its initial position is

$$1.25 + 1.25\cos\tfrac{1}{3}\pi = 1.25 + 0.625 = 1.875\,\text{m}$$

The tension is always perpendicular to the direction of motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Then

$$\tfrac{1}{2}m \times 7^2 + 0 + 0 = \tfrac{1}{2}mv^2 + m \times 9.8 \times 1.875$$

$$49 = v^2 + 36.75$$

$$v^2 = 12.25$$

$$v = 3.5$$

The speed of P is $3.5\,\text{m s}^{-1}$.

(b) The radial acceleration is

$$\frac{v^2}{r} = \frac{12.25}{1.25} = 9.8\,\mathrm{m\,s^{-2}}$$

It is directed along PO, towards the centre O.

(c) Let the tangential component of the acceleration be $a\,\mathrm{m\,s^{-2}}$. The tension acts along the string, so only the weight has a component along the tangent. Since OP makes an angle of $\frac{1}{3}\pi$ with the upward vertical, the component of the weight along the tangent is $mg\sin\frac{1}{3}\pi$, opposing the motion as P rises.

Take the direction of motion as positive. Using Newton's second law:

$$-m \times 9.8 \sin\tfrac{1}{3}\pi = ma$$

$$a = -9.8 \times \frac{\sqrt{3}}{2} = -4.9\sqrt{3}$$

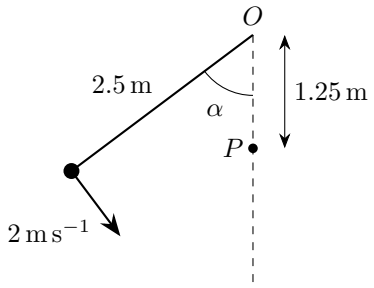
From part (b), the radial component is $9.8\,\mathrm{m\,s^{-2}}$, and it is perpendicular to the tangential component. The magnitude of the resultant acceleration is

$$\sqrt{9.8^2 + \left(4.9\sqrt{3}\right)^2} = \sqrt{4 \times 4.9^2 + 3 \times 4.9^2}$$

$$= 4.9\sqrt{7} \approx 13.0\,\mathrm{m\,s^{-2}}$$

Final answers with $g = 10$: (a) $3.39\,\mathrm{m\,s^{-1}}$; (b) $9.2\,\mathrm{m\,s^{-2}}$; (c) $\frac{\sqrt{3991}}{5} \approx 12.6\,\mathrm{m\,s^{-2}}$

2. In this question use $g = 10\,\text{m s}^{-2}$.
A sphere of mass 0.5 kg is attached to one end of a light inextensible string of length 2.5 m. The other end of the string is attached to a fixed point O. A small smooth peg P is fixed 1.25 m vertically below O. The sphere is held with the string taut and making an angle α with the downward vertical, where $\cos\alpha = \frac{3}{5}$. It is then projected with speed $2\,\text{m s}^{-1}$, perpendicular to the string, so that it moves downwards, as shown in the diagram below. When the string becomes vertical it meets the peg, and the sphere then moves on a circle with centre P.



(a) Show that the speed of the sphere immediately before the string meets the peg is $2\sqrt{6}\,\text{m s}^{-1}$. **[3]**

(b) State one modelling assumption that you have made about the sphere in part (a). **[1]**

(c) Find the tension in the string immediately before the string meets the peg and the tension in the string immediately after it meets the peg. **[4]**

(d) In an experiment with this set-up, the speed of the sphere immediately before the string meets the peg is measured as $4.6\,\text{m s}^{-1}$.

(i) Use this measurement to find the work done against air resistance as the sphere moves from its initial position to the point directly below O. **[2]**

(ii) Find the tension in the string immediately after it meets the peg in this experiment. **[2]**

Solution

(a) The sphere starts $2.5\cos\alpha$ m below O and is 2.5 m below O when the string is vertical, so the drop in height is

$$2.5 - 2.5 \times \frac{3}{5} = 2.5 - 1.5 = 1\,\text{m}$$

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

with state 0 the initial position and state 1 the point directly below O. Take the final position as the zero level for gravitational potential energy. The tension is perpendicular to the direction of motion,

so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. With $v\,\text{m s}^{-1}$ the speed immediately before the string meets the peg:

$$\frac{1}{2} \times 0.5 \times 2^2 + 0.5 \times 10 \times 1 + 0 = \frac{1}{2} \times 0.5 \times v^2 + 0$$

$$1 + 5 = 0.25v^2$$

$$v^2 = 24$$

$$v = \sqrt{24} = 2\sqrt{6}$$

So the speed is $2\sqrt{6}\,\text{m s}^{-1}$, as required.

(b) The sphere is modelled as a particle, so its size is ignored.

(c) Immediately before the string meets the peg, the sphere moves on a circle of radius 2.5 m with centre O, and its acceleration is towards O, vertically upwards. Take the direction towards the centre of the circle as positive.

Using Newton's second law, with acceleration $\frac{v^2}{r}$, where $r = 2.5$:

$$T_1 - 0.5 \times 10 = 0.5 \times \frac{24}{2.5}$$

$$T_1 - 5 = 4.8$$

$$T_1 = 9.8$$

The tension immediately before the string meets the peg is 9.8 N.

The peg is smooth and the tension is perpendicular to the motion, so the speed is still $2\sqrt{6}\,\text{m s}^{-1}$ immediately after the string meets the peg. The sphere now moves on a circle with centre P and radius $2.5 - 1.25 = 1.25$ m.

Using Newton's second law, with acceleration $\frac{v^2}{r}$, where $r = 1.25$:

$$T_2 - 0.5 \times 10 = 0.5 \times \frac{24}{1.25}$$

$$T_2 - 5 = 9.6$$

$$T_2 = 14.6$$

The tension immediately after the string meets the peg is 14.6 N.

(d) (i) Using the work–energy principle with the same states and zero level as in part (a), the only other force doing work is now air resistance, so $\sum W_{\text{other}}$ is the work done by air resistance:

$$\frac{1}{2} \times 0.5 \times 2^2 + 0.5 \times 10 \times 1 + \sum W_{\text{other}} = \frac{1}{2} \times 0.5 \times 4.6^2 + 0$$

$$6 + \sum W_{\text{other}} = 5.29$$

$$\sum W_{\text{other}} = -0.71$$

The work done by air resistance is -0.71 J, so the work done against air resistance is 0.71 J.

(ii) Immediately after the string meets the peg, the speed is $4.6\,\text{m s}^{-1}$ and the radius is 1.25 m. Take the direction towards P as positive.

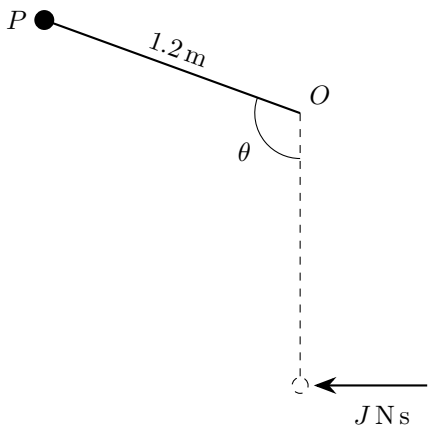
Using Newton's second law, with acceleration $\frac{v^2}{r}$, where $r = 1.25$:

$$T - 0.5 \times 10 = 0.5 \times \frac{4.6^2}{1.25}$$

$$T - 5 = 8.464$$

$$T = 13.464$$

The tension is approximately 13.5 N.



3. A particle P of mass 2.5 kg is attached to one end of a light inextensible string of length 1.2 m. The other end of the string is attached to a fixed point O.
P is hanging at rest vertically below O when it receives a horizontal impulse of magnitude J N s. In the subsequent motion, the angle that the string makes with the downward vertical through O is denoted by θ, as shown in the diagram above.
The string first becomes slack when $\theta = \frac{2}{3}\pi$.

(a) Find the value of J. **[6]**

(b) Find the tension in the string at the first instant when $\theta = \frac{1}{3}\pi$. **[6]**

Solution

(a) Let the speed of P be v_1 m s^{-1} when $\theta = \frac{2}{3}\pi$, and let the tension in the string be T N. At this point P is above the level of O and the string makes an angle $\frac{1}{3}\pi$ with the upward vertical, so the component of the weight towards O is $2.5g\cos\frac{1}{3}\pi$.

Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v_1^2}{r}$, where $r = 1.2$:

$$T + 2.5g\cos\tfrac{1}{3}\pi = \frac{2.5v_1^2}{1.2}$$

The string first becomes slack here, so $T = 0$ and

$$2.5 \times 9.8 \times \tfrac{1}{2} = \frac{2.5v_1^2}{1.2}$$

$$v_1^2 = 5.88$$

Let the speed of P immediately after the impulse be v m s^{-1}. The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is the lowest point, immediately after the impulse, and state 1 is the position where $\theta = \frac{2}{3}\pi$. Take the initial position as the zero level for gravitational potential energy. At state 1, P is $1.2 - 1.2\cos\frac{2}{3}\pi = 1.8$ m above this level. The tension is perpendicular to the motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$.

Substituting into the principle:

$$\tfrac{1}{2} \times 2.5v^2 + 0 + 0 = \tfrac{1}{2} \times 2.5 \times 5.88 + 2.5 \times 9.8 \times 1.8$$

$$v^2 = 5.88 + 35.28$$

$$= 41.16$$

$$v = \frac{7\sqrt{21}}{5}$$

Take the direction of the impulse as positive. Using impulse = change in momentum:

$$J = mv - mu$$

Here $m = 2.5$, $u = 0$ since P starts at rest, and $v = \frac{7\sqrt{21}}{5}$, so

$$J = 2.5 \times \frac{7\sqrt{21}}{5} - 0 = \frac{7\sqrt{21}}{2} \approx 16.0$$

(b) Let the speed of P be $v_2\,\mathrm{m\,s^{-1}}$ when $\theta = \frac{1}{3}\pi$. Here P is $1.2 - 1.2\cos\frac{1}{3}\pi = 0.6\,\mathrm{m}$ above the lowest point.

Apply the work–energy principle with state 0 the lowest point, as in part (a), and state 1 now the position where $\theta = \frac{1}{3}\pi$, with the same zero level. Again $\sum W_{\text{other}} = 0$, so

$$\tfrac{1}{2} \times 2.5 \times 41.16 + 0 + 0 = \tfrac{1}{2} \times 2.5v_2^2 + 2.5 \times 9.8 \times 0.6$$

$$v_2^2 = 41.16 - 11.76$$

$$= 29.4$$

At this point P is below the level of O and the string makes an angle $\frac{1}{3}\pi$ with the downward vertical, so the component of the weight towards O is $-2.5g\cos\frac{1}{3}\pi$.

Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v_2^2}{r}$, where $r = 1.2$:

$$T - 2.5g\cos\tfrac{1}{3}\pi = \frac{2.5v_2^2}{1.2}$$

$$T = \frac{2.5 \times 29.4}{1.2} + 2.5 \times 9.8 \times \tfrac{1}{2}$$

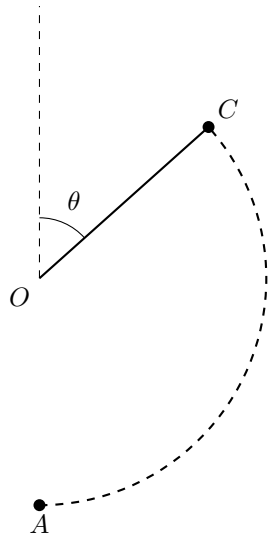
$$= 61.25 + 12.25$$

$$= 73.5$$

The tension in the string is $73.5\,\mathrm{N}$.

Final answers with $g = 10$: (a) $J = \frac{5\sqrt{42}}{2} \approx 16.2$; (b) $75\,\mathrm{N}$

4. A small sphere, of mass m, is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O.
The sphere is at rest at the point A, directly below O, when it is struck, giving it a horizontal impulse of magnitude $2m\sqrt{ag}$, where g is the acceleration due to gravity.
The sphere then moves on a circular path in a vertical plane containing O until the string becomes slack at the point C. At C the string makes an angle θ with the upward vertical through O, as shown in the diagram below.



(a) Find the exact value of $\cos\theta$ and hence find, in terms of a, the height of C above A. **[7]**

(b) With reference to any modelling assumptions that you have made, explain why giving the height of C above A as an inequality would be more appropriate, and state this inequality. **[2]**

Solution

(a) Take the direction of the impulse as positive. Let the speed of the sphere at A just after the impulse be v_A. Using impulse = change in momentum:

$$I = mv - mu$$

The sphere starts at rest, so

$$2m\sqrt{ag} = mv_A - m \times 0 \quad \Longrightarrow \quad v_A = 2\sqrt{ag}$$

Let the speed of the sphere at C be v_C. The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is at A, just after the impulse, and state 1 is at C. Take the initial position as the zero level for gravitational potential energy. O is a height a above A, and C is a height $a\cos\theta$ above O, so

C is a height $a + a\cos\theta$ above A. The tension acts along the string, perpendicular to the direction of motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Substituting:

$$\tfrac{1}{2}m(2\sqrt{ag})^2 + 0 + 0 = \tfrac{1}{2}mv_C^2 + mg(a + a\cos\theta)$$

$$2mag = \tfrac{1}{2}mv_C^2 + mga(1 + \cos\theta)$$

$$v_C^2 = 4ag - 2ag(1 + \cos\theta)$$

$$v_C^2 = 2ag(1 - \cos\theta)$$

At C, the component of the weight along the string, towards O, is $mg\cos\theta$. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v_C^2}{a}$:

$$T + mg\cos\theta = \frac{mv_C^2}{a}$$

The string becomes slack at C, so $T = 0$ there, which gives

$$v_C^2 = ag\cos\theta$$

Equating the two expressions for v_C^2:

$$ag\cos\theta = 2ag(1 - \cos\theta)$$

$$\cos\theta = 2 - 2\cos\theta$$

$$3\cos\theta = 2$$

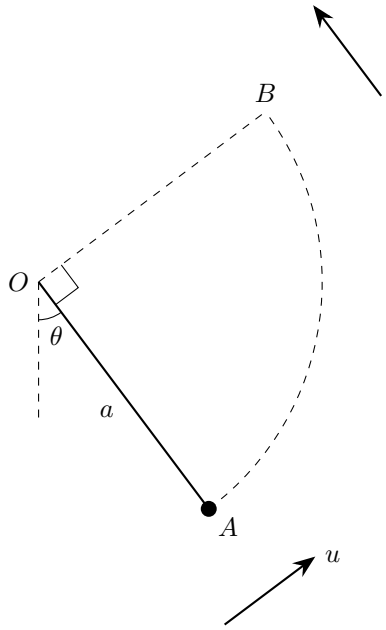
$$\cos\theta = \frac{2}{3}$$

Hence the height of C above A is

$$a + a\cos\theta = a + \frac{2}{3}a = \frac{5a}{3}$$

(b) The model treats the sphere as a particle and ignores air resistance, so no mechanical energy is lost as the sphere moves. In reality air resistance does negative work on the sphere, so at each point of the path its speed is less than the model predicts. Then $v_C^2 < 2ag(1 - \cos\theta)$, and with $v_C^2 = ag\cos\theta$ when the string becomes slack this gives $3\cos\theta < 2$, so $\cos\theta < \frac{2}{3}$: the string becomes slack sooner, at a lower point. Letting h be the height of C above A, it is more appropriate to give

$$h < \frac{5a}{3}$$



5. A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O.
The particle is held with the string taut at the point A, where OA makes an angle θ with the downward vertical through O and $\tan\theta = \frac{3}{4}$. The particle is then projected with speed u perpendicular to OA and begins to move upwards in part of a vertical circle, as shown in the diagram above.
The string goes slack when the particle reaches the point B, where angle AOB is a right angle.

(a) Find the tension in the string when the particle is at A, giving your answer in terms of m and g. **[8]**

(b) Find, in terms of a, the greatest height above O reached by the particle in its subsequent motion. **[3]**

Solution

(a) Since $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$. Let the speed of the particle at B be v_B.

Angle AOB is a right angle, so OB makes an angle θ with the horizontal through O, with B above O. The string at B therefore makes an angle $90° - \theta$ with the upward vertical, and the component of the weight towards O is $mg\cos(90° - \theta) = mg\sin\theta$. The string goes slack at B, so the tension there is zero.

