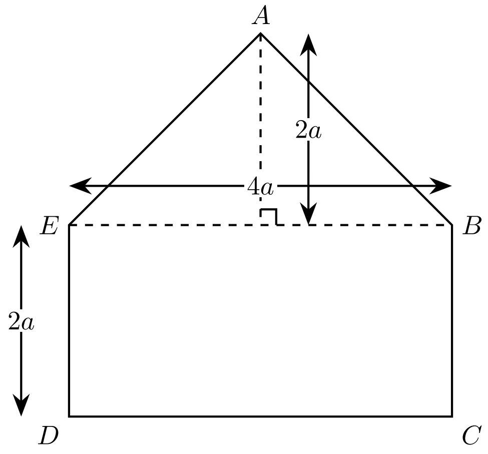
Questions



1. A uniform plane lamina is in the shape of the pentagon $ABCDE$, as shown shaded in the diagram above.

- $BCDE$ is a rectangle
- $BE = 4a$ and $DE = 2a$
- triangle ABE is isosceles with perpendicular height $2a$

(a) Show that the distance of the centre of mass of the lamina from CD is $\frac{14a}{9}$. **[5]**

The mass of the lamina is M.

The lamina is suspended by two light vertical strings, one attached to the lamina at D and the other attached to the lamina at E.

The lamina hangs freely in equilibrium, with DE horizontal.

(b) Using part (a), find, in terms of M and g, the tension in the string attached at E. **[2]**

Solution

(a) Split the lamina into rectangle $BCDE$ and triangle ABE.

Because the density is constant and the materials are the same, we can use areas as weights.

Let $\bar{y}$ be the distance of the centre of mass from CD.

For rectangle $BCDE$,

$$\text{area} = BE \times DE = 4a \times 2a = 8a^2$$

Its centre of mass is halfway up the rectangle, so its distance from CD is

$$a$$

For triangle ABE,

$$\text{area} = \frac{1}{2} \times 4a \times 2a = 4a^2$$

The centre of mass of a triangle is $\frac{1}{3}$ of the perpendicular height from the base. So its distance above BE is

$$\frac{1}{3}(2a) = \frac{2a}{3}$$

Hence its distance from CD is

$$2a + \frac{2a}{3} = \frac{8a}{3}$$

Now use moments of area about CD:

$$(8a^2 + 4a^2)\bar{y} = 8a^2(a) + 4a^2\left(\frac{8a}{3}\right)$$

$$12a^2\bar{y} = 8a^3 + \frac{32a^3}{3}$$

$$12a^2\bar{y} = \frac{56a^3}{3}$$

$$\bar{y} = \frac{56a^3}{36a^2} = \frac{14a}{9}$$

So the distance of the centre of mass from CD is $\dfrac{14a}{9}$.

(b) When the lamina hangs with DE horizontal, the line CD is vertical through D. So, from part (a), the centre of mass is horizontally $\dfrac{14a}{9}$ from D.

Also, $DE = 2a$, so the string at E acts at a horizontal distance $2a$ from D.

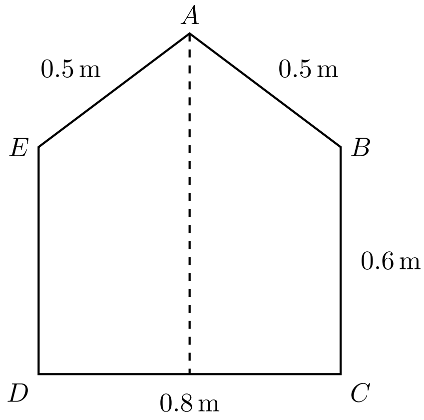
Taking moments about D:

$$T_E(2a) = Mg\left(\frac{14a}{9}\right)$$

Hence

$$T_E = \frac{Mg\left(\frac{14a}{9}\right)}{2a} = \frac{7Mg}{9}$$

So the tension in the string attached at E is $\dfrac{7Mg}{9}$.



2. $ABCDE$ is a uniform lamina formed by joining an isosceles triangle ABE to the top of a rectangle $BCDE$, where $AB = AE = 0.5\,\text{m}$, $BC = DE = 0.6\,\text{m}$ and $CD = BE = 0.8\,\text{m}$, as shown in the diagram. The centre of mass of the lamina is G.

(a) Explain why G lies on the dashed line of symmetry. **[1]**

(b) Show that the perpendicular distance of G from CD is $0.38\,\text{m}$. **[4]**

The lamina is freely suspended from the point B.

(c) Determine the acute angle that CD makes with the horizontal, stating which of C or D is higher. **[4]**

Solution

(a) The lamina is uniform and symmetric about the dashed line AM. Therefore its centre of mass must lie on this line, so G lies on the dashed line of symmetry.

(b) Let M be the midpoint of CD, so AM is the line of symmetry.

For rectangle $BCDE$,

$$\text{area} = 0.8 \times 0.6 = 0.48$$

and its centre is halfway up the rectangle, so it is

$$0.3\,\text{m}$$

above CD.

For triangle ABE, the base is $BE = 0.8\,\text{m}$, so

$$BM = EM = 0.4\,\text{m}$$

Hence its height is

$$\begin{aligned} AM &= \sqrt{AB^2 - BM^2} \\ &= \sqrt{0.5^2 - 0.4^2} \\ &= \sqrt{0.25 - 0.16} \\ &= 0.3\,\text{m} \end{aligned}$$

So the area of the triangle is

$$\frac{1}{2} \times 0.8 \times 0.3 = 0.12$$

The centre of mass of a triangle is one-third of the height from its base, so the triangle's centre of mass is

$$0.6 + \frac{0.3}{3} = 0.7\,\text{m}$$

above CD.

Because the density is constant and the materials are the same, we can use areas as weights:

$$Ay = A_1y_1 + A_2y_2$$

So

$$\begin{aligned}(0.48 + 0.12)y &= 0.48(0.3) + 0.12(0.7)\\ 0.6y &= 0.144 + 0.084\\ 0.6y &= 0.228\\ y &= 0.38\end{aligned}$$

So the perpendicular distance of G from CD is $0.38\,\text{m}$.

