FURTHER MECHANICS / CIRCULAR MOTION & OSCILLATIONS

Motion in a Horizontal Circle

Worked solutions 15 questions · 146 marks

EXAM BOARDS Edexcel FM2 9FM0/4C · AQA 7367/3M · OCR A Y543 · OCR MEI Mech Major Y421 CIE 9231 Paper 3

Instructions

Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question gives a value. Where a question leaves g unspecified and a final answer depends on it, the solutions also give that answer for $g = 10\,\mathrm{m\,s^{-2}}$.

CIE (9231 Paper 3): papers take $g = 10\,\mathrm{m\,s^{-2}}$; use the $g = 10$ final answers where the solutions give them.

Questions

Question 1	*10 marks*	2
Question 2	*11 marks*	4
Question 3	*9 marks*	6
Question 4	*11 marks*	8
Question 5	*12 marks*	10
Question 6	*8 marks*	12
Question 7	*7 marks*	14
Question 8	*7 marks*	16
Question 9	*7 marks*	18
Question 10	*9 marks*	20
Question 11	*11 marks*	22
Question 12	*12 marks*	24
Question 13	*10 marks*	26
Question 14	*11 marks*	29
Question 15	*11 marks*	31

1. A cyclist rides round a circular track which is banked at a constant angle to the horizontal.
The cyclist and bicycle are modelled as a particle moving at constant speed in a horizontal circle, taking 22 s to complete each lap.
At this speed there is no sideways frictional force on the bicycle, and the magnitude of the normal reaction of the track on the particle is $\frac{25}{24}$ times the weight of the particle.

(a) Find the radius of the circle. **[7]**

(b) Identify one limitation of this model. **[1]**

The cyclist now slows down, so that each lap of the same horizontal circle takes 30 s, again at constant speed. The cyclist and bicycle are still modelled as a particle.

(c) Describe the extra force that will now be acting on the particle, stating its direction and giving a reason for this direction. **[2]**

Solution

(a) Let the track be banked at an angle θ to the horizontal, let the particle have mass m, and let the normal reaction be R, so that $R = \frac{25}{24}mg$. Let the radius of the circle be r m.

Each lap is an angle of 2π radians in 22 s, so the angular speed is

$$\omega = \frac{2\pi}{22} = \frac{\pi}{11}\,\text{rad}\,\text{s}^{-1}$$

There is no friction and no vertical acceleration, so resolving vertically gives

$$R\cos\theta = mg$$

Substituting $R = \frac{25}{24}mg$:

$$\frac{25}{24}mg\cos\theta = mg \quad \Longrightarrow \quad \cos\theta = \frac{24}{25}$$

Since θ is acute, $\sin^2\theta + \cos^2\theta = 1$ gives

$$\sin\theta = \sqrt{1 - \left(\frac{24}{25}\right)^2} = \frac{7}{25}$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $r\omega^2$:

$$R\sin\theta = mr\omega^2$$

Substituting the values of R, $\sin\theta$ and ω:

$$\frac{25}{24}mg \times \frac{7}{25} = mr\left(\frac{\pi}{11}\right)^2$$

$$\frac{7}{24}g = \frac{\pi^2 r}{121}$$

$$r = \frac{7 \times 121 \times g}{24\pi^2}$$

$$r = \frac{847 \times 9.8}{24\pi^2}$$

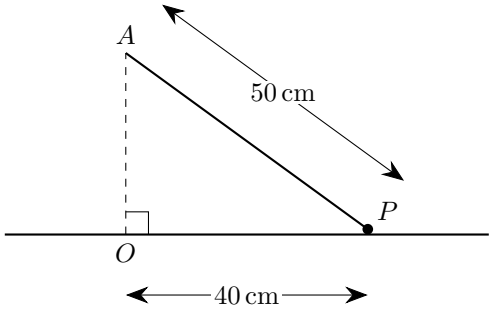
The radius of the circle is approximately 35.0 m.

(b) The cyclist and bicycle have size, so they are not a particle: the weight and the normal reaction do not act at a single point on the track.

(c) The extra force is friction between the tyres and the track, acting sideways up the slope, along the line of greatest slope of the track.

The lap takes longer, so the speed is lower and a smaller force towards the centre is needed than in part (a). With no friction, the normal reaction would supply more force towards the centre than this, so the particle would tend to slide down the slope. Friction opposes this tendency, so it acts up the slope.

Final answers with $g = 10$: (a) 35.8 m



2. A particle P of mass 0.8 kg is attached to one end of a light inextensible string of length 50 cm. The other end of the string is attached to a fixed point A that is vertically above the point O on a smooth horizontal table.
The particle remains in contact with the table, with the string taut, and moves with constant speed in a horizontal circle with centre O and radius 40 cm, as shown in the diagram above.
The tension in the string and the normal reaction between P and the table are equal in magnitude.

(a) (i) Find the tension in the string. **[3]**

(ii) Find the speed of P. **[3]**

The string will break if the tension in it exceeds 15 N. The speed of P is now gradually increased.

(b) Show that P loses contact with the table before the string breaks, and find the speed of P at the instant it loses contact. **[5]**

Solution

By Pythagoras in triangle OAP:

$$OA = \sqrt{50^2 - 40^2} = 30\,\text{cm}$$

Let θ be the angle between the string and the vertical, so $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$. Let the tension in the string be T and the normal reaction be R.

(a) (i) Resolving vertically, with $R = T$:

$$T\cos\theta + R = 0.8g$$

$$\frac{3}{5}T + T = 0.8 \times 9.8$$

$$\frac{8}{5}T = 7.84$$

$$T = 4.9\,\text{N}$$

(ii) Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$, where v is the speed of P and $r = 0.4\,\text{m}$:

$$T\sin\theta = \frac{0.8v^2}{0.4}$$

$$4.9 \times \frac{4}{5} = 2v^2$$

$$v^2 = 1.96$$

$$v = 1.4\,\text{m}\,\text{s}^{-1}$$

(b) Consider P while it is in contact with the table. Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$:

$$T\sin\theta = \frac{0.8v^2}{0.4} \quad \Longrightarrow \quad T = \frac{5}{2}v^2$$

so T increases as v increases. Resolving vertically gives

$$T\cos\theta + R = 0.8g \quad \Longrightarrow \quad R = 0.8g - \frac{3}{5}T$$

so R decreases as T increases, and P loses contact with the table when $R = 0$. At that instant

$$T = \frac{5}{3} \times 0.8 \times 9.8 = \frac{196}{15} \approx 13.1\,\text{N}$$

Since $13.1 < 15$, the tension has not reached the breaking value when $R = 0$, so P loses contact with the table before the string breaks.

