

FURTHER MECHANICS / CENTRE OF MASS

Centre of Mass of Composite Solids

Worked solutions

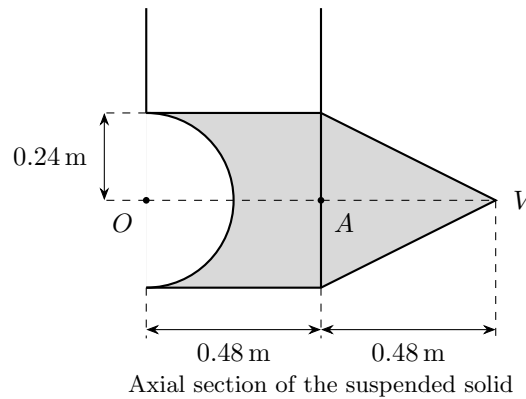
6 questions · 60 marks

Instructions

Unless a question states otherwise, take $g = 9.8 \text{ m s}^{-2}$ wherever a numerical value of g is required. Solutions also give final answers for $g = 10 \text{ m s}^{-2}$ where these differ.

Questions

Question 1	10 marks	2
Question 2	10 marks	4
Question 3	10 marks	6
Question 4	9 marks	8
Question 5	9 marks	10
Question 6	12 marks	12



1. A solid cylinder has radius 0.24 m and length 0.48 m. A hemispherical cavity of radius 0.24 m is cut into one end of the cylinder, with its circular opening coinciding with that end face. A solid cone of base radius 0.24 m and height 0.48 m is joined to the opposite end of the cylinder so that their circular faces coincide.

The resulting solid S has uniform density throughout. Let O be the centre of the opening of the cavity, A the centre of the cylinder-cone joint and V the vertex of the cone. The points O , A and V lie on the common axis, in that order. The centre of mass of S is G .

You may use the following results for uniform solids:

- A hemisphere of radius r has volume $\frac{2}{3}\pi r^3$ and its centre of mass is $\frac{3r}{8}$ from its plane face towards its curved surface.
- A cone of base radius r and height h has volume $\frac{1}{3}\pi r^2 h$ and its centre of mass is $\frac{h}{4}$ from its base towards its vertex.

- (i) Explain why G lies on the line OV . [1]
- (ii) Find the distance OG . [6]

The solid is suspended by two light vertical strings. One is attached to the uppermost point of the circular rim centred at O , and the other to the uppermost point of the circular joint centred at A . The solid hangs in equilibrium with OV horizontal, as shown in the diagram above.

- (iii) Given that the weight of S is W , find the tension in each string in terms of W . [3]

Solution

- (i) The solid has uniform density and is rotationally symmetric about OV , so its centre of mass lies on OV .
- (ii) Let $r = 0.24$ m and let the density be ρ . Both the cylinder's length and the cone's height are $2r$. Measure distances along the axis from O towards V .

The centre of mass of a uniform solid cylinder lies halfway along its axis. The centre of mass of a uniform solid hemisphere of radius R lies on its axis, a distance $\frac{3R}{8}$ from its plane face towards the curved surface. The centre of mass of a uniform solid cone of height H lies on its axis, a distance $\frac{H}{4}$ from its base towards its vertex.

Using mass = density $\times$ volume:

Component	Mass magnitude	Axial distance from O
Complete cylinder	$M_1 = \rho\pi r^2(2r) = 2\rho\pi r^3$	$x_1 = r$
Removed hemisphere	$M_2 = \frac{2}{3}\rho\pi r^3$	$x_2 = \frac{3r}{8}$
Cone	$M_3 = \frac{1}{3}\rho\pi r^2(2r) = \frac{2}{3}\rho\pi r^3$	$x_3 = 2r + \frac{2r}{4} = \frac{5r}{2}$

For a composite solid, the centre-of-mass coordinate is

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

Removed material is represented by negative mass. Here the removed hemisphere has signed mass $-M_2$, so

$$OG = \frac{M_1 x_1 - M_2 x_2 + M_3 x_3}{M_1 - M_2 + M_3}$$

Substituting and cancelling $\rho\pi r^3$:

$$\begin{aligned} OG &= \frac{2r - \frac{2}{3}\left(\frac{3r}{8}\right) + \frac{2}{3}\left(\frac{5r}{2}\right)}{2 - \frac{2}{3} + \frac{2}{3}} \\ &= \frac{r}{2} \left(2 - \frac{1}{4} + \frac{5}{3} \right) \\ &= \frac{41r}{24} \\ &= \frac{41(0.24)}{24} \\ &= 0.41 \text{ m} \end{aligned}$$

(iii) Let T_O and T_A be the tensions in the strings attached to the rims centred at O and A , respectively.