(c) From part (b), G is $0.38\,\text{m}$ above CD, while B is $0.6\,\text{m}$ above CD. So G is

$$0.6 - 0.38 = 0.22\,\text{m}$$

below B.

Also, since G lies on the line of symmetry AM, its horizontal distance from B is half the width of the rectangle:

$$BM = \frac{CD}{2} = 0.4\,\text{m}$$

When the lamina is freely suspended from B, the centre of mass G hangs vertically below B. Therefore the lamina must rotate through an angle θ where

$$\begin{aligned}\tan\theta &= \frac{\text{horizontal distance of } G \text{ from } B}{\text{vertical distance of } G \text{ below } B}\\ &= \frac{0.4}{0.22}\\ &= \frac{20}{11}\end{aligned}$$

Hence

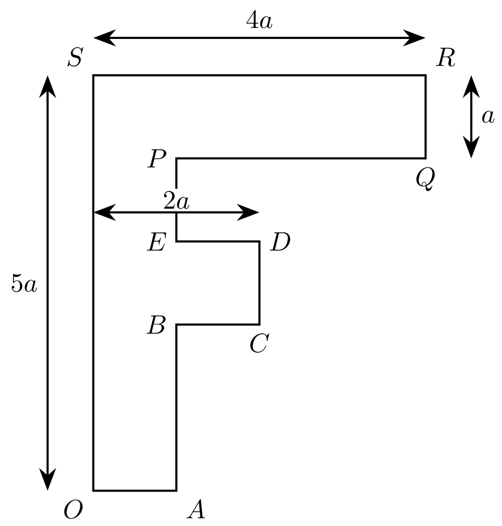
$$\theta = \tan^{-1}\left(\frac{20}{11}\right) \approx 61.2^\circ$$

Since G is to the left of B, the lamina rotates clockwise, so D is higher than C.

So the acute angle that CD makes with the horizontal is

$$\tan^{-1}\left(\frac{20}{11}\right) \approx 61.2^\circ$$

with D higher than C.



3. A letter F from a shop sign is modelled as a uniform plane lamina in the shape shown in the diagram above.

The left-hand vertical strip has width a and height $5a$. The top horizontal strip has length $4a$ and thickness a. The middle horizontal strip has length $2a$ and thickness a.

Using the model,

(a) find, in terms of a, the distance of the centre of mass of the letter F, from

(i) OS

(ii) OA

[6]

The letter F is freely suspended from O and hangs in equilibrium. The angle between OS and the downward vertical is α.

Using the model,

(b) find the exact value of $\tan\alpha$. **[2]**

Solution

(a) Take O as the origin, with OA as the x-axis and OS as the y-axis.

To avoid overlap, split the letter F into three rectangles:

Part	Area	x-coordinate of centre	y-coordinate of centre
vertical strip	$5a^2$	$\frac{a}{2}$	$\frac{5a}{2}$
top extension	$3a^2$	$\frac{5a}{2}$	$\frac{9a}{2}$
middle extension	a^2	$\frac{3a}{2}$	$\frac{5a}{2}$

Because the density is constant and the material is the same, we can use areas as weights.

The total area is

$$A = 5a^2 + 3a^2 + a^2 = 9a^2$$

(i) If the centre of mass is $(\bar{x}, \bar{y})$, then the distance from OS is $\bar{x}$.

$$\begin{aligned} A\bar{x} &= 5a^2\left(\frac{a}{2}\right) + 3a^2\left(\frac{5a}{2}\right) + a^2\left(\frac{3a}{2}\right) \\ &= \frac{5a^3}{2} + \frac{15a^3}{2} + \frac{3a^3}{2} \\ &= \frac{23a^3}{2} \end{aligned}$$

So

$$\bar{x} = \frac{\frac{23a^3}{2}}{9a^2} = \frac{23a}{18}$$

Hence the distance of the centre of mass from OS is $\dfrac{23a}{18}$.

(ii) The distance from OA is $\bar{y}$.

$$\begin{aligned} A\bar{y} &= 5a^2\left(\frac{5a}{2}\right) + 3a^2\left(\frac{9a}{2}\right) + a^2\left(\frac{5a}{2}\right) \\ &= \frac{25a^3}{2} + \frac{27a^3}{2} + \frac{5a^3}{2} \\ &= \frac{57a^3}{2} \end{aligned}$$

So

$$\bar{y} = \frac{\frac{57a^3}{2}}{9a^2} = \frac{57a}{18} = \frac{19a}{6}$$

Hence the distance of the centre of mass from OA is $\dfrac{19a}{6}$.

(b) Let the centre of mass be G.

When the lamina is freely suspended from O, it hangs with G vertically below O. So the downward vertical is the line OG.

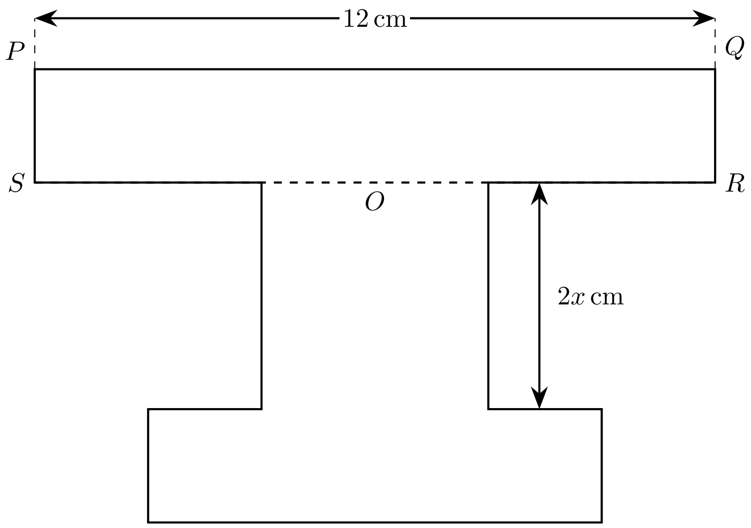
Therefore α is the angle between OS and OG. Using the coordinates of G found in part (a),

$$\tan\alpha = \frac{\text{distance of } G \text{ from } OS}{\text{distance of } G \text{ from } OA} = \frac{\bar{x}}{\bar{y}}$$

Hence

$$\begin{aligned} \tan\alpha &= \frac{\frac{23a}{18}}{\frac{19a}{6}} \\ &= \frac{23a}{18} \times \frac{6}{19a} \\ &= \frac{23}{57} \end{aligned}$$

So the exact value is $\tan\alpha = \dfrac{23}{57}$.