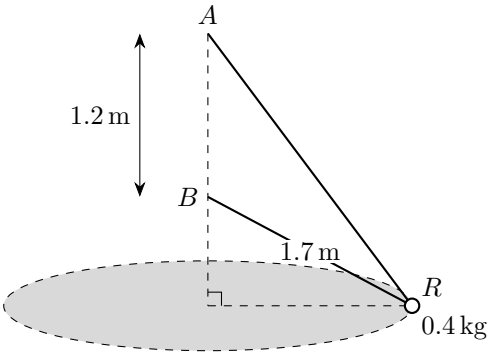
At the instant it loses contact, using $T = \frac{5}{2}v^2$:

$$\frac{5}{2}v^2 = \frac{196}{15}$$

$$v^2 = \frac{392}{75}$$

$$v = \frac{14\sqrt{6}}{15} \approx 2.29\,\text{m s}^{-1}$$

Final answers with $g = 10$: (a)(i) 5 N; (a)(ii) $1.41\,\text{m s}^{-1}$; (b) $\frac{4\sqrt{3}}{3} \approx 2.31\,\text{m s}^{-1}$



3. The diagram above shows a light inextensible string of length 4.2 m passing through a small smooth ring R of mass 0.4 kg. The ends of the string are attached to fixed points A and B, where A is 1.2 m vertically above B.
The ring moves with constant speed in a horizontal circle whose centre lies on the line through A and B, vertically below B. The distance BR is 1.7 m and both sections of the string are taut.

(a) Show that the radius of the circle is 1.5 m. **[2]**

(b) Find the tension in the string. **[3]**

(c) Find the time taken for R to complete one revolution. **[4]**

Solution

(a) Let O be the centre of the circle. Working with lengths in metres, let $BO = h$ and let the radius be r. The string has length 4.2 m, so $AR = 4.2 - 1.7 = 2.5$ m, and $AO = h + 1.2$. Using Pythagoras in triangles BOR and AOR, which are right-angled at O:

$$h^2 + r^2 = 1.7^2$$

$$(h + 1.2)^2 + r^2 = 2.5^2$$

Subtracting the first equation from the second:

$$2.4h + 1.44 = 3.36$$

$$h = 0.8$$

Substituting into the first equation:

$$r^2 = 1.7^2 - 0.8^2 = 2.25 \quad \Longrightarrow \quad r = 1.5$$

So the radius of the circle is 1.5 m, as required.

(b) The ring is smooth, so the tension T is the same in both sections of the string. Let α and β be the angles that AR and BR make with the vertical. From part (a), $AO = 2$ m and $BO = 0.8$ m, so

$$\cos\alpha = \frac{2}{2.5} = \frac{4}{5}, \qquad \cos\beta = \frac{0.8}{1.7} = \frac{8}{17}$$

The ring has no vertical acceleration, so resolving vertically:

$$T\cos\alpha + T\cos\beta = 0.4g$$

$$T\left(\frac{4}{5} + \frac{8}{17}\right) = 0.4g$$

$$\frac{108}{85}T = 0.4g$$

$$T = \frac{17g}{54} \approx 3.09\,\text{N}$$

(c) Let the angular speed of the ring be ω. From the right-angled triangles in part (a):

$$\sin\alpha = \frac{1.5}{2.5} = \frac{3}{5}, \qquad \sin\beta = \frac{1.5}{1.7} = \frac{15}{17}$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $r\omega^2$ and $r = 1.5$:

$$T\sin\alpha + T\sin\beta = 0.4 \times 1.5\,\omega^2$$

$$T\left(\frac{3}{5} + \frac{15}{17}\right) = 0.6\omega^2$$

$$\frac{126}{85}T = 0.6\omega^2$$

Substituting $T = \frac{17g}{54}$ from part (b):

$$\frac{126}{85} \times \frac{17g}{54} = 0.6\omega^2$$

$$\frac{7g}{15} = 0.6\omega^2$$

$$\omega^2 = \frac{7g}{9}$$

The time taken for one revolution is

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{9}{7g}} = \frac{6\pi}{\sqrt{7g}} \approx 2.28\,\text{s}$$

Final answers with $g = 10$: (b) $\frac{85}{27} \approx 3.15\,\text{N}$; (c) $\frac{6\pi}{\sqrt{70}} \approx 2.25\,\text{s}$

4. A particle moves with constant speed on a circular path of radius $2\sqrt{2}\,\text{m}$ in a horizontal plane.
The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are perpendicular and lie in this plane.
The centre of the circle has position vector $(3\mathbf{i}+3\mathbf{j})\,\text{m}$ relative to the origin.
At time $t=0$, the particle is at the point with position vector $(5\mathbf{i}+\mathbf{j})\,\text{m}$ and its velocity is $-\frac{\pi}{2}(\mathbf{i}+\mathbf{j})\,\text{m s}^{-1}$.

(a) Calculate the angular speed of the particle. **[2]**

(b) Find an expression for the position vector of the particle at time t seconds. **[3]**

(c) Find an expression for the acceleration of the particle at time t seconds. **[3]**

(d) State the magnitude of the acceleration of the particle. **[1]**

(e) Find the time when the acceleration of the particle is first directed towards the origin. **[2]**

Solution

(a) The speed is

$$v=\left|-\frac{\pi}{2}(\mathbf{i}+\mathbf{j})\right|=\frac{\pi}{2}\sqrt{1^2+1^2}=\frac{\sqrt{2}\pi}{2}\,\text{m s}^{-1}$$

The speed equals the radius multiplied by the angular speed ω, so

$$2\sqrt{2}\,\omega=\frac{\sqrt{2}\pi}{2}\quad\Longrightarrow\quad\omega=\frac{\pi}{4}\,\text{rad s}^{-1}$$

(b) At $t=0$, the position of the particle relative to the centre is

$$(5\mathbf{i}+\mathbf{j})-(3\mathbf{i}+3\mathbf{j})=2\mathbf{i}-2\mathbf{j}$$

Taking $\mathbf{i}$ to the right and $\mathbf{j}$ upwards, this vector has length $2\sqrt{2}$ and lies at an angle of $\frac{\pi}{4}$ below the $\mathbf{i}$ direction. The velocity at $t=0$ has negative $\mathbf{i}$ and $\mathbf{j}$ components, so the particle moves clockwise.

Let θ be the angle from the $\mathbf{i}$ direction to the radius through the particle, measured clockwise. It starts at $\frac{\pi}{4}$ and increases at the rate $\omega=\frac{\pi}{4}$ from part (a), so

$$\theta=\frac{\pi t}{4}+\frac{\pi}{4}$$

The position relative to the centre is $2\sqrt{2}\cos\theta\,\mathbf{i}-2\sqrt{2}\sin\theta\,\mathbf{j}$. Adding the position vector of the centre gives

$$\mathbf{r}=\left[\left(3+2\sqrt{2}\cos\left(\frac{\pi t}{4}+\frac{\pi}{4}\right)\right)\mathbf{i}+\left(3-2\sqrt{2}\sin\left(\frac{\pi t}{4}+\frac{\pi}{4}\right)\right)\mathbf{j}\right]\text{m}$$

(c) Differentiate the expression from part (b) twice with respect to t, each differentiation giving a factor of $\frac{\pi}{4}$ from the chain rule:

$$\mathbf{v}=\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=-\frac{\sqrt{2}\pi}{2}\sin\left(\frac{\pi t}{4}+\frac{\pi}{4}\right)\mathbf{i}-\frac{\sqrt{2}\pi}{2}\cos\left(\frac{\pi t}{4}+\frac{\pi}{4}\right)\mathbf{j}$$

$$\mathbf{a}=\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=\frac{\sqrt{2}\pi^2}{8}\left[-\cos\left(\frac{\pi t}{4}+\frac{\pi}{4}\right)\mathbf{i}+\sin\left(\frac{\pi t}{4}+\frac{\pi}{4}\right)\mathbf{j}\right]\text{m s}^{-2}$$