Since OV is horizontal and the strings are vertical, their horizontal distances from O are 0 and 0.48 m. The weight acts vertically through G , at horizontal distance 0.41 m from O .

Taking moments about O :

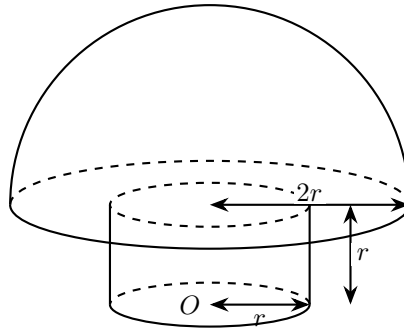
$$0.48T_A - 0.41W = 0$$

$$T_A = \frac{41W}{48}$$

Resolving vertically:

$$T_O + T_A = W$$

$$T_O = W - \frac{41W}{48} = \frac{7W}{48}$$



2. A composite solid S consists of a uniform solid cylinder of radius r and height r , and a uniform solid hemisphere of radius $2r$. Both components have the same density. The upper circular face of the cylinder is joined centrally to the plane face of the hemisphere, with their axes coincident and the curved surface of the hemisphere uppermost, as shown in the diagram above. The point O is the centre of the lower circular face of the cylinder.

- (a) Find the distance from O to the centre of mass of S , giving your answer as an exact multiple of r . [4]

The lower circular face of S rests on a rough plane. The coefficient of friction between S and the plane is μ . Initially the plane is horizontal. Its inclination θ to the horizontal is then increased slowly, with S remaining at rest until it either slides or begins to topple. No external force other than gravity and the contact force acts on S .

- (b) Find the angle at which S would begin to topple if sliding were prevented, giving your answer to the nearest 0.1° . Hence find the range of values of μ for which S slides before it begins to topple. [6]

Solution

- (a) By symmetry, the centre of mass G lies on the common axis. Let the common density be ρ . The masses of the cylinder and hemisphere are

$$m_c = \rho \pi r^2(r) = \rho \pi r^3$$

$$m_h = \rho \left(\frac{2}{3} \pi (2r)^3 \right) = \frac{16}{3} \rho \pi r^3$$

The centre of mass of a uniform solid cylinder lies halfway along its axis. The centre of mass of a uniform solid hemisphere of radius R lies on its axis, a distance $\frac{3R}{8}$ from its plane face towards the curved surface. Thus their distances above O are

$$z_c = \frac{r}{2}, \quad z_h = r + \frac{3(2r)}{8} = \frac{7r}{4}$$

For a composite solid, the centre-of-mass height is

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

Using the masses and heights above:

$$\begin{aligned}
 OG &= \frac{m_c z_c + m_h z_h}{m_c + m_h} \\
 &= \frac{\rho \pi r^3 \left(\frac{r}{2}\right) + \frac{16}{3} \rho \pi r^3 \left(\frac{7r}{4}\right)}{\rho \pi r^3 + \frac{16}{3} \rho \pi r^3} \\
 &= \frac{\left(\frac{1}{2} + \frac{28}{3}\right)r}{1 + \frac{16}{3}} \\
 &= \frac{59r}{38}
 \end{aligned}$$

- (b) Let M be the total mass and θ_t the inclination at impending toppling. The solid topples about the downhill edge P of its lower circular face, where the contact force acts at this instant.

Resolve the weight parallel and perpendicular to the plane. Their moment arms about P are OG and r , respectively. Taking moments about P :

$$(Mg \sin \theta_t) OG = (Mg \cos \theta_t) r$$

$$\frac{59r}{38} \sin \theta_t = r \cos \theta_t$$

$$\tan \theta_t = \frac{38}{59}$$

$$\theta_t = \arctan\left(\frac{38}{59}\right) \approx 32.8^\circ \quad (1 \text{ d.p.})$$

Before motion begins, let R be the normal reaction and F the frictional force acting up the plane. Resolving perpendicular and parallel to the plane gives

$$R = Mg \cos \theta, \quad F = Mg \sin \theta$$

At the inclination θ_s for impending sliding, friction is limiting, so

$$F = \mu R$$

$$Mg \sin \theta_s = \mu Mg \cos \theta_s$$

$$\tan \theta_s = \mu$$

Since $\tan \theta$ increases for $0 \leq \theta < 90^\circ$, sliding occurs before toppling when $\theta_s < \theta_t$. Equality would give simultaneous sliding and toppling. Hence, using the toppling angle found above and $\mu \geq 0$, the required range is

$$0 \leq \mu < \frac{38}{59}$$

3. A solid ornament consists of a cylinder of radius a and height a , surmounted by a cone of base radius a and height h . Their circular faces coincide, and both parts are made from the same material of uniform density. The point O is the centre of the lower circular face of the cylinder.