Take the direction towards O as positive. Using Newton's second law at B, with acceleration $\frac{v_B^2}{a}$:

$$mg\sin\theta = \frac{mv_B^2}{a}$$

$$v_B^2 = ga\sin\theta = \frac{3ga}{5}$$

The point A is $a\cos\theta = \frac{4a}{5}$ below O, and B is $a\sin\theta = \frac{3a}{5}$ above O, so B is $\frac{7a}{5}$ above A.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is the particle at A and state 1 is the particle at B. Take the initial position as the

zero level for gravitational potential energy. The tension is perpendicular to the direction of motion throughout. No other force does work, so $\sum W_{\text{other}} = 0$. Then

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv_B^2 + mg\left(\frac{7a}{5}\right)$$

$$u^2 = \frac{3ga}{5} + \frac{14ga}{5} = \frac{17ga}{5}$$

At A, the component of the weight along the string is $mg\cos\theta$, directed away from O. Let the tension at A be T. Take the direction towards O as positive. Using Newton's second law at A, with acceleration $\frac{u^2}{a}$:

$$T - mg\cos\theta = \frac{mu^2}{a}$$

$$T = \frac{4mg}{5} + \frac{m}{a} \times \frac{17ga}{5}$$

$$T = \frac{21mg}{5}$$

(b) After B the string is slack, so the particle moves freely under gravity. At B its velocity is perpendicular to OB, so it makes an angle θ with the upward vertical, and its vertical component is

$$v_B\cos\theta = \frac{4}{5}v_B$$

The horizontal component of velocity is not zero, so the particle is at its greatest height when the vertical component of velocity is zero.

Take upwards as positive. For the vertical motion from B, with initial velocity v_0, final velocity v, acceleration α and displacement s:

$$v^2 = v_0^2 + 2\alpha s$$

With $v = 0$, $v_0 = \frac{4}{5}v_B$ and $\alpha = -g$:

$$0 = \frac{16}{25} \times \frac{3ga}{5} - 2gs$$

$$s = \frac{24a}{125}$$

So the greatest height above O is

$$\frac{3a}{5} + \frac{24a}{125} = \frac{99a}{125}$$

6. A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in complete vertical circles with centre O. The points A and B are on the path of P, both on the same side of the vertical through O. OA makes an acute angle θ with the downward vertical through O and OB makes an angle θ with the upward vertical through O. The tensions in the string when P is at A and when P is at B are T_A and T_B respectively.

(a) Show that $T_A - T_B = 6mg\cos\theta$. **[8]**

The speed of P at the lowest point of its path is $\sqrt{6ag}$, and it is now given that $5T_A = 11T_B$.

(b) Find the value of θ and find, in terms of a and g, the speed of P when it is at B. **[5]**

Solution

(a) Let the speeds of P at A and B be v_A and v_B. At A, the weight has component $mg\cos\theta$ along OA directed away from O; at B, it has component $mg\cos\theta$ along BO directed towards O.

Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{a}$:

$$\text{At } A: \quad T_A - mg\cos\theta = \frac{mv_A^2}{a}$$

$$\text{At } B: \quad T_B + mg\cos\theta = \frac{mv_B^2}{a}$$

A is $a\cos\theta$ below O and B is $a\cos\theta$ above O, so B is $2a\cos\theta$ above A.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

with state 0 the particle at A and state 1 the particle at B. The tension is perpendicular to the motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Take the initial position as the zero level for gravitational potential energy. Then

$$\tfrac{1}{2}mv_A^2 = \tfrac{1}{2}mv_B^2 + mg(2a\cos\theta)$$

$$v_A^2 - v_B^2 = 4ag\cos\theta$$

Subtracting the two equations of motion:

$$\begin{aligned} T_A - T_B - 2mg\cos\theta &= \frac{m\left(v_A^2 - v_B^2\right)}{a} \\ &= \frac{m(4ag\cos\theta)}{a} \\ &= 4mg\cos\theta \\ T_A - T_B &= 6mg\cos\theta \end{aligned}$$

as required.

(b) Let L be the lowest point of the path. B is $a + a\cos\theta$ above L. Applying the work–energy principle

with state 0 at L and state 1 at B, taking L as the zero level for gravitational potential energy, with $\sum W_{\text{other}} = 0$ as before:

$$\begin{aligned} \tfrac{1}{2}m(6ag) &= \tfrac{1}{2}mv_B^2 + mga(1+\cos\theta) \\ v_B^2 &= 6ag - 2ag(1+\cos\theta) \\ &= 4ag - 2ag\cos\theta \end{aligned}$$

Using the equation of motion at B from part (a):

$$\begin{aligned} T_B &= \frac{mv_B^2}{a} - mg\cos\theta \\ &= m(4g - 2g\cos\theta) - mg\cos\theta \\ &= mg(4 - 3\cos\theta) \end{aligned}$$

Using part (a), $T_A = T_B + 6mg\cos\theta$, so $5T_A = 11T_B$ gives

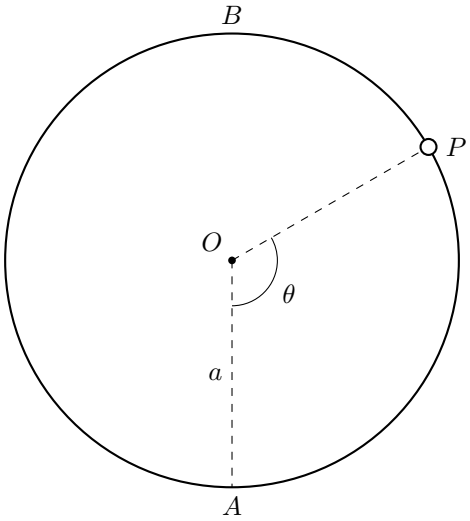
$$\begin{aligned} 5T_B + 30mg\cos\theta &= 11T_B \\ T_B &= 5mg\cos\theta \end{aligned}$$

Equating the two expressions for T_B gives

$$\begin{aligned} 4 - 3\cos\theta &= 5\cos\theta \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3} \end{aligned}$$

that is, $\theta = 60°$. Then

$$\begin{aligned} v_B^2 &= 4ag - 2ag \times \tfrac{1}{2} = 3ag \\ v_B &= \sqrt{3ag} \end{aligned}$$



7. The diagram above shows a small bead P of mass m threaded on a smooth thin wire. The wire is in the form of a circle of radius a and centre O, and it is fixed in a vertical plane. The points A and B are the lowest and highest points of the wire respectively.
The bead is at A when it is projected along the wire with speed $\sqrt{5ga}$. While P moves from A to B, the angle AOP is denoted by θ.

(a) Find, in terms of m, g and θ, the magnitude of the reaction of the wire on the bead. Hence find the value of θ when this magnitude is $\frac{3}{2}mg$. **[8]**

(b) Find, in terms of g, a and θ, the angular velocity and the angular acceleration of P. **[4]**

(c) Hence find, in terms of g, the magnitude of the acceleration of P when θ takes the value found in part (a). Give your answer in exact form. **[4]**

Solution

(a) Let state 0 be the bead at A, and state 1 the bead at P, where its speed is v. The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

Take the initial position as the zero level for gravitational potential energy. At P the bead is a height $a - a\cos\theta$ above A. The wire is smooth, so its reaction is perpendicular to the motion. No other force does work, so $\sum W_{\text{other}} = 0$, and

$$\frac{1}{2}m(5ga) + 0 + 0 = \frac{1}{2}mv^2 + mga(1 - \cos\theta)$$

so

$$\begin{aligned} v^2 &= 5ga - 2ga(1 - \cos\theta) \\ &= ga(3 + 2\cos\theta) \end{aligned}$$

Let R be the reaction of the wire on the bead, taken towards O. The weight has a component $mg\cos\theta$

directed away from O. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{a}$:

$$\begin{aligned} R - mg\cos\theta &= \frac{mv^2}{a} \\ R &= mg\cos\theta + mg(3 + 2\cos\theta) \\ &= 3mg(1 + \cos\theta) \end{aligned}$$

Since $0 \leq \theta \leq \pi$, $1 + \cos\theta \geq 0$, so $R \geq 0$ and the magnitude of the reaction is $3mg(1 + \cos\theta)$.