4. A template T consists of a uniform plane lamina made from three rectangles, as shown in the diagram. The upper rectangle $PQRS$ has $PQ = 12$ cm and $PS = 2$ cm. The point O is the midpoint of SR. A rectangle of width 4 cm and height $2x$ cm is attached centrally below SR, and a rectangle of width 8 cm and height 2 cm is attached centrally below this rectangle. The whole template is symmetrical about the line through O perpendicular to SR, where $x > 1$.

(a) Show that the centre of mass of T is a distance

$$\frac{x^2 + 4x - 1}{x + 5} \text{ cm}$$

below SR. **[7]**

The template T is freely suspended from the point P and hangs in equilibrium.

Given that $x = 2$ and that θ is the angle that PQ makes with the horizontal,

(b) show that

$$\tan\theta = \frac{42}{25}$$

[4]

Solution

(a) Let the centre of mass be G.

Since the template is symmetrical about the line through O perpendicular to SR, the point G lies on this line.

Take distances from SR, positive downwards.

Because the density is constant and the materials are the same, we can use areas as weights in

$$A\bar{y} = A_1y_1 + A_2y_2 + A_3y_3.$$

For the three rectangles:

$$\begin{aligned}
\text{upper rectangle: } & A_1 = 12 \times 2 = 24, & y_1 &= -1 \\
\text{middle rectangle: } & A_2 = 4 \times 2x = 8x, & y_2 &= x \\
\text{lower rectangle: } & A_3 = 8 \times 2 = 16, & y_3 &= 2x + 1
\end{aligned}$$

So

$$\begin{aligned}(24+8x+16)\bar{y} &= 24(-1)+8x(x)+16(2x+1)\\ &= -24+8x^2+32x+16\\ &= 8x^2+32x-8\\ &= 8(x^2+4x-1).\end{aligned}$$

Also,

$$24+8x+16 = 8x+40 = 8(x+5).$$

Hence

$$\bar{y} = \frac{8(x^2+4x-1)}{8(x+5)} = \frac{x^2+4x-1}{x+5}.$$

Since $x > 1$, this value is positive, so G is below SR.

Therefore the centre of mass is

$$\frac{x^2+4x-1}{x+5}\,\text{cm}$$

below SR.

(b) From part (a), when $x = 2$,

$$OG = \frac{2^2+4(2)-1}{2+5} = \frac{11}{7}\,\text{cm}.$$

So G is $\frac{11}{7}$ cm below SR on the line of symmetry through O.

The horizontal distance from P to this line is half of PQ, so

$$\text{horizontal distance } = \frac{12}{2} = 6\,\text{cm}.$$

The vertical distance from P down to G is

$$PS + OG = 2 + \frac{11}{7} = \frac{25}{7}\,\text{cm}.$$

Let α be the angle between PQ and PG. Then

$$\tan\alpha = \frac{25/7}{6} = \frac{25}{42}.$$

When the template hangs freely from P, the centre of mass is vertically below P, so PG is vertical. Hence PQ makes an angle θ with the horizontal where

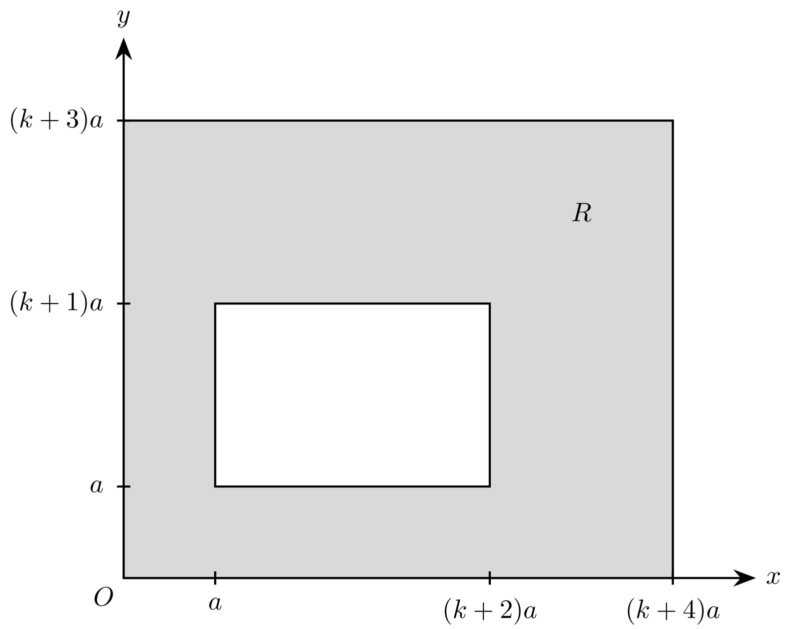
$$\theta = 90^\circ - \alpha.$$

Therefore

$$\tan\theta = \tan(90^\circ-\alpha) = \frac{1}{\tan\alpha} = \frac{42}{25}.$$

So

$$\tan\theta = \frac{42}{25}.$$



5. A uniform lamina R is bounded externally by the rectangle with vertices $(0,0)$, $((k+4)a, 0)$, $((k+4)a, (k+3)a)$ and $(0, (k+3)a)$, where $k > 0$.

A rectangular hole with vertices (a, a), $((k+2)a, a)$, $((k+2)a, (k+1)a)$ and $(a, (k+1)a)$ is removed, as shown in the diagram.