The centre of mass of the ornament is on its axis of symmetry, at a distance $\frac{9a}{10}$ above O .

- (i) Show that $h = 2a$. [4]

A hemispherical cavity of radius $\frac{a}{2}$ is then cut into the ornament through the lower face of the cylinder. The centre of the circular opening is O , and the cavity extends upwards into the cylinder.

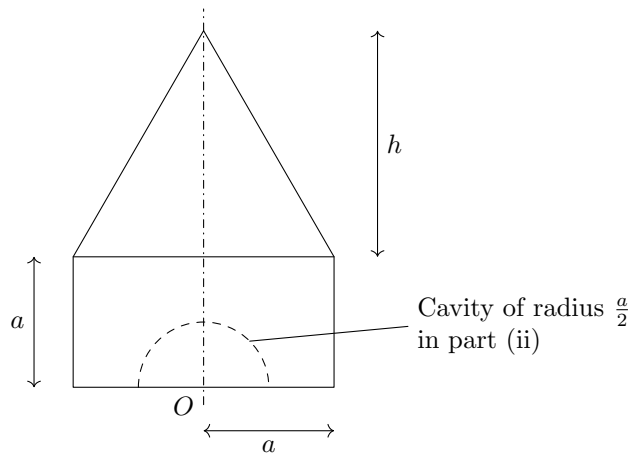
- (ii) Find the distance of the centre of mass of the remaining ornament from O . [4]

The remaining ornament is placed with its flat annular base on a rough plane. The inclination of the plane to the horizontal is slowly increased. The friction is sufficient to prevent sliding.

- (iii) Find the angle of inclination at which the ornament is on the point of toppling, giving your answer to the nearest degree. [2]

Solution

- (i) A schematic axial section is shown below, with the cavity introduced in part (ii) shown dashed:



Measure heights along the axis from O , and let the uniform density be ρ . For a composite solid, the centre-of-mass height is

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

The centre of mass of a uniform solid cylinder lies halfway along its axis. The centre of mass of a uniform solid cone of height H lies on its axis, a distance $\frac{H}{4}$ from its base towards its vertex. Thus the component heights are $\frac{a}{2}$ and $a + \frac{h}{4}$, respectively.

The cylinder and cone have masses $\rho\pi a^3$ and $\frac{1}{3}\rho\pi a^2 h$. Using the given centre of mass and cancelling $\rho\pi a^2$:

$$\frac{9a}{10}\left(a + \frac{h}{3}\right) = \frac{a^2}{2} + \frac{h}{3}\left(a + \frac{h}{4}\right)$$

$$54a^2 + 18ah = 30a^2 + 20ah + 5h^2$$

$$5h^2 + 2ah - 24a^2 = 0$$

$$(h - 2a)(5h + 12a) = 0$$

Since $h > 0$, the negative root is inadmissible, so

$$h = 2a$$

- (ii) The centre of mass of a uniform solid hemisphere of radius R lies on its axis, a distance $\frac{3R}{8}$ from its plane face towards the curved surface. Hence the centre of mass of the removed material is at height

$$\frac{3}{8}\left(\frac{a}{2}\right) = \frac{3a}{16}$$

Using $h = 2a$ from part (i), the original mass M and removed mass m are

$$M = \rho\left(\pi a^3 + \frac{1}{3}\pi a^2(2a)\right) = \frac{5}{3}\rho\pi a^3$$

$$m = \rho\left(\frac{2}{3}\pi\left(\frac{a}{2}\right)^3\right) = \frac{1}{12}\rho\pi a^3$$