When the magnitude is $\frac{3}{2}mg$:

$$\begin{aligned} 3mg(1 + \cos\theta) &= \frac{3}{2}mg \\ 1 + \cos\theta &= \frac{1}{2} \\ \cos\theta &= -\frac{1}{2} \end{aligned}$$

For $0 \leq \theta \leq \pi$, this gives

$$\theta = \frac{2\pi}{3}$$

(b) The speed of P is $v = a\omega$, where ω is the angular velocity, so, using part (a):

$$\omega^2 = \frac{v^2}{a^2} = \frac{g(3 + 2\cos\theta)}{a}$$

The bead moves in the direction of increasing θ, so the angular velocity is

$$\omega = \sqrt{\frac{g(3 + 2\cos\theta)}{a}}$$

Take the direction of increasing θ as positive. Along the tangent, the reaction has no component and the weight has component $-mg\sin\theta$. Using Newton's second law, with acceleration $a\frac{\mathrm{d}\omega}{\mathrm{d}t}$:

$$\begin{aligned} -mg\sin\theta &= ma\frac{\mathrm{d}\omega}{\mathrm{d}t} \\ \frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{g\sin\theta}{a} \end{aligned}$$

The angular acceleration is $-\frac{g\sin\theta}{a}$, that is, of magnitude $\frac{g\sin\theta}{a}$ in the direction of decreasing θ.

(c) Using part (b) with $\theta = \frac{2\pi}{3}$, the component of the acceleration towards O is

$$\begin{aligned} a\omega^2 &= g\left(3 + 2\cos\frac{2\pi}{3}\right) \\ &= g(3 - 1) \\ &= 2g \end{aligned}$$

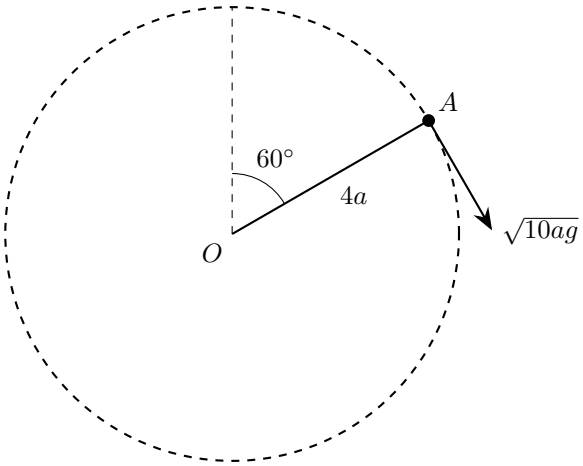
and the component along the tangent is

$$\begin{aligned} a\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -g\sin\frac{2\pi}{3} \\ &= -\frac{\sqrt{3}}{2}g \end{aligned}$$

These two components are perpendicular, so the magnitude of the acceleration is

$$\sqrt{(2g)^2 + \left(\frac{\sqrt{3}}{2}g\right)^2} = g\sqrt{4 + \frac{3}{4}}$$

$$= \frac{\sqrt{19}}{2}g$$



8. A particle of mass m is attached to one end of a light inextensible string of length $4a$. The other end of the string is attached to a fixed point O. The particle is held at the point A, above the level of O, with the string taut and OA making an angle of $60°$ with the upward vertical.
The particle is then projected downwards with speed $\sqrt{10ag}$, perpendicular to OA, in the vertical plane containing OA, as shown in the diagram above.
In an initial model of the motion of the particle, it is assumed that the string does not break.

(a) Using this model, show that the particle performs complete vertical circles. **[8]**

In a refined model it is assumed that the string will break if the tension in it exceeds $5mg$.

(b) Using this refined model, show that the string breaks before the particle first reaches its lowest point, and find the angle between the string and the downward vertical, and the speed of the particle, at the instant the string breaks. **[7]**

Solution

(a) Let P be the position of the particle when the string makes an angle θ with the upward vertical, with θ measured from the upward vertical in the direction of motion, so that $\theta = 60°$ at A and $\theta = 180°$ at the lowest point. Then P is a height $4a\cos\theta$ above O, and A is a height $4a\cos 60° = 2a$ above O. Let the speed at P be v.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is the particle at A and state 1 is the particle at P. Take the initial position as the zero level for gravitational potential energy. The tension is perpendicular to the direction of motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Then

$$\frac{1}{2}m(10ag) + 0 + 0 = \frac{1}{2}mv^2 + mg(4a\cos\theta - 2a)$$

$$10ag = v^2 + 8ag\cos\theta - 4ag$$

$$v^2 = 14ag - 8ag\cos\theta$$

Take the direction towards O as positive. The component of the weight towards O is $mg\cos\theta$. Using Newton's second law, with acceleration $\frac{v^2}{4a}$:

$$T + mg\cos\theta = \frac{m(14ag - 8ag\cos\theta)}{4a}$$

$$= \frac{7mg}{2} - 2mg\cos\theta$$

$$T = \frac{7mg}{2} - 3mg\cos\theta$$

Since $\cos\theta \leq 1$ for every position of the particle, the tension satisfies

$$T \geq \frac{7mg}{2} - 3mg = \frac{mg}{2} > 0$$

so the tension is least at the highest point, where it is still positive, and the string never becomes slack. Also $v^2 = 14ag - 8ag\cos\theta \geq 6ag > 0$, so the particle never comes to rest. The string therefore stays taut throughout the motion and the particle performs complete vertical circles, as required.

(b) From part (a), the tension when the string makes an angle θ with the upward vertical is

$$T = \frac{7mg}{2} - 3mg\cos\theta$$

As the particle moves from A to the lowest point, θ increases from $60°$ to $180°$, so $\cos\theta$ decreases and T increases. At A, $T = \frac{7mg}{2} - \frac{3mg}{2} = 2mg < 5mg$. At the lowest point, $\cos\theta = -1$, so

$$T = \frac{7mg}{2} + 3mg = \frac{13mg}{2} > 5mg$$

So the tension reaches $5mg$ at some point between A and the lowest point, and the string breaks there, before the particle first reaches its lowest point. At that instant:

$$\frac{7mg}{2} - 3mg\cos\theta = 5mg \quad \Longrightarrow \quad \cos\theta = -\frac{1}{2} \quad \Longrightarrow \quad \theta = 120°$$

The angle between the string and the downward vertical is $180° - 120° = 60°$.

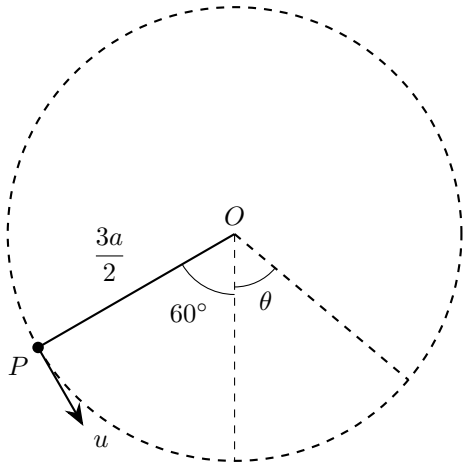
Using part (a), the speed at this instant is given by

$$v^2 = 14ag - 8ag\left(-\frac{1}{2}\right)$$

$$= 18ag$$

$$v = 3\sqrt{2ag}$$

At the instant the string breaks, the string makes an angle of $60°$ with the downward vertical and the speed of the particle is $3\sqrt{2ag}$.



9. A small ball P of mass m is attached to one end of a string of length $\frac{3a}{2}$. The other end of the string is attached to a fixed point O.
The ball is held with the string taut and OP at an angle of $60°$ to the downward vertical through O. It is then projected with speed u perpendicular to OP, so that it moves in the vertical plane containing O and initially moves downwards, as shown in the diagram above.
When the string is taut and OP makes an angle θ with the downward vertical through O, the tension in the string is T.
The ball is modelled as a particle and the string as being light and inextensible.

(a) Show that

$$T = 3mg\cos\theta - mg + \frac{2mu^2}{3a}$$ **[6]**

(b) Given that P moves in a complete vertical circle with centre O, find, in terms of a and g, the minimum possible value of u. **[2]**

(c) Given instead that $u = 2\sqrt{ag}$, find, in terms of a and g, the speed of P at the instant when the string becomes slack. **[4]**

(d) State how the modelling assumption that the string is inextensible has been used in your working. **[1]**

Solution

(a) Let the speed of P be v when OP makes an angle θ with the downward vertical. At that instant P is a distance $\frac{3a}{2}\cos\theta$ below O, and initially it is $\frac{3a}{2}\cos 60° = \frac{3a}{4}$ below O.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is the point of projection and state 1 is the position where OP makes an angle θ with the downward vertical. Take the initial position as the zero level for gravitational potential energy. In state 1, P is then at a height $\frac{3a}{4} - \frac{3a}{2}\cos\theta$ above this level (negative when P is below it). The

tension is always perpendicular to the direction of motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Hence

$$\tfrac{1}{2}mu^2 + 0 + 0 = \tfrac{1}{2}mv^2 + mg\left(\frac{3a}{4} - \frac{3a}{2}\cos\theta\right)$$

$$u^2 = v^2 + \frac{3ag}{2} - 3ag\cos\theta$$

$$v^2 = u^2 - \frac{3ag}{2} + 3ag\cos\theta$$

Take the direction from P towards O as positive. The component of the weight in this direction is $-mg\cos\theta$. Using Newton's second law, with acceleration $\frac{v^2}{r}$, where $r = \frac{3a}{2}$:

$$T - mg\cos\theta = \frac{mv^2}{\frac{3a}{2}}$$

$$T = mg\cos\theta + \frac{2mv^2}{3a}$$

Substituting the expression for v^2:

$$T = mg\cos\theta + \frac{2m}{3a}\left(u^2 - \frac{3ag}{2} + 3ag\cos\theta\right)$$

$$= mg\cos\theta + \frac{2mu^2}{3a} - mg + 2mg\cos\theta$$

$$= 3mg\cos\theta - mg + \frac{2mu^2}{3a}$$

as required.