Determine the range of values of k for which the centre of mass of R lies in the hole. **[10]**

Solution

Let the outer rectangle be R_1 and the rectangular hole be R_2.
Because the density is constant and the materials are the same, areas can be used as weights.
For R_1,

$$A_1 = (k+4)(k+3)a^2, \qquad (x_1, y_1) = \left(\frac{(k+4)a}{2}, \frac{(k+3)a}{2}\right)$$

For the hole R_2,

$$A_2 = \big((k+2)a - a\big)\big((k+1)a - a\big) = k(k+1)a^2$$

and its centre is

$$(x_2, y_2) = \left(\frac{a + (k+2)a}{2}, \frac{a + (k+1)a}{2}\right) = \left(\frac{(k+3)a}{2}, \frac{(k+2)a}{2}\right)$$

So the area of the remaining lamina R is

$$A = A_1 - A_2 = \big((k+4)(k+3) - k(k+1)\big)a^2 = 6(k+2)a^2$$

Using subtraction of moments,

$$A\bar{x} = A_1x_1 - A_2x_2, \qquad A\bar{y} = A_1y_1 - A_2y_2$$

Hence

$$\begin{aligned}
\bar{x} &= \frac{(k+4)(k+3)a^2 \cdot \frac{(k+4)a}{2} - k(k+1)a^2 \cdot \frac{(k+3)a}{2}}{6(k+2)a^2} \\
&= \frac{a(k+3)\big((k+4)^2 - k(k+1)\big)}{12(k+2)} \\
&= \frac{a(k+3)(7k+16)}{12(k+2)}
\end{aligned}$$

and

$$\begin{aligned}\bar{y} &= \frac{(k+4)(k+3)a^2 \cdot \frac{(k+3)a}{2} - k(k+1)a^2 \cdot \frac{(k+2)a}{2}}{6(k+2)a^2} \\ &= \frac{a\big((k+4)(k+3)^2 - k(k+1)(k+2)\big)}{12(k+2)} \\ &= \frac{a(7k^2+31k+36)}{12(k+2)}\end{aligned}$$

For the centre of mass to lie inside the hole, we need

$$a < \bar{x} < (k+2)a, \qquad a < \bar{y} < (k+1)a$$

Since $a > 0$ and $k > 0$, we can divide through by a.
First, for the x-coordinate,

$$\begin{aligned}\bar{x} > a &\iff \frac{(k+3)(7k+16)}{12(k+2)} > 1 \\ &\iff (k+3)(7k+16) > 12(k+2) \\ &\iff 7k^2+25k+24 > 0\end{aligned}$$

which is true for all $k > 0$.
Also,

$$\begin{aligned}\bar{x} < (k+2)a &\iff \frac{(k+3)(7k+16)}{12(k+2)} < k+2 \\ &\iff (k+3)(7k+16) < 12(k+2)^2 \\ &\iff 5k^2+11k > 0\end{aligned}$$

which is also true for all $k > 0$.
Now for the y-coordinate,

$$\begin{aligned}\bar{y} > a &\iff \frac{7k^2+31k+36}{12(k+2)} > 1 \\ &\iff 7k^2+31k+36 > 12(k+2) \\ &\iff 7k^2+19k+12 > 0\end{aligned}$$

which is true for all $k > 0$.
Finally,

$$\begin{aligned}\bar{y} < (k+1)a &\iff \frac{7k^2+31k+36}{12(k+2)} < k+1 \\ &\iff 7k^2+31k+36 < 12(k+2)(k+1) \\ &\iff 5k^2+5k-12 > 0\end{aligned}$$

Solving $5k^2+5k-12=0$,

$$k = \frac{-5 \pm \sqrt{25+240}}{10} = \frac{-5 \pm \sqrt{265}}{10}$$

Since the quadratic opens upwards, $5k^2+5k-12 > 0$ when

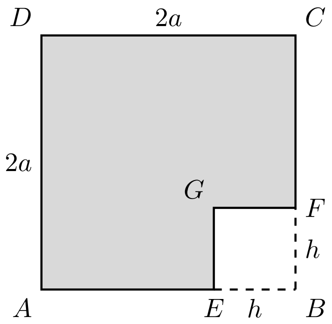
$$k < \frac{-5-\sqrt{265}}{10} \quad \text{or} \quad k > \frac{-5+\sqrt{265}}{10}$$

But $k > 0$, so the only possible range is

$$k > \frac{\sqrt{265}-5}{10}$$

Therefore the centre of mass of R lies in the hole when

$$\boxed{k > \frac{\sqrt{265}-5}{10}} \approx 1.13$$



6. A uniform lamina is formed by removing the square $EBGF$ from a square lamina $ABCD$, where $AB = BC = CD = DA = 2a$ and $EB = BF = h$, with $0 < h < 2a$ (see diagram).

(a) Find the distance of the centre of mass of the remaining lamina from AD and from AB, giving your answers in terms of a and h. **[5]**

The lamina is placed vertically on its edge AE on a horizontal plane.

(b) Find, in terms of a, the set of values of h for which the lamina remains in equilibrium. **[3]**

Solution

(a) Let the centre of mass of the remaining lamina be $(\bar{x}, \bar{y})$, where $\bar{x}$ is the distance from AD and $\bar{y}$ is the distance from AB.