By symmetry, the remaining centre of mass is still on the axis. Removed material is represented by negative mass. Using the general formula from part (i), with signed masses M and $-m$, its height $\bar{z}$ satisfies

$$(M - m)\bar{z} = M\left(\frac{9a}{10}\right) - m\left(\frac{3a}{16}\right)$$

$$\left(\frac{5}{3} - \frac{1}{12}\right)\bar{z} = \frac{5}{3}\left(\frac{9a}{10}\right) - \frac{1}{12}\left(\frac{3a}{16}\right)$$

$$\frac{19}{12}\bar{z} = \frac{95a}{64}$$

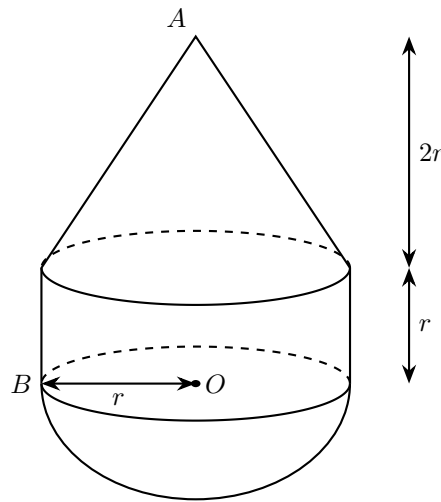
$$\bar{z} = \frac{15a}{16}$$

- (iii) Let the inclination be θ . At the point of toppling, the vertical line through the centre of mass passes through the downhill outer rim of the annular base. The right-angled triangle formed by O , the centre of mass and this rim point gives

$$\tan \theta = \frac{a}{\bar{z}} = \frac{a}{15a/16} = \frac{16}{15}$$

Therefore the required inclination is

$$\theta = \arctan\left(\frac{16}{15}\right) \approx 47^\circ \quad (\text{nearest degree})$$



4. A composite solid is made from a solid hemisphere, a solid cylinder and a solid right circular cone, all made from the same material of uniform density. Each component has radius r . The cylinder has height r and the cone has height $2r$.

The plane face of the hemisphere is joined to one circular face of the cylinder, and the base of the cone is joined to the other circular face of the cylinder. Their axes coincide, as shown in the diagram above. The centre of the common face of the hemisphere and cylinder is O , and B lies on its circumference, with $OB = r$.

- (a) Find the distance of the centre of mass of the composite solid from O , stating on which side of O it lies. [5]

The cone is now removed and replaced by a solid right circular cone of base radius r and height hr , where $h > 0$, made from the same material. The hemisphere and cylinder are unchanged. This new composite solid is suspended by a light string attached to B and hangs freely in equilibrium. The acute angle between OB and the vertical is 45° .

- (b) Find the exact value of h . [4]

Solution

- (a) Take the common axis as the x -axis, with origin O and positive direction towards A . By symmetry, the centre of mass lies on this axis. Let the common density be ρ .

The centre of mass of a uniform solid hemisphere of radius R lies on its axis, a distance $\frac{3R}{8}$ from its plane face towards the curved surface. The centre of mass of a uniform solid cylinder lies halfway along its axis. The centre of mass of a uniform solid cone of height H lies on its axis, a distance $\frac{H}{4}$ from its base towards its vertex. Using mass = density $\times$ volume gives

Component	Mass	Axial coordinate
Hemisphere	$\frac{2}{3}\rho\pi r^3$	$-\frac{3r}{8}$
Cylinder	$\rho\pi r^3$	$\frac{r}{2}$
Cone	$\frac{2}{3}\rho\pi r^3$	$r + \frac{2r}{4} = \frac{3r}{2}$

For a composite solid, the centre-of-mass coordinate is

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

Substituting the masses and positions:

$$\begin{aligned} \left(\frac{2}{3} + 1 + \frac{2}{3}\right) \rho \pi r^3 \bar{x} &= \frac{2}{3} \rho \pi r^3 \left(-\frac{3r}{8}\right) + \rho \pi r^3 \left(\frac{r}{2}\right) + \frac{2}{3} \rho \pi r^3 \left(\frac{3r}{2}\right) \\ \frac{7}{3} \rho \pi r^3 \bar{x} &= \left(-\frac{1}{4} + \frac{1}{2} + 1\right) \rho \pi r^4 \\ \bar{x} &= \frac{15r}{28} \end{aligned}$$

This coordinate is positive, so the required distance and side are

$$\frac{15r}{28}, \quad \text{on the cylinder and cone side of } O$$

- (b) Let the new centre of mass be G' , with signed axial coordinate x_1 . The replacement cone has mass $\frac{h}{3} \rho \pi r^3$ and its centre of mass is at $r + \frac{hr}{4}$.