(b) For a complete circle the string must stay taut, so $T \geq 0$ for all θ. From part (a), T is least when $\cos\theta = -1$, at the top of the circle, so the condition is

$$-3mg - mg + \frac{2mu^2}{3a} \geq 0$$

$$u^2 \geq 6ag$$

The minimum possible value of u is $\sqrt{6ag}$.

(c) With $u = 2\sqrt{ag}$, $u^2 = 4ag$. The string becomes slack when $T = 0$. Using part (a):

$$3mg\cos\theta - mg + \frac{2m(4ag)}{3a} = 0$$

$$3g\cos\theta + \frac{5g}{3} = 0$$

$$\cos\theta = -\frac{5}{9}$$

Substituting into the expression for v^2 from part (a):

$$\begin{aligned} v^2 &= 4ag - \frac{3ag}{2} + 3ag\left(-\frac{5}{9}\right) \\ &= 4ag - \frac{3ag}{2} - \frac{5ag}{3} \\ &= \frac{5ag}{6} \\ v &= \sqrt{\frac{5ag}{6}} \end{aligned}$$

(d) Since the string is inextensible, its length stays $\frac{3a}{2}$ while it is taut, so P moves on a circle of constant radius $\frac{3a}{2}$: this radius was used for the heights in the energy equation and in the acceleration $\frac{v^2}{r}$.

10. A particle P of mass m is attached to one end of a light rod of length L. The other end of the rod is attached to a fixed point O so that the rod is free to rotate in a vertical plane about O.
The particle is hanging at rest vertically below O when it is projected horizontally with speed u. The particle first comes to instantaneous rest at the point A.

(a) Explain why the acceleration of P at A is perpendicular to OA. **[1]**

It is given that $u^2 = 3gL$.

(b) (i) Find the magnitude of the acceleration of P at A, giving your answer as an exact multiple of g. **[4]**

(ii) Find the magnitude of the force exerted by the rod on P at A, stating whether the rod is in tension or in thrust. **[3]**

(c) Find, in terms of m and g, the tension in the rod at the instant when OP first makes an angle of $60°$ with the downward vertical. **[6]**

Solution

(a) P moves in a circle of radius L about O, so the component of its acceleration along PO, towards O, is $\frac{v^2}{L}$, where v is its speed. At A the particle is instantaneously at rest, so $v = 0$ and this component is zero. The acceleration therefore has no component along OA, so it is perpendicular to OA.

(b) (i) Let θ be the angle between OP and the downward vertical, and let v be the speed of P. When OP makes angle θ with the downward vertical, P is a height $L - L\cos\theta = L(1 - \cos\theta)$ above its lowest point.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is the lowest point and state 1 is the position at angle θ. Take the initial position as the zero level for gravitational potential energy. The force from the rod acts along the rod, perpendicular to the motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Hence

$$\tfrac{1}{2}mu^2 + 0 + 0 = \tfrac{1}{2}mv^2 + mgL(1 - \cos\theta)$$

With $u^2 = 3gL$, this gives

$$v^2 = 3gL - 2gL(1 - \cos\theta) = gL(1 + 2\cos\theta)$$

At A, $v = 0$:

$$1 + 2\cos\theta = 0 \quad \Longrightarrow \quad \cos\theta = -\frac{1}{2} \quad \Longrightarrow \quad \theta = 120°$$

So OA makes an angle of $120°$ with the downward vertical, that is, $60°$ with the upward vertical.

By part (a), the acceleration at A is perpendicular to OA, along the tangent to the circle. The force from the rod acts along OA, so only the weight has a component along the tangent. The angle between the weight and OA is $60°$, so the component of the weight along the tangent is $mg\sin 60°$. Take the direction along the tangent in which θ decreases as positive, and let α be the acceleration of P in that direction. Using Newton's second law:

$$m\alpha = mg\sin 60° \quad \Longrightarrow \quad \alpha = \frac{\sqrt{3}}{2}g$$

The magnitude of the acceleration of P at A is $\frac{\sqrt{3}}{2}g$.

(ii) Let T be the tension in the rod, so the rod exerts a force T on P directed from A towards O. Since A is above the level of O with OA at $60°$ to the upward vertical, the component of the weight in the direction from A towards O is $mg\cos 60°$.

Take the direction from P towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{L}$:

$$T + mg\cos 60° = \frac{mv^2}{L}$$

At A, $v = 0$, so

$$T + \frac{mg}{2} = 0 \quad \Longrightarrow \quad T = -\frac{mg}{2}$$

The tension is negative, so the rod is in thrust, and the magnitude of the force exerted by the rod on P at A is $\frac{mg}{2}$.

(c) Using the work–energy principle as in part (b)(i), with state 1 now the position where $\theta = 60°$:

$$v^2 = gL(1 + 2\cos 60°) = gL(1 + 1) = 2gL$$

Here P is below the level of O, and the component of the weight along PO acts away from O, with magnitude $mg\cos 60°$.

Take the direction from P towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{L}$:

$$T - mg\cos 60° = \frac{mv^2}{L}$$

$$T - \frac{mg}{2} = \frac{m \times 2gL}{L} = 2mg$$

$$T = \frac{5mg}{2}$$

11. A light inextensible string of length a has one end attached to a fixed point O. The other end of the string is attached to a small stone of mass m.
The stone is held with the string taut and horizontal. The stone is then projected vertically upwards with speed $2\sqrt{ag}$.
The stone is modelled as a particle and air resistance is modelled as being negligible.

(a) Assuming that the string does not break, use the model to find, in terms of m and g, the tension in the string when the stone is at the highest point of the circle. **[6]**

The string will break if the tension in it reaches $6mg$.

(b) Use the model to find, in degrees, the total angle through which the string has turned, from the instant the stone is projected to the instant the string breaks. **[8]**

(c) Use the model to find the magnitude of the acceleration of the stone at the instant just before the string breaks, giving your answer as a multiple of g, and the angle between the direction of this acceleration and the string. **[7]**

Solution

(a) Let the speed at the highest point be v. The highest point of the circle is a distance a above O, so it is a above the point of projection.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is the instant of projection and state 1 is the highest point. Take the initial position as the zero level for gravitational potential energy. The tension is always perpendicular to the motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Substituting:

$$\tfrac{1}{2}m(2\sqrt{ag})^2 + 0 + 0 = \tfrac{1}{2}mv^2 + mga$$

$$2mag = \tfrac{1}{2}mv^2 + mga$$

$$v^2 = 2ag$$

At the highest point, both the tension and the weight act vertically downwards, towards O. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{a}$:

$$T + mg = \frac{mv^2}{a}$$

$$T + mg = \frac{m(2ag)}{a} = 2mg$$

$$T = mg$$

The tension at the highest point is mg.

(b) Let θ be the angle through which the string has turned since projection, so that the stone is at height $a\sin\theta$ above O, and let its speed be v and the tension be T.