Because the density is constant and the materials are the same:

$$\text{area of } ABCD = 4a^2, \qquad \text{centre } (a, a)$$

$$\text{area of removed square } EBGF = h^2, \qquad \text{centre } \left(2a - \frac{h}{2}, \frac{h}{2}\right)$$

So the area of the remaining lamina is

$$4a^2 - h^2$$

For the distance from AD, use areas as weights:

$$(4a^2 - h^2)\bar{x} = 4a^2(a) - h^2\left(2a - \frac{h}{2}\right)$$

$$\begin{aligned}\bar{x} &= \frac{4a^3 - 2ah^2 + \frac{1}{2}h^3}{4a^2 - h^2} \\ &= \frac{a(4a^2 - h^2) - \frac{1}{2}h^2(2a - h)}{(2a - h)(2a + h)} \\ &= a - \frac{h^2}{2(2a + h)}\end{aligned}$$

For the distance from AB:

$$(4a^2 - h^2)\bar{y} = 4a^2(a) - h^2\left(\frac{h}{2}\right)$$

$$\begin{aligned}\bar{y} &= \frac{4a^3 - \frac{1}{2}h^3}{4a^2 - h^2} \\ &= \frac{a(4a^2 - h^2) + \frac{1}{2}h^2(2a - h)}{(2a - h)(2a + h)} \\ &= a + \frac{h^2}{2(2a + h)}\end{aligned}$$

Therefore the centre of mass is at distance

$$a - \frac{h^2}{2(2a+h)}$$

from AD, and at distance

$$a + \frac{h^2}{2(2a+h)}$$

from AB

(b) When the lamina is placed vertically on edge AE, it will remain in equilibrium if the vertical line through the centre of mass meets the base AE.

Now

$$AE = AB - EB = 2a - h$$

In this position, the horizontal position of the centre of mass along the base is $\bar{x}$. So we require

$$0 \leq \bar{x} \leq 2a - h$$

Since $\bar{x} > 0$, it is enough to solve

$$a - \frac{h^2}{2(2a+h)} \leq 2a - h$$

$$\begin{aligned} -h^2 &\leq 2(2a+h)(a-h) \\ -h^2 &\leq 4a^2 - 2ah - 2h^2 \\ h^2 + 2ah - 4a^2 \leq 0 \end{aligned}$$

The roots of $h^2 + 2ah - 4a^2 = 0$ are

$$h = \frac{-2a \pm \sqrt{4a^2 + 16a^2}}{2} = -a \pm a\sqrt{5}$$

So

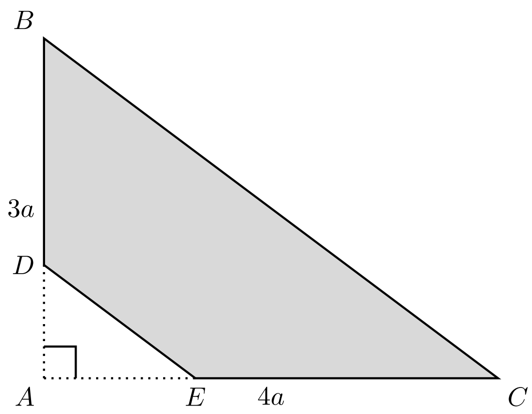
$$-a(1+\sqrt{5}) \leq h \leq a(\sqrt{5}-1)$$

Given that $0 < h < 2a$, this reduces to

$$0 < h \leq a(\sqrt{5}-1)$$

Therefore the lamina remains in equilibrium for

$$0 < h \leq a(\sqrt{5}-1)$$



7. A uniform lamina is formed from the right-angled triangle ABC by removing the triangle ADE, as shown in the diagram. The triangle ABC has $AB = 3a$, $AC = 4a$ and $\angle BAC = 90°$. The point D lies on AB and satisfies $AD : AB = 1 : k$. The point E lies on AC, and DE is parallel to BC.

(a) Taking A as the origin, with AC as the x-axis and AB as the y-axis, find the coordinates of the centre of mass of the remaining lamina $DBCE$ in terms of a and k. **[4]**

When the lamina $DBCE$ is freely suspended from C, the edge CE makes an angle θ with the downward vertical, where $\tan\theta = \dfrac{21}{52}$.

(b) Find the value of k. **[3]**

Solution

(a) Let the centre of mass of the remaining lamina be $G(\bar{x}, \bar{y})$.

Since

$$\frac{AD}{AB} = \frac{1}{k}$$

we have

$$AD = \frac{3a}{k}$$

so

$$D = \left(0, \frac{3a}{k}\right)$$

Also, $DE \parallel BC$, so triangles ADE and ABC are similar. Hence

$$\frac{AE}{AC} = \frac{AD}{AB} = \frac{1}{k}$$

so

$$AE = \frac{4a}{k}$$

and therefore

$$E = \left(\frac{4a}{k}, 0\right)$$

For a uniform triangular lamina, the centre of mass is at the centroid, so it is the average of the vertex coordinates.

For $\triangle ABC$:

$$A_1 = \frac{1}{2}(3a)(4a) = 6a^2$$

and its centre of mass is

$$\left(\frac{0+0+4a}{3}, \frac{0+3a+0}{3}\right) = \left(\frac{4a}{3}, a\right)$$

For $\triangle ADE$:

$$A_2 = \frac{1}{2}\left(\frac{3a}{k}\right)\left(\frac{4a}{k}\right) = \frac{6a^2}{k^2}$$

and its centre of mass is

$$\left(\frac{0+0+\frac{4a}{k}}{3}, \frac{0+\frac{3a}{k}+0}{3}\right) = \left(\frac{4a}{3k}, \frac{a}{k}\right)$$

Because the density is constant and the material is the same,

$$A_R\bar{x} = A_1x_1 - A_2x_2, \qquad A_R\bar{y} = A_1y_1 - A_2y_2$$

where A_R is the area of the remaining lamina.

Now

$$A_R = 6a^2 - \frac{6a^2}{k^2} = 6a^2\left(1 - \frac{1}{k^2}\right)$$

So

$$\begin{aligned}
\bar{x} &= \frac{6a^2 \cdot \frac{4a}{3} - \frac{6a^2}{k^2} \cdot \frac{4a}{3k}}{6a^2\left(1 - \frac{1}{k^2}\right)} \\
&= \frac{\frac{4a}{3}\left(1 - \frac{1}{k^3}\right)}{1 - \frac{1}{k^2}} \\
&= \frac{4a}{3} \cdot \frac{k^3 - 1}{k(k^2 - 1)} \\
&= \frac{4a}{3} \cdot \frac{(k-1)(k^2+k+1)}{k(k-1)(k+1)} \\
&= \frac{4a(k^2+k+1)}{3k(k+1)}
\end{aligned}$$

and

$$\begin{aligned}
\bar{y} &= \frac{6a^2 \cdot a - \frac{6a^2}{k^2} \cdot \frac{a}{k}}{6a^2\left(1 - \frac{1}{k^2}\right)} \\
&= \frac{a\left(1 - \frac{1}{k^3}\right)}{1 - \frac{1}{k^2}} \\
&= a \cdot \frac{k^3 - 1}{k(k^2 - 1)} \\
&= \frac{a(k^2+k+1)}{k(k+1)}
\end{aligned}$$

So the centre of mass of $DBCE$ is

$$\left(\frac{4a(k^2+k+1)}{3k(k+1)}, \frac{a(k^2+k+1)}{k(k+1)}\right)$$

(b) When the lamina is suspended from C, the centre of mass lies vertically below C. So the line CG is the downward vertical.