(d) Since $\cos^2\theta + \sin^2\theta = 1$, the expression from part (c) gives

$$|\mathbf{a}| = \frac{\sqrt{2}\pi^2}{8}\,\mathrm{m\,s^{-2}} \approx 1.74\,\mathrm{m\,s^{-2}}$$

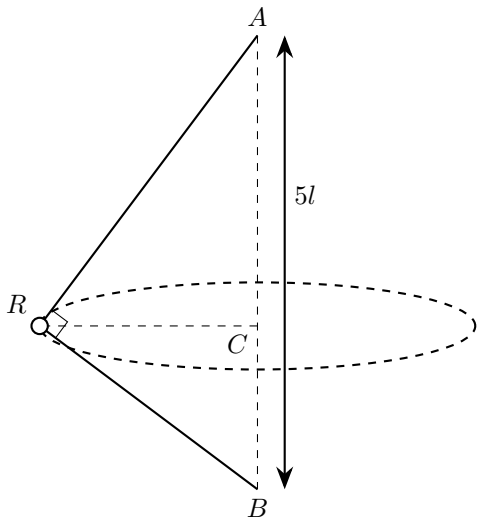
(e) The acceleration always points from the particle towards the centre. It is therefore directed towards the origin when the particle, the centre and the origin lie on one straight line, with the centre and the origin on the same side of the particle.

The line through the origin and the centre meets the circle at the points with position vectors $(\mathbf{i}+\mathbf{j})\,\mathrm{m}$ and $(5\mathbf{i}+5\mathbf{j})\,\mathrm{m}$. At the first, the centre and the origin are on opposite sides of the particle, so the particle must be at the second, which is $2\mathbf{i}+2\mathbf{j}$ relative to the centre. Using part (b):

$$2\sqrt{2}\cos\theta = 2, \qquad -2\sqrt{2}\sin\theta = 2$$

So $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$. Since θ starts at $\frac{\pi}{4}$ and increases, the first such value is $\theta = \frac{7\pi}{4}$, so

$$\frac{\pi t}{4} + \frac{\pi}{4} = \frac{7\pi}{4} \quad \Longrightarrow \quad t = 6$$



5. One end of a light inextensible string of length $7l$ is attached to a fixed point A. A small smooth ring R of mass m is threaded on the string, and the other end of the string is attached to a fixed point B, vertically below A, with $AB = 5l$.
The ring moves with constant speed in a horizontal circle whose centre C lies on AB. Both sections of the string are taut and angle $ARB = 90°$, as shown in the diagram above.

(a) Show that AR is either $3l$ or $4l$. **[2]**

(b) By resolving vertically for the ring, show that $AR = 4l$ and find, in terms of m and g, the tension in the string. **[5]**

(c) Find, in terms of g and l, the time taken for the ring to complete one revolution. **[5]**

Solution

(a) Let $AR = x$. The string has length $7l$, so $BR = 7l - x$. Using Pythagoras in triangle ARB, where angle $ARB = 90°$:

$$x^2 + (7l - x)^2 = (5l)^2$$

$$2x^2 - 14lx + 24l^2 = 0$$

$$x^2 - 7lx + 12l^2 = 0$$

$$(x - 3l)(x - 4l) = 0$$

Hence AR is either $3l$ or $4l$, as required.

(b) The ring is smooth, so the tension T is the same in both sections of the string. Let angle $RAB = \alpha$ and angle $RBA = \beta$. Triangle ARB has a right angle at R, so $\cos\alpha = \frac{AR}{5l}$ and $\cos\beta = \frac{BR}{5l}$.

The section AR pulls the ring upwards and the section BR pulls it downwards. The ring has no vertical acceleration, so resolving vertically gives

$$T\cos\alpha = T\cos\beta + mg$$

$$\frac{T \times AR}{5l} = \frac{T \times BR}{5l} + mg$$

$$T(AR - BR) = 5mgl$$

Since $T > 0$ and $5mgl > 0$, this requires $AR > BR$. If $AR = 3l$, then $BR = 4l$ and $AR < BR$, which is not possible. Hence $AR = 4l$, as required, and $BR = 3l$.

Substituting these lengths:

$$T(4l - 3l) = 5mgl \quad \Longrightarrow \quad T = 5mg$$

(c) In triangle ARB, $\sin\alpha = \frac{BR}{AB} = \frac{3}{5}$ and $\sin\beta = \frac{AR}{AB} = \frac{4}{5}$. The radius of the circle is

$$r = CR = AR\sin\alpha = 4l \times \frac{3}{5} = \frac{12l}{5}$$

Let ω be the angular speed of the ring. The horizontal components of the tension in both sections act towards C.

Take the direction towards C as positive. Using Newton's second law, with acceleration $r\omega^2$:

$$T\sin\alpha + T\sin\beta = m \times \frac{12l}{5} \times \omega^2$$

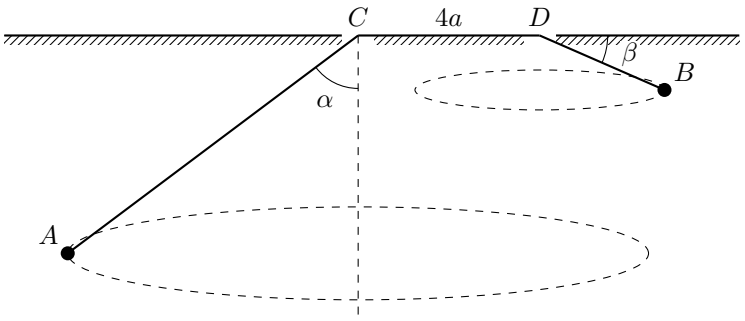
$$5mg\left(\frac{3}{5} + \frac{4}{5}\right) = \frac{12ml\omega^2}{5}$$

$$7mg = \frac{12ml\omega^2}{5}$$

$$\omega^2 = \frac{35g}{12l}$$

The time taken for one revolution is

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{12l}{35g}}$$



6. A light inextensible string AB of length $15a$ passes through two small holes C and D in a smooth horizontal table, where $CD = 4a$.
A particle of mass $3m$ is attached at the end A and moves in a horizontal circle, centred vertically below C, with constant angular speed ω.
A particle of mass $2m$ is attached at the end B and moves in a horizontal circle, centred vertically below D, with constant angular speed 2ω.
AC makes an angle α with the downward vertical, where $\cos\alpha = \frac{3}{5}$, and DB makes an angle β with the horizontal, as shown in the diagram above.