Using the work–energy principle again, with state 1 now the position where the string has turned through θ, and the same zero level:

$$\tfrac{1}{2}m(4ag) = \tfrac{1}{2}mv^2 + mga\sin\theta$$

$$v^2 = 4ag - 2ag\sin\theta$$

The component of the weight towards O is $mg\sin\theta$. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{a}$:

$$T + mg\sin\theta = \frac{mv^2}{a}$$

$$T + mg\sin\theta = 4mg - 2mg\sin\theta$$

$$T = 4mg - 3mg\sin\theta$$

For $0° \le \theta \le 180°$, $\sin\theta \ge 0$, so $T \le 4mg$ and the string cannot break while the stone is above O. The string breaks when $T = 6mg$:

$$4mg - 3mg\sin\theta = 6mg$$

$$\sin\theta = -\tfrac{2}{3}$$

The first value of θ greater than $180°$ with $\sin\theta = -\frac{2}{3}$ is

$$\theta = 180° + \sin^{-1}\tfrac{2}{3} \approx 221.8°$$

(c) Just before the string breaks, the string is at angle φ below the horizontal, where $\varphi = \theta - 180°$, so

$$\sin\varphi = \tfrac{2}{3}, \qquad \cos\varphi = \sqrt{1 - \tfrac{4}{9}} = \tfrac{\sqrt{5}}{3}$$

From part (b), the speed then satisfies

$$v^2 = 4ag - 2ag\left(-\tfrac{2}{3}\right) = \tfrac{16}{3}ag$$

so the component of the acceleration along the string, towards O, is

$$\frac{v^2}{a} = \frac{16g}{3}$$

Perpendicular to the string, the tension has no component, and the component of the weight is $mg\cos\varphi$, in the direction of motion since the stone is moving downwards. Take the direction of motion as positive, and let α be the component of the acceleration in this direction. Using Newton's second law:

$$m\alpha = mg\cos\varphi$$

$$\alpha = \frac{\sqrt{5}}{3}g$$

The two components are perpendicular, so the magnitude of the acceleration is

$$\sqrt{\left(\frac{16g}{3}\right)^2 + \left(\frac{\sqrt{5}\,g}{3}\right)^2} = \frac{g}{3}\sqrt{256+5}$$

$$= \frac{g}{3}\sqrt{261}$$

$$= \sqrt{29}\,g \approx 5.39g$$

Let β be the angle between the acceleration and the string. Then

$$\tan\beta = \frac{\frac{\sqrt{5}}{3}g}{\frac{16}{3}g} = \frac{\sqrt{5}}{16}$$

$$\beta \approx 8.0^\circ$$

The acceleration has magnitude $\sqrt{29}\,g \approx 5.39g$ and makes an angle of 8.0° with the string.

12. A light inextensible rope, of length 2 m, has one end attached to a fixed point A on the horizontal ceiling of a gym, 5 m above the horizontal ground. The point O on the ground is vertically below A. A child of mass 30 kg holds the other end of the rope, and the child may be modelled as a particle attached to the end of the rope.
Initially the rope is taut and makes an angle α with the downward vertical, where $\tan\alpha = \frac{3}{4}$. The child is given a speed of $2.8\,\mathrm{m\,s^{-1}}$ perpendicular to the rope, moving downwards towards the vertical through A.

(a) Show that, while the rope is taut, the speed $v\,\mathrm{m\,s^{-1}}$ of the child when the rope makes an angle θ with the downward vertical satisfies $v^2 = 7.84 + 4g(\cos\theta - 0.8)$. Hence find the greatest angle that the rope makes with the downward vertical in the subsequent motion. **[6]**

(b) On the other side of the vertical through A, at the instant when the rope again makes an angle α with the downward vertical, the child lets go of the rope and then moves freely under gravity. Find the distance from O to the point where the child lands. **[5]**

(c) The child returns to the initial position and is again given a speed of $2.8\,\mathrm{m\,s^{-1}}$ perpendicular to the rope, moving downwards towards the vertical through A. This time the child holds on to the rope until first coming to instantaneous rest. The rope breaks if the tension in it exceeds 600 N. Find the greatest and the least tension in the rope during this motion, and determine whether the rope breaks. **[6]**

Solution

(a) Since $\tan\alpha = \frac{3}{4}$, $\cos\alpha = \frac{4}{5} = 0.8$ and $\sin\alpha = \frac{3}{5} = 0.6$. When the rope makes an angle θ with the downward vertical, the child is $2\cos\theta$ m below A, so initially the child is $2 \times 0.8 = 1.6$ m below A.

Let state 0 be the initial position and state 1 the position where the rope makes an angle θ with the downward vertical. The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

The tension is perpendicular to the child's motion, so it does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Take the initial position as the zero level for gravitational potential energy. In state 1 the child is $(2\cos\theta - 1.6)$ m below this level, so

$$\tfrac{1}{2} \times 30 \times 2.8^2 + 0 + 0 = \tfrac{1}{2} \times 30v^2 - 30g(2\cos\theta - 1.6)$$

$$117.6 = 15v^2 - 60g\cos\theta + 48g$$

$$v^2 = 7.84 + 4g\cos\theta - 3.2g$$

$$v^2 = 7.84 + 4g(\cos\theta - 0.8)$$

as required.

While the child is below the level of A, the rope stays taut. The child comes to instantaneous rest when $v = 0$:

$$0 = 7.84 + 4 \times 9.8(\cos\theta - 0.8)$$

$$\cos\theta - 0.8 = -\frac{7.84}{39.2} = -0.2$$

$$\cos\theta = 0.6$$

The child then swings back, so the greatest angle is

$$\theta = \cos^{-1} 0.6 \approx 53.1^\circ$$

(b) On the other side, when the rope makes angle α with the downward vertical, $\cos\theta = 0.8$, so from part (a) $v^2 = 7.84$ and $v = 2.8$.

The velocity is perpendicular to the rope, so it makes angle α with the horizontal, directed upwards and away from the vertical through A. Its components are

$$\text{horizontal: } 2.8\cos\alpha = 2.8 \times 0.8 = 2.24\,\text{m s}^{-1}$$

$$\text{vertical (upwards): } 2.8\sin\alpha = 2.8 \times 0.6 = 1.68\,\text{m s}^{-1}$$

At release the child is 1.6 m below A, so $5 - 1.6 = 3.4$ m above the ground, and $2\sin\alpha = 1.2$ m horizontally from the vertical through A, that is, from O.

Take upwards as positive. Vertically, the child lands when its displacement is -3.4 m. Using

$$s = ut + \tfrac{1}{2}at^2$$

with $u = 1.68$ and $a = -9.8$:

$$-3.4 = 1.68t - 4.9t^2$$

$$4.9t^2 - 1.68t - 3.4 = 0$$

Taking the positive root:

$$t = \frac{1.68 + \sqrt{1.68^2 + 4 \times 4.9 \times 3.4}}{9.8} \approx 1.0219\,\text{s}$$

The horizontal velocity is constant, so the horizontal distance moved after release is $2.24t \approx 2.2890$ m. The child moves away from the vertical through A, so the distance from O to the landing point is

$$1.2 + 2.2890 \approx 3.49\,\text{m}$$

(c) The motion is the same as in part (a), so $v^2 = 7.84 + 4g(\cos\theta - 0.8)$ holds throughout. Take the direction towards A as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$, where $r = 2$:

$$T - 30g\cos\theta = \frac{30v^2}{2}$$

$$T = 30g\cos\theta + 15(7.84 + 4g\cos\theta - 3.2g)$$

$$T = 90g\cos\theta + 117.6 - 48g$$

$$T = 882\cos\theta - 352.8$$

The rope swings from angle α on one side, through the vertical, to the point of instantaneous rest found in part (a), where $\cos\theta = 0.6$. So during this motion $0.6 \le \cos\theta \le 1$, and T increases as $\cos\theta$ increases.

The greatest tension is when the rope is vertical, $\cos\theta = 1$:

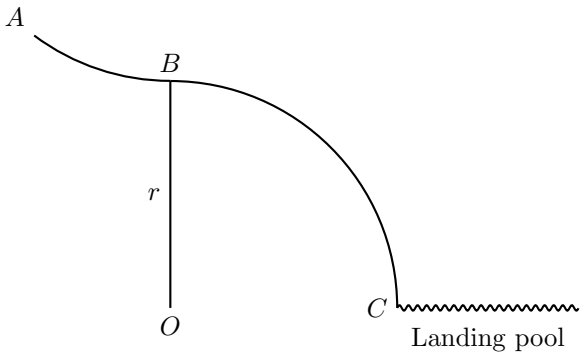
$$T_{\max} = 882 - 352.8 = 529.2 \approx 529\,\text{N}$$

The least tension is at instantaneous rest, $\cos\theta = 0.6$:

$$T_{\min} = 882 \times 0.6 - 352.8 = 176.4 \approx 176\,\text{N}$$

Since $529.2 < 600$, the tension never exceeds 600 N, so the rope does not break.