The angle between CE and CG is unchanged when the lamina rotates. In the original coordinates, CE is horizontal, so

$$\tan\theta = \frac{\bar{y}}{4a - \bar{x}}$$

Using the result from part (a),

$$\begin{aligned}
4a - \bar{x} &= 4a - \frac{4a(k^2+k+1)}{3k(k+1)} \\
&= \frac{4a\big(3k(k+1) - (k^2+k+1)\big)}{3k(k+1)} \\
&= \frac{4a(2k^2+2k-1)}{3k(k+1)}
\end{aligned}$$

Hence

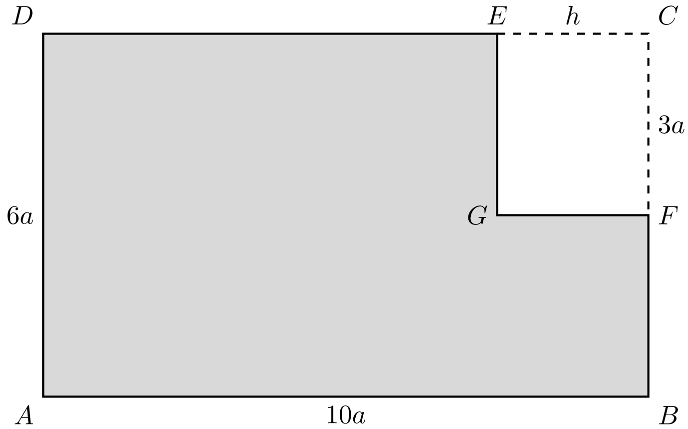
$$\begin{aligned}\tan\theta &= \frac{\frac{a(k^2+k+1)}{k(k+1)}}{\frac{4a(2k^2+2k-1)}{3k(k+1)}}\\ &= \frac{3(k^2+k+1)}{4(2k^2+2k-1)}\end{aligned}$$

Given that $\tan\theta = \dfrac{21}{52}$,

$$\begin{aligned}\frac{3(k^2+k+1)}{4(2k^2+2k-1)} &= \frac{21}{52}\\ 156(k^2+k+1) &= 84(2k^2+2k-1)\\ 13(k^2+k+1) &= 7(2k^2+2k-1)\\ 13k^2+13k+13 &= 14k^2+14k-7\\ k^2+k-20 &= 0\\ (k-4)(k+5) &= 0\end{aligned}$$

Since $k > 1$, it follows that

$$k = 4$$



8. $ABCD$ is a uniform rectangular lamina with $AB = 10a$ and $BC = 6a$. Point E lies on DC with $CE = h$, where $0 < h < 10a$, and point F lies on BC with $CF = 3a$. The rectangle $CEGF$ is removed, where G is the fourth vertex of the rectangle (see diagram).

(a) Show that the distance of the centre of mass of the remaining lamina from AD is

$$\frac{h^2 - 20ah + 200a^2}{2(20a - h)}$$

and find a corresponding expression for the distance of the centre of mass from AB. **[5]**

When the lamina is suspended from the point D, the edge DA makes an angle θ with the downward vertical, where $\tan\theta = \frac{13}{12}$.

(b) Find, in terms of a, the two possible values of h. **[3]**

Solution

(a) Take A as the origin, with AB as the x-axis and AD as the y-axis.

Then the whole rectangle $ABCD$ has area

$$60a^2$$

and centre $(5a, 3a)$.

The removed rectangle $CEGF$ has width h and height $3a$, so its area is

$$3ah$$

and its centre is

$$\left(10a - \frac{h}{2}, \frac{9a}{2}\right)$$

Because the density is constant and the material is the same, areas can be used as weights.

So the area of the remaining lamina is

$$60a^2 - 3ah = 3a(20a - h)$$

If the centre of mass of the remaining lamina is $(\bar{x}, \bar{y})$, then

$$(60a^2 - 3ah)\bar{x} = 60a^2(5a) - 3ah\left(10a - \frac{h}{2}\right)$$

Hence

$$\begin{aligned}
3a(20a-h)\bar{x} &= 300a^3 - 30a^2h + \frac{3}{2}ah^2 \\
\bar{x} &= \frac{300a^3 - 30a^2h + \frac{3}{2}ah^2}{3a(20a-h)} \\
&= \frac{100a^2 - 10ah + \frac{1}{2}h^2}{20a-h} \\
&= \frac{h^2 - 20ah + 200a^2}{2(20a-h)}
\end{aligned}$$

So the distance from AD is

$$\frac{h^2 - 20ah + 200a^2}{2(20a-h)}$$

Similarly,

$$(60a^2 - 3ah)\bar{y} = 60a^2(3a) - 3ah\left(\frac{9a}{2}\right)$$

So

$$\begin{aligned}
3a(20a-h)\bar{y} &= 180a^3 - \frac{27}{2}a^2h \\
\bar{y} &= \frac{180a^3 - \frac{27}{2}a^2h}{3a(20a-h)} \\
&= \frac{120a^2 - 9ah}{2(20a-h)} \\
&= \frac{3a(40a-3h)}{2(20a-h)}
\end{aligned}$$

Therefore the corresponding distance from AB is

$$\frac{3a(40a-3h)}{2(20a-h)}$$

(b) When the lamina is suspended from D, the centre of mass lies vertically below D.