(a) Find the lengths AC and DB in terms of a. **[4]**

(b) Find ω in terms of a and g, and find the vertical distance of B below the table in terms of a. **[4]**

Solution

(a) The table is smooth, so the tension has the same value T throughout the string. Let $AC = x$ and $DB = y$. The string has length $15a$ and $CD = 4a$, so

$$x + y = 15a - 4a = 11a$$

The particle at A moves in a circle of radius $x\sin\alpha$, and the particle at B moves in a circle of radius $y\cos\beta$. Take the direction towards the centre of each circle as positive. Using Newton's second law for the particle at A, with acceleration $(x\sin\alpha)\omega^2$:

$$T\sin\alpha = 3m(x\sin\alpha)\omega^2$$

$$T = 3mx\omega^2$$

Using Newton's second law for the particle at B, with acceleration $(y\cos\beta)(2\omega)^2$:

$$T\cos\beta = 2m(y\cos\beta)(2\omega)^2$$

$$T = 8my\omega^2$$

Equating the two expressions for T gives

$$3mx\omega^2 = 8my\omega^2 \quad \Longrightarrow \quad 3x = 8y$$

Substituting $y = 11a - x$:

$$3x = 8(11a - x) \quad \Longrightarrow \quad 11x = 88a \quad \Longrightarrow \quad x = 8a$$

Then $y = 11a - 8a = 3a$, so

$$AC = 8a, \qquad DB = 3a$$

(b) The particle at A has no vertical acceleration. Resolving vertically at A, with $\cos\alpha = \frac{3}{5}$:

$$T\cos\alpha = 3mg \quad \Longrightarrow \quad \frac{3}{5}T = 3mg \quad \Longrightarrow \quad T = 5mg$$

Substituting $T = 5mg$ and $x = 8a$ into $T = 3mx\omega^2$ from part (a):

$$5mg = 3m(8a)\omega^2$$

$$\omega^2 = \frac{5g}{24a}$$

$$\omega = \sqrt{\frac{5g}{24a}}$$

Resolving vertically at B:

$$T\sin\beta = 2mg \quad \Longrightarrow \quad 5mg\sin\beta = 2mg \quad \Longrightarrow \quad \sin\beta = \frac{2}{5}$$

The vertical distance of B below the table is

$$DB\sin\beta = 3a \times \frac{2}{5} = \frac{6a}{5}$$

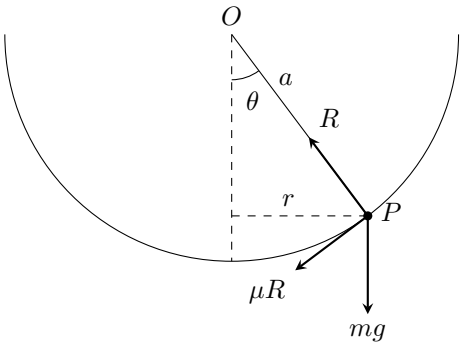
7. A hollow hemispherical bowl of radius a has a rough inner surface and is fixed with its axis vertical.
A particle P of mass m moves with constant speed $\sqrt{\frac{6}{5}ga}$ in a horizontal circle on the inner surface of the bowl, at a height $\frac{1}{5}a$ above the lowest point of the bowl.
The coefficient of friction between P and the bowl is μ, and P is on the point of slipping up the surface of the bowl.
Find the value of μ. **[7]**

Solution

Let O be the centre of the bowl. P is at a height $\frac{1}{5}a$ above the lowest point, so it is $a - \frac{1}{5}a = \frac{4}{5}a$ below O. Let θ be the angle between OP and the downward vertical, and let r be the radius of the circle. Since $OP = a$, this gives

$$\cos\theta = \frac{4}{5}, \qquad \sin\theta = \frac{3}{5}, \qquad r = a\sin\theta = \frac{3a}{5}$$

P is on the point of slipping up the surface, so friction acts down the surface and takes its limiting value μR, where R is the normal reaction, which acts towards O. The diagram below shows the forces on P.



The speed v of P satisfies $v^2 = \frac{6}{5}ga$, so

$$\frac{v^2}{r} = \frac{6}{5}ga \times \frac{5}{3a} = 2g$$

There is no vertical acceleration. Resolving vertically:

$$R\cos\theta - \mu R\sin\theta = mg$$

$$\frac{4}{5}R - \frac{3}{5}\mu R = mg$$

$$R(4 - 3\mu) = 5mg$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$:

$$R\sin\theta + \mu R\cos\theta = m \times 2g$$

$$\frac{3}{5}R + \frac{4}{5}\mu R = 2mg$$

$$R(3 + 4\mu) = 10mg$$

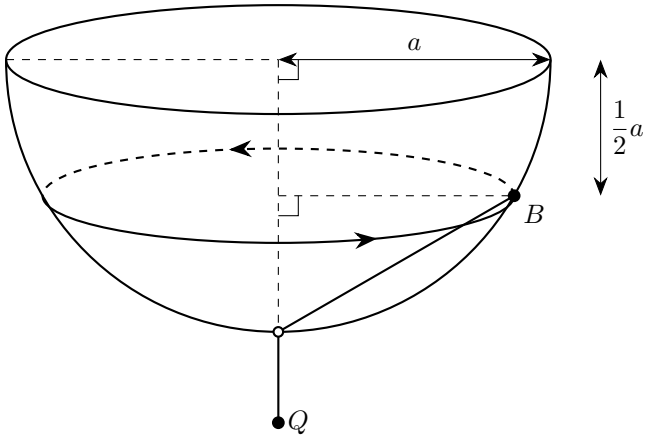
Dividing the second result by the first eliminates R, m and g:

$$\frac{3+4\mu}{4-3\mu} = 2$$

$$3+4\mu = 8-6\mu$$

$$10\mu = 5$$

$$\mu = \frac{1}{2}$$



8. A hemispherical shell of radius a is fixed with its rim uppermost and horizontal. There is a small smooth hole at the lowest point of the shell.
A small bead, B, of mass m, is moving with constant angular speed, ω, in a horizontal circle on the smooth inner surface of the shell. The centre of the path of B is at a distance $\frac{1}{2}a$ vertically below the level of the rim of the hemisphere, as shown in the diagram above.
A light inextensible string is attached to B and passes through the hole. A particle, Q, of mass $2m$ is attached to the other end of the string and hangs at rest vertically below the hole.

(a) Find, in terms of m and g, the magnitude of the normal reaction of the shell on B. **[4]**

(b) Find the magnitude of ω, giving your answer in terms of a and g. **[3]**

Solution

(a) Let O be the centre of the rim, H the hole and C the centre of the path of B. The normal reaction, R, acts along BO towards O, and the tension, T, acts along BH towards H.

Since $OB = a$ and $OC = \frac{1}{2}a$, the angle OBC satisfies

$$\sin\angle OBC = \frac{\frac{1}{2}a}{a} = \frac{1}{2} \quad \Longrightarrow \quad \angle OBC = 30^\circ$$

Also $OH = a$, so $CH = \frac{1}{2}a = OC$. Since BC is perpendicular to OH, the triangles OBC and HBC are congruent, so $\angle HBC = 30^\circ$. Hence R acts at 30° above the horizontal and T at 30° below the horizontal.

The particle Q hangs at rest, so $T = 2mg$.