Final answers with $g = 10$: (a) $\cos^{-1} 0.604 \approx 52.8°$; (b) 3.46 m; (c) greatest 538 N, least 181 N, the rope does not break



13. The diagram above shows part of a water park slide, ABC, in the shape of two circular arcs, AB and BC, each of radius r.
The point A is at a height of $\frac{r}{5}$ above B. The arc BC has centre O, and B is vertically above O. The two arcs meet at B, where the slide is horizontal.
A child is pushed from A down the slide towards B with speed $\sqrt{\frac{gr}{5}}$, where g is the acceleration due to gravity. The child moves past B and then loses contact with the slide at a point D on the arc BC, where angle BOD is θ.
The slide is modelled as smooth, and the child is modelled as a particle.

(a) Show that $\cos\theta = \frac{13}{15}$. **[6]**

(b) The child lands in a pool whose surface is level with C. Find, in terms of g and r, the speed of the child as it reaches the surface of the pool. **[2]**

(c) A refined model takes air resistance into account. State, with a reason, whether the child would then lose contact with the slide above or below the point D. **[2]**

(d) Explain why, according to the model, the child must lose contact with the slide before reaching C, whatever its speed at A. **[1]**

Solution

(a) Let the child have mass m and speed v at D. The point D is $r - r\cos\theta$ below B, so D is $\frac{6r}{5} - r\cos\theta$ below A.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

with state 0 the child at A and state 1 the child at D. Take the final position as the zero level for gravitational potential energy. The slide is smooth, so the normal reaction is perpendicular to the motion and does no work. No other force does work, so $\sum W_{\text{other}} = 0$. Substituting:

$$\tfrac{1}{2}m\left(\frac{gr}{5}\right) + mg\left(\frac{6r}{5} - r\cos\theta\right) = \tfrac{1}{2}mv^2$$

$$v^2 = \frac{gr}{5} + \frac{12gr}{5} - 2gr\cos\theta$$

$$v^2 = \frac{13gr}{5} - 2gr\cos\theta$$

Let the normal reaction of the slide on the child be R. At D, the component of the weight towards O

is $mg\cos\theta$. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$:

$$mg\cos\theta - R = \frac{mv^2}{r}$$

The child loses contact at D, so $R = 0$ there, which gives

$$gr\cos\theta = v^2$$

$$gr\cos\theta = \frac{13gr}{5} - 2gr\cos\theta$$

$$3\cos\theta = \frac{13}{5}$$

$$\cos\theta = \frac{13}{15}$$

as required.

(b) Let the speed of the child as it reaches the surface of the pool be V. After leaving the slide, only gravity acts on the child. The pool surface is level with C, which is r below B, so it is $\frac{r}{5} + r = \frac{6r}{5}$ below A.

Using the work–energy principle from A to the pool surface, taking the pool surface as the zero level for gravitational potential energy, with $\sum W_{\text{other}} = 0$:

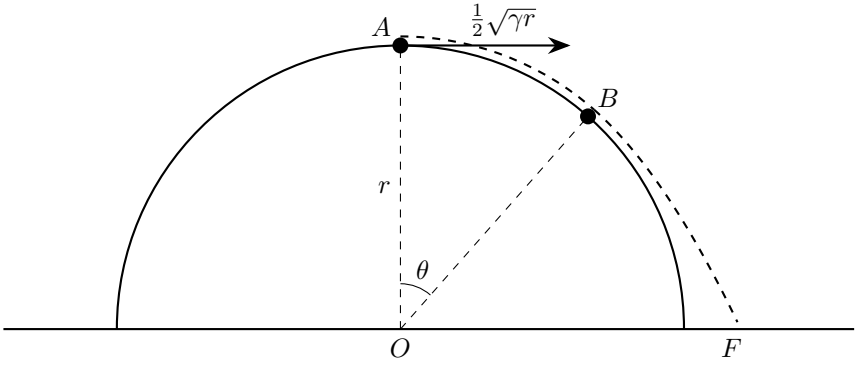
$$\tfrac{1}{2}m\left(\frac{gr}{5}\right) + mg\left(\frac{6r}{5}\right) = \tfrac{1}{2}mV^2$$

$$V^2 = \frac{gr}{5} + \frac{12gr}{5} = \frac{13gr}{5}$$

$$V = \sqrt{\frac{13gr}{5}}$$

(c) Air resistance does negative work on the child, so at every point of the arc BC the child's speed is less than in the original model. Since $R = mg\cos\theta - \frac{mv^2}{r}$, a smaller speed means that R is still positive when the child reaches D: there $v^2 < gr\cos\theta$. The child therefore stays on the slide beyond D, and loses contact **below** the point D.

(d) At a point P of the arc BC with angle $BOP = \alpha$, the radial equation of part (a) gives $R = mg\cos\alpha - \frac{mv^2}{r}$. At C, $\alpha = 90°$, so $\cos\alpha = 0$ and the weight has no component towards O. Then $R = -\frac{mv^2}{r} < 0$ for any speed $v > 0$, which is impossible, so the child must lose contact before reaching C, whatever its speed at A.



14. The flat surface of a smooth solid hemisphere of radius r is fixed to a horizontal plane on a planet where the acceleration due to gravity is γ. The point O is the centre of the flat surface of the hemisphere, and A is the highest point of its curved surface.
A particle P is placed at A and projected horizontally with speed $\frac{1}{2}\sqrt{\gamma r}$, so that it begins to move on the curved surface. P leaves the surface at the point B, where OB makes an angle θ with the upward vertical through O, and then first hits the plane at the point F, as shown in the diagram above.

(a) Show that $\cos\theta = \frac{3}{4}$. **[6]**

(b) Find an exact expression for the distance OF. **[6]**

(c) Explain why, if P were instead projected horizontally from A with a speed greater than $\sqrt{\gamma r}$, it would leave the surface immediately. **[1]**

Solution

(a) Let the mass of P be m and its speed at B be v. The surface is smooth, so the only force on P other than its weight is the normal reaction, which is perpendicular to the motion.

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is P at A and state 1 is P at B. Take the final position as the zero level for gravitational potential energy. A is a height r above O and B is a height $r\cos\theta$ above O, so A is a height $r(1-\cos\theta)$ above B. No other force does work, so $\sum W_{\text{other}} = 0$. Substituting:

$$\frac{1}{2}m\left(\frac{1}{2}\sqrt{\gamma r}\right)^2 + m\gamma r(1-\cos\theta) + 0 = \frac{1}{2}mv^2$$

$$v^2 = \frac{\gamma r}{4} + 2\gamma r(1-\cos\theta)$$

Let the normal reaction at B be R. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$:

$$m\gamma\cos\theta - R = \frac{mv^2}{r}$$

P leaves the surface at B, so $R = 0$ there, which gives

$$v^2 = \gamma r\cos\theta$$

Equating the two expressions for v^2 and dividing by γr:

$$\cos\theta = \frac{1}{4} + 2 - 2\cos\theta$$

$$3\cos\theta = \frac{9}{4}$$

$$\cos\theta = \frac{3}{4}$$

as required.

(b) From part (a), $\cos\theta = \frac{3}{4}$, so $\sin\theta = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$, and the speed at B is given by

$$v^2 = \frac{3}{4}\gamma r \quad \Longrightarrow \quad v = \frac{\sqrt{3}}{2}\sqrt{\gamma r}$$

At B, P is a height $r\cos\theta = \frac{3r}{4}$ above the plane and a horizontal distance $r\sin\theta = \frac{\sqrt{7}}{4}r$ from O. Its velocity is along the tangent at B, at an angle θ below the horizontal, so its components are

$$\text{horizontal:} \quad v\cos\theta = \frac{\sqrt{3}}{2}\sqrt{\gamma r} \times \frac{3}{4} = \frac{3\sqrt{3}}{8}\sqrt{\gamma r}$$

$$\text{vertical (downwards):} \quad v\sin\theta = \frac{\sqrt{3}}{2}\sqrt{\gamma r} \times \frac{\sqrt{7}}{4} = \frac{\sqrt{21}}{8}\sqrt{\gamma r}$$

After leaving the surface, P moves freely under gravity. Let t be the time taken from B to F, and take downwards as positive. Vertically, using

$$s = ut + \frac{1}{2}\alpha t^2$$

with $s = \frac{3r}{4}$, $u = \frac{\sqrt{21}}{8}\sqrt{\gamma r}$ and $\alpha = \gamma$:

$$\frac{3r}{4} = \frac{\sqrt{21}}{8}\sqrt{\gamma r}\, t + \frac{1}{2}\gamma t^2$$

$$4\gamma t^2 + \sqrt{21\gamma r}\, t - 6r = 0$$

The discriminant is $b^2 - 4ac = 21\gamma r - 4 \times 4\gamma \times (-6r) = 117\gamma r$. Since $t > 0$, the quadratic formula gives

$$t = \frac{-\sqrt{21\gamma r} + \sqrt{117\gamma r}}{8\gamma} = \frac{3\sqrt{13} - \sqrt{21}}{8}\sqrt{\frac{r}{\gamma}}$$