From part (a), the horizontal distance of the centre of mass from D is $\bar{x}$, and the vertical distance below D is

$$\begin{aligned}
6a - \bar{y} &= 6a - \frac{3a(40a-3h)}{2(20a-h)} \\
&= \frac{12a(20a-h) - 3a(40a-3h)}{2(20a-h)} \\
&= \frac{3a(40a-h)}{2(20a-h)}
\end{aligned}$$

Hence

$$\tan\theta = \frac{\bar{x}}{6a-\bar{y}} = \frac{13}{12}$$

Substituting the expressions from part (a):

$$\begin{aligned}
\frac{\frac{h^2 - 20ah + 200a^2}{2(20a-h)}}{\frac{3a(40a-h)}{2(20a-h)}} &= \frac{13}{12} \\
\frac{h^2 - 20ah + 200a^2}{3a(40a-h)} &= \frac{13}{12}
\end{aligned}$$

So

$$\begin{aligned}12(h^2-20ah+200a^2) &= 39a(40a-h)\\ 12h^2-240ah+2400a^2 &= 1560a^2-39ah\\ 12h^2-201ah+840a^2 &= 0\end{aligned}$$

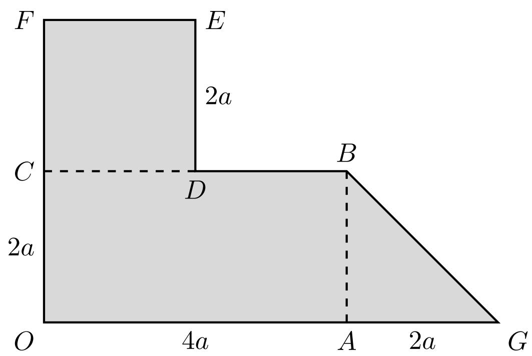
Solving this quadratic for h:

$$\begin{aligned}h &= \frac{201a \pm \sqrt{201^2a^2-4(12)(840)a^2}}{24}\\ &= \frac{201a \pm 9a}{24}\end{aligned}$$

Hence

$$h = 8a \quad \text{or} \quad h = \frac{35a}{4}$$

Both values satisfy $0 < h < 10a$.



9. The uniform plane lamina shown in the diagram is formed from the rectangle $OABC$, the square $CDEF$ and the triangle ABG. The rectangle has dimensions $4a$ by $2a$, the square has side $2a$ and $AG = 2a$. Take O as the origin, with OA as the positive x-axis and OC as the positive y-axis.

In part (a) you may use, without proof, any of the centre of mass formulae given in the formulae booklet.

(a) Show that the distance of the centre of mass of triangle ABG from AB is $\dfrac{2a}{3}$. **[3]**

(b) Find the coordinates of the centre of mass of the lamina. **[4]**

The lamina is freely suspended from F and hangs in equilibrium with FC at an angle $\theta°$ to the downward vertical.

(c) Find the value of θ. **[4]**

Solution

(a) Let the centre of mass of triangle ABG be T.

For a triangular lamina, the centroid has coordinates equal to the mean of the coordinates of its vertices.

$$T = \left(\frac{4a + 4a + 6a}{3}, \frac{0 + 2a + 0}{3}\right) = \left(\frac{14a}{3}, \frac{2a}{3}\right)$$

The line AB is vertical, with equation $x = 4a$. So the perpendicular distance of T from AB is

$$\frac{14a}{3} - 4a = \frac{14a - 12a}{3} = \frac{2a}{3}$$

Hence the distance of the centre of mass of triangle ABG from AB is $\dfrac{2a}{3}$.

(b) Let the centre of mass of the whole lamina be $(\bar{x}, \bar{y})$.

Because the density is constant and the materials are the same, areas can be used as weights.

For the three parts:

Shape	Area	Centre of mass
Rectangle $OABC$	$8a^2$	$(2a, a)$
Square $CDEF$	$4a^2$	$(a, 3a)$
Triangle ABG	$2a^2$	$\left(\frac{14a}{3}, \frac{2a}{3}\right)$

So the total area is

$$8a^2 + 4a^2 + 2a^2 = 14a^2$$

Using $A\bar{x} = A_1x_1 + A_2x_2 + A_3x_3$,

$$\begin{aligned} 14a^2\bar{x} &= 8a^2(2a) + 4a^2(a) + 2a^2\left(\frac{14a}{3}\right) \\ &= 16a^3 + 4a^3 + \frac{28a^3}{3} \\ &= \frac{88a^3}{3} \end{aligned}$$

Hence

$$\bar{x} = \frac{88a^3/3}{14a^2} = \frac{44a}{21}$$

Using $A\bar{y} = A_1y_1 + A_2y_2 + A_3y_3$,

$$\begin{aligned} 14a^2\bar{y} &= 8a^2(a) + 4a^2(3a) + 2a^2\left(\frac{2a}{3}\right) \\ &= 8a^3 + 12a^3 + \frac{4a^3}{3} \\ &= \frac{64a^3}{3} \end{aligned}$$

Hence

$$\bar{y} = \frac{64a^3/3}{14a^2} = \frac{32a}{21}$$

So the coordinates of the centre of mass of the lamina are

$$\left(\frac{44a}{21}, \frac{32a}{21}\right)$$

(c) Let the centre of mass be M.

When the lamina is freely suspended from F, the centre of mass lies vertically below F in equilibrium. So the downward vertical is along the line FM.

Now

$$F = (0, 4a), \qquad M = \left(\frac{44a}{21}, \frac{32a}{21}\right)$$

Hence

$$\overrightarrow{FM} = \left(\frac{44a}{21}, \frac{32a}{21} - 4a\right) = \left(\frac{44a}{21}, -\frac{52a}{21}\right)$$

The line FC is vertical in the lamina, so after turning through angle θ, it makes angle θ with the downward vertical. Therefore θ is the angle between FC and FM.