The bead has no vertical acceleration, so resolving vertically gives

$$R\sin 30^\circ = T\sin 30^\circ + mg$$

$$\frac{1}{2}R = \frac{1}{2}(2mg) + mg$$

$$R = 4mg$$

(b) The radius of the path is

$$r = BC = a\cos 30^\circ = \frac{\sqrt{3}}{2}a$$

Take the direction towards C as positive. Using Newton's second law, with acceleration $r\omega^2$:

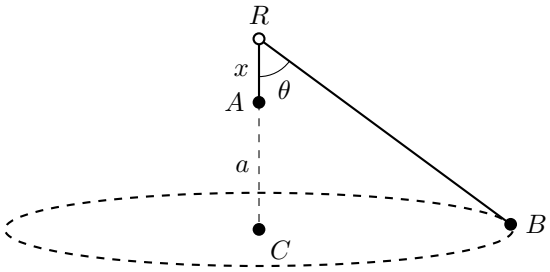
$$R\cos 30^\circ + T\cos 30^\circ = mr\omega^2$$

Substituting $R = 4mg$ from part (a), $T = 2mg$ and the value of r:

$$(4mg + 2mg) \times \frac{\sqrt{3}}{2} = m \times \frac{\sqrt{3}}{2} a\,\omega^2$$

$$6g = a\omega^2$$

$$\omega = \sqrt{\frac{6g}{a}}$$



9. A light inextensible string of length $3a$ is threaded through a fixed smooth ring R.
One end of the string is attached to a particle A of mass $5m$, and the other end is attached to a particle B of mass $3m$.
The particle A hangs in equilibrium at a distance x vertically below the ring, and the angle between AR and BR is θ, as shown in the diagram above.
The particle B moves with constant speed in a horizontal circle whose centre C is a distance a vertically below A.

(a) Show that $\cos\theta = \frac{3}{5}$ and find x in terms of a. **[4]**

(b) Find the speed of B in terms of a and g. **[3]**

Solution

(a) The ring is smooth, so the tension T is the same throughout the string.

Resolving vertically for A, which is in equilibrium:

$$T = 5mg$$

Resolving vertically for B, which has no vertical acceleration:

$$T\cos\theta = 3mg$$

$$5mg\cos\theta = 3mg$$

$$\cos\theta = \frac{3}{5}$$

The string has length $3a$ and $RA = x$, so $RB = 3a - x$. Also, $RC = RA + AC = x + a$. The angle at C in triangle RCB is a right angle, so

$$\cos\theta = \frac{RC}{RB} = \frac{x+a}{3a-x}$$

$$\frac{3}{5} = \frac{x+a}{3a-x}$$

$$3(3a-x) = 5(x+a)$$

$$9a - 3x = 5x + 5a$$

$$8x = 4a$$

$$x = \frac{a}{2}$$

(b) From part (a), $RB = 3a - \frac{a}{2} = \frac{5a}{2}$. Since $\cos\theta = \frac{3}{5}$ and θ is acute, $\sin\theta = \frac{4}{5}$. The radius of the circle is

$$r = BC = RB\sin\theta = \frac{5a}{2} \times \frac{4}{5} = 2a$$

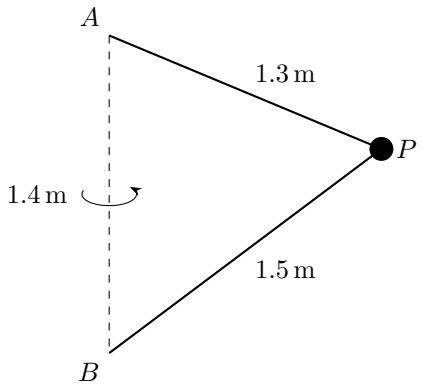
Take the direction towards C as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$, where v is the speed of B:

$$T\sin\theta = \frac{3mv^2}{2a}$$

$$5mg \times \frac{4}{5} = \frac{3mv^2}{2a}$$

$$v^2 = \frac{8ag}{3}$$

$$v = \sqrt{\frac{8ag}{3}}$$



10. A particle P of mass 0.7 kg is attached to one end of a light inextensible string of length 1.3 m. The other end of this string is attached to the fixed point A.
A second light inextensible string, of length 1.5 m, has one end attached to P and the other end attached to the fixed point B. The point B is vertically below A with $AB = 1.4$ m, as shown in the diagram above. The particle moves in a horizontal circle with constant angular speed ω rad s^{-1}, with both strings taut.

(a) Show that the tension in the string BP is $\frac{3}{8}\left(\omega^2 - 2g\right)$ N, where g m s^{-2} is the acceleration due to gravity. **[7]**

(b) The string BP will break if the tension in it exceeds 15 N.
Hence find the range of values of ω for which the tension in BP is positive and BP does not break. **[2]**

Solution

(a) Let C be the centre of the circle, which lies on AB with PC horizontal. Let $AC = x$ m and let the radius CP be r m. Using Pythagoras in triangles ACP and BCP:

$$x^2 + r^2 = 1.3^2$$

$$(1.4 - x)^2 + r^2 = 1.5^2$$

Subtracting the first equation from the second gives

$$1.96 - 2.8x = 0.56$$

$$x = 0.5$$

so $r = \sqrt{1.3^2 - 0.5^2} = 1.2$ and $BC = 0.9$ m.

Let α be angle PAB and β be angle PBA. Then

$$\cos\alpha = \frac{0.5}{1.3} = \frac{5}{13}, \qquad \sin\alpha = \frac{1.2}{1.3} = \frac{12}{13}, \qquad \cos\beta = \frac{0.9}{1.5} = \frac{3}{5}, \qquad \sin\beta = \frac{1.2}{1.5} = \frac{4}{5}$$

Let the tensions in AP and BP be T_A N and T_B N. The particle has no vertical acceleration, so resolving vertically gives

$$T_A \cos\alpha = T_B \cos\beta + 0.7g$$

$$\frac{5}{13}T_A = \frac{3}{5}T_B + 0.7g$$

Take the direction towards C as positive. Using Newton's second law, with acceleration $r\omega^2$:

$$T_A \sin\alpha + T_B \sin\beta = 0.7 \times 1.2\,\omega^2$$

$$\frac{12}{13}T_A + \frac{4}{5}T_B = 0.84\,\omega^2$$

Multiplying the vertical equation by $\frac{12}{5}$ gives

$$\frac{12}{13}T_A = 1.44T_B + 1.68g$$

Substituting into the horizontal equation:

$$1.44T_B + 1.68g + 0.8T_B = 0.84\,\omega^2$$

$$2.24T_B = 0.84\,\omega^2 - 1.68g$$

$$2.24T_B = 0.84\big(\omega^2 - 2g\big)$$

$$T_B = \frac{3}{8}\big(\omega^2 - 2g\big)$$

as required.

(b) Using part (a), the tension is positive and does not exceed 15 N when

$$0 < \frac{3}{8}\big(\omega^2 - 2g\big) \leq 15$$

$$0 < \omega^2 - 2g \leq 40$$

$$2g < \omega^2 \leq 40 + 2g$$

With $g = 9.8$, and since $\omega > 0$:

$$\sqrt{19.6} < \omega \leq \sqrt{59.6}$$

$$4.43 < \omega \leq 7.72$$

Final answers with $g = 10$: (b) $4.47 < \omega \leq 7.75$

11. A light inextensible string has length $35a$. One end of the string is attached to a fixed point A and the other end is attached to a fixed point B, where A is vertically above B and $AB = 7a$.
A small ball of mass m is attached to a point P on the string, where $AP = 20a$.
The ball moves in a horizontal circle with constant speed v, with both AP and BP taut.
The string will break if the tension in either part of it exceeds $\frac{10mg}{7}$.
Model the ball as a particle and assume that the string does not break.