The horizontal component of velocity is constant, so the horizontal distance travelled from B to F is

$$\frac{3\sqrt{3}}{8}\sqrt{\gamma r} \times \frac{3\sqrt{13} - \sqrt{21}}{8}\sqrt{\frac{r}{\gamma}} = \frac{3\sqrt{3}\left(3\sqrt{13} - \sqrt{21}\right)}{64}r$$

$$= \frac{9\sqrt{39} - 3\sqrt{63}}{64}r$$

$$= \frac{9\sqrt{39} - 9\sqrt{7}}{64}r$$

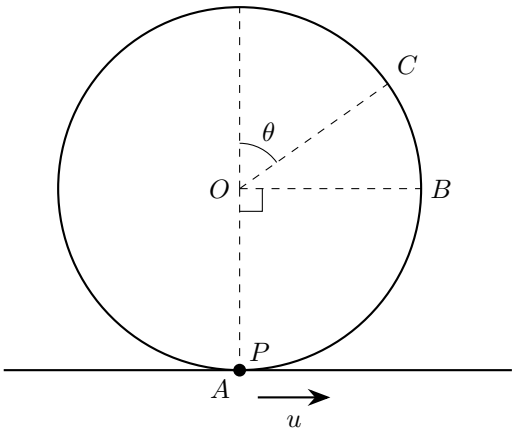
Adding the horizontal distance of B from O:

$$\begin{aligned} OF &= \frac{\sqrt{7}}{4}r + \frac{9\sqrt{39} - 9\sqrt{7}}{64}r \\ &= \frac{16\sqrt{7} + 9\sqrt{39} - 9\sqrt{7}}{64}r \\ &= \frac{7\sqrt{7} + 9\sqrt{39}}{64}r \end{aligned}$$

(c) Let the speed of projection be U, and the normal reaction at A be R. Take the direction towards O as positive. Using Newton's second law at A, with acceleration $\frac{U^2}{r}$:

$$m\gamma - R = \frac{mU^2}{r} \quad \Longrightarrow \quad R = m\left(\gamma - \frac{U^2}{r}\right)$$

If $U > \sqrt{\gamma r}$, then $\frac{U^2}{r} > \gamma$, so R would be negative. The surface can only push on P, so the weight alone cannot provide the force towards O needed to keep P on the circular path, and P leaves the surface immediately.



15. The diagram above shows a hollow sphere, with centre O and internal radius a, which is fixed to a horizontal surface.
A particle P of mass m is projected horizontally with speed u from the lowest point A of the inner surface of the sphere. The particle moves in a vertical circle with centre O on the smooth inner surface of the sphere, passing through the point B, on the inner surface of the sphere, where OB is horizontal.
The particle leaves the inner surface of the sphere at the point C, where OC makes an angle θ with the upward vertical and $\tan\theta = \sqrt{2}$.

(a) Show that $u^2 = \left(2+\sqrt{3}\right)ag$. **[6]**

(b) Find, in terms of m and g, the magnitude of the normal reaction exerted on P by the surface of the sphere when P is at B. **[6]**

(c) Show that, after leaving the surface of the sphere at C, the particle passes through O. **[7]**

Solution

(a) At C the particle leaves the surface, so the normal reaction there is zero, and the only force on P is its weight, whose component towards O is $mg\cos\theta$. Since $\tan\theta = \sqrt{2}$, a right-angled triangle with opposite side $\sqrt{2}$ and adjacent side 1 has hypotenuse $\sqrt{3}$, so

$$\cos\theta = \frac{1}{\sqrt{3}}, \qquad \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

Let v_C be the speed of P at C. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v_C^2}{a}$:

$$mg\cos\theta = \frac{mv_C^2}{a}$$

$$v_C^2 = ag\cos\theta = \frac{ag}{\sqrt{3}}$$

The work–energy principle is

$$KE_0 + GPE_0 + \sum W_{\text{other}} = KE_1 + GPE_1$$

where state 0 is P at A and state 1 is P at C. Take the initial position as the zero level for gravitational

potential energy. O is a above A and C is $a\cos\theta$ above O, so C is $a\left(1+\frac{1}{\sqrt{3}}\right)$ above A. The normal reaction is perpendicular to the direction of motion. No other force does work, so $\sum W_{\text{other}} = 0$. Then

$$\frac{1}{2}mu^2 + 0 + 0 = \frac{1}{2}mv_C^2 + mga\left(1+\frac{1}{\sqrt{3}}\right)$$

$$\begin{aligned} u^2 &= \frac{ag}{\sqrt{3}} + 2ag + \frac{2ag}{\sqrt{3}} \\ &= 2ag + \frac{3ag}{\sqrt{3}} \\ &= 2ag + \sqrt{3}ag \\ &= \left(2+\sqrt{3}\right)ag \end{aligned}$$

as required.

(b) Let v_B be the speed of P at B. Apply the work–energy principle with state 0 as P at A and state 1 as P at B, which is a above A, taking the initial position as the zero level for gravitational potential energy. As in part (a), no other force does work, so $\sum W_{\text{other}} = 0$. Using part (a):

$$\begin{aligned} \frac{1}{2}mu^2 + 0 + 0 &= \frac{1}{2}mv_B^2 + mga \\ v_B^2 &= u^2 - 2ag \\ &= \left(2+\sqrt{3}\right)ag - 2ag \\ &= \sqrt{3}ag \end{aligned}$$

At B, OB is horizontal, so the normal reaction R acts horizontally towards O, and the weight, acting vertically, has no component towards O. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v_B^2}{a}$:

$$\begin{aligned} R &= \frac{mv_B^2}{a} \\ &= \frac{m \times \sqrt{3}ag}{a} \\ &= \sqrt{3}mg \end{aligned}$$

(c) After leaving the surface at C, the only force on P is its weight, so its horizontal velocity stays constant and its vertical acceleration is g downwards.

At C the velocity of P is along the tangent to the circle, perpendicular to OC. Since OC makes an angle θ with the vertical, the velocity makes an angle θ with the horizontal, directed upwards and towards the vertical line through O. Its components are

$$\text{horizontal: } v_C\cos\theta = \frac{v_C}{\sqrt{3}}, \qquad \text{vertical: } v_C\sin\theta = \frac{\sqrt{2}\,v_C}{\sqrt{3}}$$

The point C is a horizontal distance $a\sin\theta = \frac{\sqrt{2}\,a}{\sqrt{3}}$ from the vertical line through O, and a height

$a\cos\theta = \frac{a}{\sqrt{3}}$ above O. Let t be the time after leaving C at which P reaches the vertical line through O. Then

$$\frac{v_C}{\sqrt{3}}\,t = \frac{\sqrt{2}\,a}{\sqrt{3}}$$

$$t = \frac{\sqrt{2}\,a}{v_C}$$

Take upwards as positive. For the vertical motion, using

$$s = v_0 t + \frac{1}{2}\alpha t^2$$

with $v_0 = \frac{\sqrt{2}\,v_C}{\sqrt{3}}$ and $\alpha = -g$, the vertical displacement y of P from C at this time is

$$\begin{aligned} y &= \frac{\sqrt{2}\,v_C}{\sqrt{3}}\,t - \frac{1}{2}gt^2 \\ &= \frac{\sqrt{2}\,v_C}{\sqrt{3}} \times \frac{\sqrt{2}\,a}{v_C} - \frac{1}{2}g \times \frac{2a^2}{v_C^2} \\ &= \frac{2a}{\sqrt{3}} - \frac{ga^2}{v_C^2} \end{aligned}$$

From part (a), $v_C^2 = \frac{ag}{\sqrt{3}}$, so $\frac{ga^2}{v_C^2} = \sqrt{3}a$ and

$$\begin{aligned} y &= \frac{2a}{\sqrt{3}} - \sqrt{3}a \\ &= \frac{2a - 3a}{\sqrt{3}} \\ &= -\frac{a}{\sqrt{3}} \end{aligned}$$

So when P reaches the vertical line through O, it is $\frac{a}{\sqrt{3}}$ below C, which is the height of O. Hence the particle passes through O, as required.