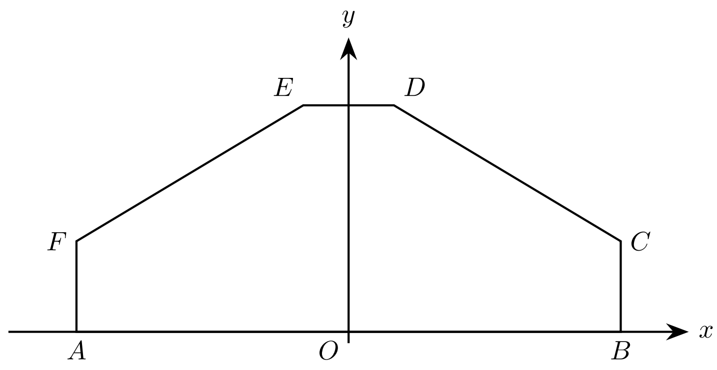
Using the horizontal and vertical components of FM,

$$\tan\theta = \frac{44a/21}{52a/21} = \frac{11}{13}$$

So

$$\theta = \arctan\left(\frac{11}{13}\right) \approx 40.2^\circ$$

Hence the value of θ is $\arctan\left(\frac{11}{13}\right)$, which is approximately 40.2°.



10. A uniform lamina $ABCDEF$ is in the shape of the hexagon shown in the diagram. In a coordinate system with origin O, the vertices have coordinates $A(-120, 0)$, $B(120, 0)$, $C(120, 40)$, $D(20, 100)$, $E(-20, 100)$ and $F(-120, 40)$. The units of the axes are centimetres.

The centre of mass of the lamina lies at $(\bar{x}, \bar{y})$.

(a) Use symmetry to write down $\bar{x}$ and show that $\bar{y} = 40$. **[5]**

The lamina is placed horizontally so that it rests on three supports, whose points of contact are at D, E and G, where G lies on the edge AB and has coordinates $(g, 0)$.

(b) Determine the range of values of g for the lamina to rest in equilibrium. **[3]**

It is now given that $g = 8$, and that the lamina has weight 100N.

(c) Determine the forces exerted on the lamina by each of the supports at D, E and G. **[4]**

Solution

(a) The hexagon is symmetric about the y-axis, so

$$\bar{x} = 0$$

To find $\bar{y}$, take the lamina as a 240×100 rectangle with two identical right-angled triangles removed. Because the density is constant and the material is the same, areas can be used as weights.

For the rectangle,

$$A_1 = 240 \times 100 = 24000, \qquad y_1 = 50$$

For each removed triangle,

$$A_2 = \frac{1}{2} \times 100 \times 60 = 3000$$

and its centroid is $\frac{1}{3}$ of the height down from the top edge $y = 100$, so

$$y_2 = 100 - \frac{60}{3} = 80$$

Hence the area of the hexagon is

$$A = 24000 - 2(3000) = 18000$$

Now take moments about the x-axis:

$$A\bar{y} = A_1 y_1 - 2A_2 y_2$$

so

$$\begin{aligned}18000\bar{y} &= 24000(50) - 2(3000)(80)\\ &= 1200000 - 480000\\ &= 720000\end{aligned}$$

Therefore

$$\bar{y} = \frac{720000}{18000} = 40$$

So the centre of mass is

$$(\bar{x}, \bar{y}) = (0, 40)$$

(b) For the lamina to rest in equilibrium, the vertical line through the centre of mass must pass inside or on the triangle formed by the three supports, that is triangle DEG.

From part (a), the centre of mass is at $(0, 40)$. So we need the point $(0, 40)$ to lie inside or on triangle DEG.

First find where the line DG has height $y = 40$. Using coordinates $D(20, 100)$ and $G(g, 0)$,

$$\frac{40-0}{100-0} = \frac{x-g}{20-g}$$

Hence

$$\begin{aligned}\frac{2}{5} &= \frac{x-g}{20-g}\\ x-g &= \frac{2}{5}(20-g)\\ x &= 8 + \frac{3g}{5}\end{aligned}$$

Now do the same for line EG, where $E(-20, 100)$:

$$\frac{40-0}{100-0} = \frac{x-g}{-20-g}$$

Hence

$$\begin{aligned}\frac{2}{5} &= \frac{x-g}{-20-g}\\ x-g &= \frac{2}{5}(-20-g)\\ x &= -8 + \frac{3g}{5}\end{aligned}$$

So at height $y = 40$, triangle DEG runs from

$$x = -8 + \frac{3g}{5} \quad \text{to} \quad x = 8 + \frac{3g}{5}$$

For $(0, 40)$ to lie on or between these values,

$$-8 + \frac{3g}{5} \le 0 \le 8 + \frac{3g}{5}$$

This gives

$$g \le \frac{40}{3} \qquad \text{and} \qquad g \ge -\frac{40}{3}$$

Therefore the required range is

$$-\frac{40}{3} \le g \le \frac{40}{3}$$

(c) Let the upward support forces at D, E and G be R_D, R_E and R_G respectively.

Since the lamina is in equilibrium, the resultant of these three upward forces is equal to the weight 100 N, and it acts through the centre of mass $(0, 40)$.

So

$$R_D + R_E + R_G = 100$$

Using the y-coordinates of the supports,

$$100 \times 40 = 100R_D + 100R_E + 0 \cdot R_G$$

so

$$R_D + R_E = 40$$

Hence

$$R_G = 100 - 40 = 60$$

Now use the x-coordinates of the supports. Since the resultant acts through $x = 0$,

$$100 \times 0 = 20R_D + (-20)R_E + 8R_G$$

So

$$\begin{aligned} 20R_D - 20R_E + 8(60) &= 0 \\ 20R_D - 20R_E &= -480 \\ R_D - R_E &= -24 \end{aligned}$$

Now solve

$$R_D + R_E = 40, \qquad R_D - R_E = -24$$

Adding gives

$$2R_D = 16$$

so

$$R_D = 8$$

and then

$$R_E = 32$$

Therefore the forces exerted on the lamina are:

$$\text{at } D: \ 8\,\text{N upward}, \qquad \text{at } E: \ 32\,\text{N upward}, \qquad \text{at } G: \ 60\,\text{N upward}$$