(a) Show that

$$9ag < v^2 \leq 15ag$$

[8]

(b) Find, in terms of a and g, the time taken for the ball to make one complete revolution when the tensions in AP and BP are equal. **[3]**

Solution

(a) Let O be the centre of the circle, which lies on the line AB extended, with $OP = r$ and O a distance h below B. Using Pythagoras in triangles OBP and OAP:

$$r^2 + h^2 = (15a)^2$$

$$r^2 + (h + 7a)^2 = (20a)^2$$

Subtracting the first equation from the second gives

$$14ah + 49a^2 = 175a^2 \quad \Longrightarrow \quad h = 9a$$

Substituting into the first equation gives

$$r^2 = 225a^2 - 81a^2 = 144a^2 \quad \Longrightarrow \quad r = 12a$$

Since $h > 0$, the ball is below B, so both parts of the string pull the ball upwards and towards O. Also $AO = 16a$. Let α be angle OAP and β be angle OBP, so that

$$\cos\alpha = \frac{16a}{20a} = \frac{4}{5}, \quad \sin\alpha = \frac{12a}{20a} = \frac{3}{5}, \quad \cos\beta = \frac{9a}{15a} = \frac{3}{5}, \quad \sin\beta = \frac{12a}{15a} = \frac{4}{5}$$

Let the tensions in AP and BP be T_A and T_B. The ball has no vertical acceleration, so resolving vertically gives

$$\frac{4}{5}T_A + \frac{3}{5}T_B = mg \quad \Longrightarrow \quad 4T_A + 3T_B = 5mg$$

Take the direction towards O as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$:

$$\frac{3}{5}T_A + \frac{4}{5}T_B = \frac{mv^2}{12a} \quad \Longrightarrow \quad 3T_A + 4T_B = \frac{5mv^2}{12a}$$

In matrix form, these equations are

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} T_A \\ T_B \end{pmatrix} = \begin{pmatrix} 5mg \\ \dfrac{5mv^2}{12a} \end{pmatrix}$$

Multiplying by the inverse matrix gives

$$\begin{pmatrix} T_A \\ T_B \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5mg \\ \dfrac{5mv^2}{12a} \end{pmatrix}$$

Reading off each tension:

$$T_A = \frac{1}{7}\left(20mg - \frac{5mv^2}{4a}\right) = \frac{5m(16ag - v^2)}{28a}$$

$$T_B = \frac{1}{7}\left(\frac{5mv^2}{3a} - 15mg\right) = \frac{5m(v^2 - 9ag)}{21a}$$

Both parts of the string are taut, so $T_A > 0$ and $T_B > 0$. These give

$$16ag - v^2 > 0 \quad \Longrightarrow \quad v^2 < 16ag$$

$$v^2 - 9ag > 0 \quad \Longrightarrow \quad v^2 > 9ag$$

The string does not break, so $T_A \leq \frac{10mg}{7}$ and $T_B \leq \frac{10mg}{7}$. These give

$$\frac{5m(16ag - v^2)}{28a} \leq \frac{10mg}{7} \quad \Longrightarrow \quad 16ag - v^2 \leq 8ag \quad \Longrightarrow \quad v^2 \geq 8ag$$

$$\frac{5m(v^2 - 9ag)}{21a} \leq \frac{10mg}{7} \quad \Longrightarrow \quad v^2 - 9ag \leq 6ag \quad \Longrightarrow \quad v^2 \leq 15ag$$

All four conditions must hold. The stricter lower bound is $v^2 > 9ag$ and the stricter upper bound is $v^2 \leq 15ag$, so

$$9ag < v^2 \leq 15ag$$

(b) Setting $T_A = T_B$ gives

$$\frac{5m(16ag - v^2)}{28a} = \frac{5m(v^2 - 9ag)}{21a}$$

$$3(16ag - v^2) = 4(v^2 - 9ag)$$

$$84ag = 7v^2$$

$$v^2 = 12ag$$

The circle has radius $12a$, so the time for one complete revolution is

$$\frac{2\pi \times 12a}{v} = \frac{2\pi \times 12a}{\sqrt{12ag}} = 2\pi\sqrt{\frac{12a}{g}} = 4\pi\sqrt{\frac{3a}{g}}$$

12. In this question use $g = 9.81\,\text{m}\,\text{s}^{-2}$.
A conical pendulum is made from an elastic string and a sphere of mass 0.5 kg.
The string has natural length 2 m and modulus of elasticity 100 N.
The sphere describes a horizontal circle at a constant speed of $5\,\text{m}\,\text{s}^{-1}$.
The angle between the elastic string and the vertical is α.

(a) Show that

$$25\cos^2\alpha = 9.81\sin^2\alpha\,(2\cos\alpha + 0.0981) \qquad \textbf{[7]}$$

(b) Use your calculator to find α, giving your answer in degrees. **[1]**

(c) Find the elastic potential energy stored in the string. **[4]**

Solution

(a) Let the tension be T N, the extension of the string x m and the radius of the circle r m. The sphere has no vertical acceleration. Resolving vertically:

$$T\cos\alpha = 0.5 \times 9.81 = 4.905$$

Hooke's law is

$$T = \frac{\lambda x}{l}$$

where x is the extension and l the natural length. Here $\lambda = 100$ and $l = 2$, so

$$T = \frac{100x}{2} = 50x$$

Substituting into the vertical equation:

$$50x\cos\alpha = 4.905 \quad \Longrightarrow \quad x = \frac{0.0981}{\cos\alpha}$$

The string has length $2 + x$, so the radius is

$$\begin{aligned} r &= (2+x)\sin\alpha \\ &= \left(2 + \frac{0.0981}{\cos\alpha}\right)\sin\alpha \\ &= \frac{\sin\alpha\,(2\cos\alpha + 0.0981)}{\cos\alpha} \end{aligned}$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$ and $v = 5$:

$$T\sin\alpha = \frac{0.5 \times 5^2}{r} = \frac{12.5}{r}$$

Since $T\cos\alpha = 4.905$, we have $T\sin\alpha = \frac{4.905\sin\alpha}{\cos\alpha}$. Substituting this and the expression for r:

$$\frac{4.905\sin\alpha}{\cos\alpha} = \frac{12.5\cos\alpha}{\sin\alpha\,(2\cos\alpha + 0.0981)}$$

$$4.905\sin^2\alpha\,(2\cos\alpha + 0.0981) = 12.5\cos^2\alpha$$

Multiplying by 2:

$$25\cos^2\alpha = 9.81\sin^2\alpha\,(2\cos\alpha + 0.0981)$$

as required.

(b) Solving the equation from part (a) numerically on a calculator, for $0° < \alpha < 90°$, gives

$$\alpha = 55.1°$$

(c) Using the unrounded value of α from part (b), the extension is

$$x = \frac{0.0981}{\cos\alpha} \approx 0.171$$

The elastic potential energy is

$$EPE = \frac{\lambda x^2}{2l}$$

where x is the extension and l the natural length. Substituting the unrounded value of x:

$$EPE = \frac{100x^2}{2\times 2} = 25x^2 \approx 0.734\,\text{J}$$

13. A hollow right circular cone C of height 3 m and base radius 4 m is fixed with its axis vertical and its vertex downwards.
One end of a light elastic string of natural length 3 m and modulus of elasticity 16 N is fixed to the vertex of C. The other end of the string is attached to a particle P of mass 5 kg.
P moves in a horizontal circle with constant speed and in contact with the smooth inner surface of C.

(a) Find the greatest speed of P for which the tension in the string is zero. **[3]**

(b) The speed of P is now $6\,\text{m}\,\text{s}^{-1}$.
Find the extension of the string and the magnitude of the normal reaction between P and C. **[7]**

Solution

(a) The slant height of the cone is $\sqrt{3^2+4^2} = 5$ m. Let θ be the angle between a slant edge and the horizontal, so that

$$\sin\theta = \frac{3}{5}, \qquad \cos\theta = \frac{4}{5}$$

The surface is smooth, so the normal reaction R is perpendicular to the surface, at an angle θ to the vertical. Let P have speed v and move in a circle of radius r. With zero tension, the only forces on P are R and its weight.

Resolving vertically:

$$R\cos\theta = 5g \quad \Longrightarrow \quad R = \frac{25g}{4}$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$:

$$R\sin\theta = \frac{5v^2}{r}$$

Substituting for R gives

$$\frac{15g}{4} = \frac{5v^2}{r} \quad \Longrightarrow \quad v^2 = \frac{3gr}{4}$$

So v increases as r increases. The tension is zero only while the string is not stretched, so P is at most 3 m from the vertex. The greatest speed therefore occurs when P is exactly 3 m from the vertex, where

$$r = 3\cos\theta = \frac{12}{5}\,\text{m}$$

Substituting this radius:

$$v^2 = \frac{3g}{4} \times \frac{12}{5} = \frac{9g}{5} = 17.64$$

$$v = 4.2\,\text{m}\,\text{s}^{-1}$$

(b) Since $6 > 4.2$, part (a) shows that the tension is not zero, so the string is stretched. Hooke's law is

$$T = \frac{\lambda x}{l}$$

where x is the extension and l the natural length. With $\lambda = 16$ and $l = 3$, this gives

$$T = \frac{16x}{3}$$

P is now $3 + x$ from the vertex, so

$$r = (3 + x)\cos\theta = \frac{4(3 + x)}{5}$$

The tension acts along the slant edge towards the vertex, so it has a component $T\sin\theta$ vertically downwards and a component $T\cos\theta$ towards the centre of the circle.

Resolving vertically:

$$R\cos\theta = 5g + T\sin\theta \quad \Longrightarrow \quad \frac{4}{5}R = 5g + \frac{3}{5}T$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $\frac{v^2}{r}$ and $v = 6$:

$$R\sin\theta + T\cos\theta = \frac{5 \times 6^2}{r}$$

$$\frac{3}{5}R + \frac{4}{5}T = \frac{225}{3 + x}$$

Multiplying the vertical equation by $\frac{3}{4}$ gives $\frac{3}{5}R = \frac{15g}{4} + \frac{9}{20}T$. Substituting this into the horizontal equation:

$$\frac{15g}{4} + \frac{9}{20}T + \frac{4}{5}T = \frac{225}{3 + x}$$

$$\frac{15g}{4} + \frac{5}{4}T = \frac{225}{3 + x}$$

Now substitute $g = 9.8$ and $T = \frac{16x}{3}$:

$$36.75 + \frac{20x}{3} = \frac{225}{3 + x}$$

Multiplying by $12(3 + x)$:

$$(441 + 80x)(3 + x) = 2700$$

$$80x^2 + 681x - 1377 = 0$$

The quadratic formula gives

$$x = \frac{-681 \pm 951}{160}$$

The extension is positive, so

$$x = \frac{27}{16} = 1.6875\,\text{m}$$

The tension is

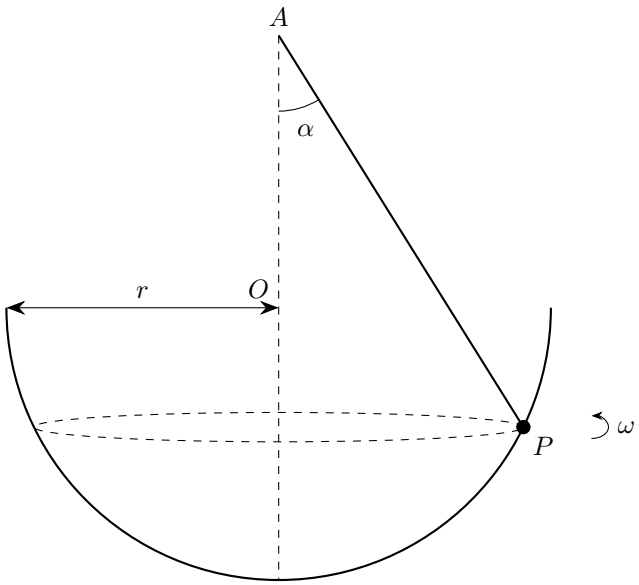
$$T = \frac{16}{3} \times \frac{27}{16} = 9\,\text{N}$$

Using the vertical equation, the normal reaction is given by

$$\frac{4}{5}R = 5 \times 9.8 + \frac{3}{5} \times 9 = 54.4$$

$$R = 68\,\text{N}$$

Final answers with $g = 10$: (a) $4.24\,\text{m}\,\text{s}^{-1}$; (b) extension $1.64\,\text{m}$, normal reaction $69.1\,\text{N}$



14. The diagram above shows a hemispherical bowl. The rim of this bowl is a circle with centre O and radius r. The bowl is fixed with its rim horizontal and uppermost.
The point A is fixed vertically above O, with $OA = r$. A particle P, of mass m, is connected to A by a light inextensible string and describes a horizontal circle with constant angular speed ω on the smooth inner surface of the bowl.
The string is taut, and AP makes an angle α with the vertical, where $0 < \alpha < \frac{\pi}{4}$.

(a) Show that the tension in the string is $2m\cos\alpha\left(g - r\omega^2\cos 2\alpha\right)$. **[7]**

(b) Find, in terms of g, r and α, the range of values of ω^2 for which this motion is possible, with the string taut and P in contact with the bowl. **[4]**

Solution

(a) Since $OA = OP = r$, triangle OAP is isosceles, so angle $OPA = \alpha$. The angle between OP and the downward vertical is an exterior angle of this triangle, so it equals $\alpha + \alpha = 2\alpha$. Hence P moves in a horizontal circle of radius $r\sin 2\alpha$.

The forces on P are its weight mg, the tension T along PA and, since the bowl is smooth, the normal reaction R along PO.

Resolving vertically:

$$T\cos\alpha + R\cos 2\alpha = mg$$

Take the direction towards the centre of the circle as positive. Using Newton's second law, with acceleration $r\omega^2\sin 2\alpha$:

$$T\sin\alpha + R\sin 2\alpha = mr\omega^2\sin 2\alpha$$

Writing $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and dividing by $\sin\alpha$, which is positive since $0 < \alpha < \frac{\pi}{4}$, gives

$$T + 2R\cos\alpha = 2mr\omega^2\cos\alpha$$

Now multiply by $\cos 2\alpha$ and substitute $R\cos 2\alpha = mg - T\cos\alpha$ from the vertical equation:

$$T\cos 2\alpha + 2\cos\alpha(mg - T\cos\alpha) = 2mr\omega^2\cos\alpha\cos 2\alpha$$

$$T\big(\cos 2\alpha - 2\cos^2\alpha\big) + 2mg\cos\alpha = 2mr\omega^2\cos\alpha\cos 2\alpha$$

Since $\cos 2\alpha = 2\cos^2\alpha - 1$, the bracket equals -1, so

$$-T + 2mg\cos\alpha = 2mr\omega^2\cos\alpha\cos 2\alpha$$

$$T = 2mg\cos\alpha - 2mr\omega^2\cos\alpha\cos 2\alpha$$

$$= 2m\cos\alpha\big(g - r\omega^2\cos 2\alpha\big)$$

as required.

(b) The string is taut when $T > 0$, and P is in contact with the bowl when $R > 0$.

Substituting the result of part (a) into $T + 2R\cos\alpha = 2mr\omega^2\cos\alpha$:

$$2R\cos\alpha = 2mr\omega^2\cos\alpha - 2m\cos\alpha\big(g - r\omega^2\cos 2\alpha\big)$$

$$R = mr\omega^2 - mg + mr\omega^2\cos 2\alpha$$

$$= mr\omega^2(1 + \cos 2\alpha) - mg$$

$$= m\big(2r\omega^2\cos^2\alpha - g\big)$$

Since $2m\cos\alpha > 0$, the condition $T > 0$ requires $g - r\omega^2\cos 2\alpha > 0$. Since $0 < 2\alpha < \frac{\pi}{2}$, $\cos 2\alpha > 0$, so

$$\omega^2 < \frac{g}{r\cos 2\alpha}$$

The condition $R > 0$ requires $2r\omega^2\cos^2\alpha > g$, so

$$\omega^2 > \frac{g}{2r\cos^2\alpha}$$

Since $2\cos^2\alpha = 1 + \cos 2\alpha > \cos 2\alpha$, the lower bound is less than the upper bound, so the range of values is

$$\frac{g}{2r\cos^2\alpha} < \omega^2 < \frac{g}{r\cos 2\alpha}$$

15. A horizontal turntable rotates with constant angular speed $\omega\,\text{rad}\,\text{s}^{-1}$ about a fixed vertical axis through its centre O.
One end of a light elastic string of natural length $l\,\text{m}$ and modulus of elasticity $\lambda\,\text{N}$ is attached to O. The other end of the string is attached to a particle A of mass $m\,\text{kg}$, which rests on the turntable.
The coefficient of friction between A and the turntable is μ.
The particle A moves with the turntable, with the string horizontal and taut, and A is on the point of slipping away from O. The extension of the string is denoted by $x\,\text{m}$.

(a) Show that

$$x = \frac{ml\left(l\omega^2 - \mu g\right)}{\lambda - ml\omega^2} \qquad \textbf{[3]}$$

(b) Using a binomial series, show that if λ is very large then

$$\lambda x \approx ml\left(l\omega^2 - \mu g\right)\left(1 + \frac{ml\omega^2}{\lambda}\right) \approx ml\left(l\omega^2 - \mu g\right) \qquad \textbf{[3]}$$

(c) By finding an approximation for the tension in the string, explain how the result obtained when λ is very large relates to the situation when the string is inextensible and of length $l\,\text{m}$. **[1]**

The nature of the surface of the turntable is such that the modelling assumption that μ is constant is justifiable provided that the angular speed does not exceed $12\,\text{rad}\,\text{s}^{-1}$.
In the case where $m = 0.25$, $l = 0.5$, $\lambda = 400$ and $\mu = 0.4$, the extension of the string is measured as being $2.0\,\text{cm}$.

(d) Use the approximation $\lambda x \approx ml\left(l\omega^2 - \mu g\right)$ to estimate the value of ω. **[2]**

(e) By comparing the radius used in part **(d)** with the true radius of the circle, explain whether your estimate is greater or less than the true value of ω, and hence whether the modelling assumption is necessarily justifiable in this situation. **[2]**

Solution

(a) Let the tension in the string be $T\,\text{N}$. Hooke's law is

$$T = \frac{\lambda x}{l}$$

where x is the extension and l the natural length.

The particle moves in a circle of radius $(l + x)\,\text{m}$. Resolving vertically, the normal reaction is mg. Since A is on the point of slipping away from O, friction is limiting and acts towards O, with magnitude μmg.

Take the direction towards O as positive. Using Newton's second law, with acceleration $(l + x)\omega^2$:

$$\frac{\lambda x}{l} + \mu mg = m(l + x)\omega^2$$

Multiplying by l and collecting the terms in x:

$$\lambda x + \mu mgl = ml^2\omega^2 + mlx\omega^2$$

$$x\big(\lambda - ml\omega^2\big) = ml\big(l\omega^2 - \mu g\big)$$

$$x = \frac{ml\big(l\omega^2 - \mu g\big)}{\lambda - ml\omega^2}$$

(b) Multiplying the result of part (a) by λ, and dividing the numerator and denominator by λ:

$$\lambda x = ml\big(l\omega^2 - \mu g\big)\left(1 - \frac{ml\omega^2}{\lambda}\right)^{-1}$$

The binomial series is

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{1\times 2}z^2 + \cdots \quad (|z| < 1,\ n \in \mathbb{R})$$

Since λ is very large, $\frac{ml\omega^2}{\lambda}$ is very small, so the series applies with $n = -1$ and $z = -\frac{ml\omega^2}{\lambda}$. Neglecting the terms in $\frac{1}{\lambda^2}$ and higher powers gives

$$\left(1 - \frac{ml\omega^2}{\lambda}\right)^{-1} \approx 1 + \frac{ml\omega^2}{\lambda}$$

Also $\frac{ml\omega^2}{\lambda}$ is negligible compared with 1 when λ is very large, so

$$\lambda x \approx ml\big(l\omega^2 - \mu g\big)\left(1 + \frac{ml\omega^2}{\lambda}\right) \approx ml\big(l\omega^2 - \mu g\big)$$

(c) Using Hooke's law with the result of part (b):

$$T = \frac{\lambda x}{l} \approx m\big(l\omega^2 - \mu g\big) = ml\omega^2 - \mu mg$$

For an inextensible string of length l m, the radius of the circle is l m. Take the direction towards O as positive. Using Newton's second law, with acceleration $l\omega^2$:

$$T + \mu mg = ml\omega^2$$

This gives the same tension, so a string with very large λ behaves like an inextensible string of length l m.

(d) The extension is 2.0 cm, so $x = 0.02$. Substituting the values, with $g = 9.8$:

$$400 \times 0.02 \approx 0.25 \times 0.5\big(0.5\omega^2 - 0.4 \times 9.8\big)$$

$$8 \approx 0.125\big(0.5\omega^2 - 3.92\big)$$

$$\omega^2 \approx 135.84$$

$$\omega \approx 11.7$$

(e) The approximation used in part (d) takes the radius of the circle to be 0.5 m, whereas the true radius is 0.52 m.

The tension depends only on the measured extension, and the limiting friction is unchanged, so the resultant force towards O has the same value whichever radius is used. This force equals $m\omega^2$ times the radius, so a larger radius requires a smaller ω. The estimate is therefore greater than the true value of ω.

The true angular speed is less than $11.7\,\mathrm{rad\,s^{-1}}$, which is below $12\,\mathrm{rad\,s^{-1}}$, so the modelling assumption is justifiable in this situation.

Final answers with $g = 10$: (d) $\omega \approx 11.7$