Using the centre-of-mass formula from part (a):

$$\begin{aligned} \frac{5+h}{3} \rho \pi r^3 x_1 &= -\frac{1}{4} \rho \pi r^4 + \frac{1}{2} \rho \pi r^4 + \frac{h}{3} \rho \pi r^3 \left(r + \frac{hr}{4}\right) \\ &= \frac{\rho \pi r^4}{12} (3 + 4h + h^2) \end{aligned}$$

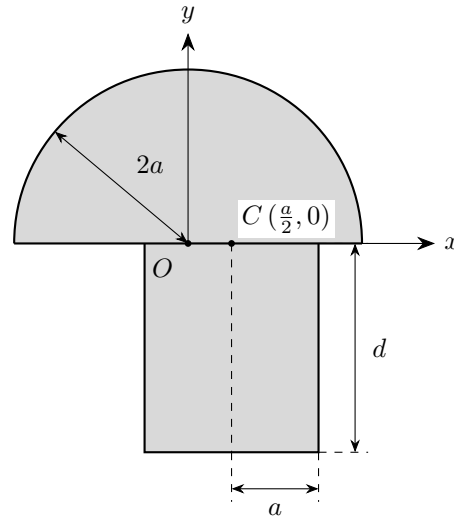
Since $h > 0$, the right-hand side is positive, so G' lies on the positive side of O .

When the solid hangs freely in equilibrium, G' is vertically below B . Also, OB is perpendicular to OG' , so the right-angled triangle BOG' gives

$$\tan 45^\circ = \frac{OG'}{OB} = \frac{x_1}{r} \implies x_1 = r$$

Substituting into the moment equation and cancelling $\rho \pi r^4$:

$$\begin{aligned} \frac{5+h}{3} &= \frac{3+4h+h^2}{12} \\ 4(5+h) &= 3+4h+h^2 \\ h^2 &= 17 \\ h &= \sqrt{17} \quad (h > 0) \end{aligned}$$



Section through the two axes

5. A composite solid consists of a solid hemisphere of radius $2a$ and a solid cylinder of radius a and height d , where $a > 0$ and $d > 0$. Both components have the same uniform density.

The cylinder is attached below the plane face of the hemisphere, as shown in the diagram above. The centre of this plane face is O . The axis of the cylinder meets the plane face at C , where $OC = \frac{a}{2}$. The x -axis is directed from O towards C , and the y -axis is directed into the hemisphere, perpendicular to its plane face.

- (a) Find the coordinates $(\bar{x}, \bar{y})$ of the centre of mass of the composite solid in terms of a and d . [5]

The solid is now placed with the lower circular face of the cylinder on a rough plane inclined at an angle θ to the horizontal, where $\tan \theta = \frac{1}{2}$. The positive x -direction points directly down the line of greatest slope, and the positive y -direction is perpendicular to the plane and away from it. The plane is sufficiently rough to prevent sliding.

- (b) Find the greatest value of d for which the solid can rest without toppling. Give your answer as an exact multiple of a . [4]

Solution

- (a) Let the common density be ρ , and let the masses of the hemisphere and cylinder be m_H and m_C . Using their standard volume formulae:

$$m_H = \rho \left(\frac{2}{3} \pi (2a)^3 \right) = \frac{16}{3} \rho \pi a^3, \quad m_C = \rho \pi a^2 d$$

The centre of mass of a uniform solid hemisphere of radius R lies on its axis, a distance $\frac{3R}{8}$ from its plane face towards the curved surface. The centre of mass of a uniform solid cylinder lies halfway along its axis. Thus the component centres are

$$H \left(0, \frac{3a}{4} \right), \quad K \left(\frac{a}{2}, -\frac{d}{2} \right)$$

For a composite solid, the centre-of-mass coordinates are

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

Substituting and cancelling the common factor $\rho\pi a^2$:

$$\bar{x} = \frac{\frac{16a}{3}(0) + d(\frac{a}{2})}{\frac{16a}{3} + d} = \frac{3ad}{2(16a + 3d)}$$

$$\bar{y} = \frac{\frac{16a}{3}(\frac{3a}{4}) + d(-\frac{d}{2})}{\frac{16a}{3} + d} = \frac{24a^2 - 3d^2}{2(16a + 3d)}$$

Hence the required coordinates are

$$(\bar{x}, \bar{y}) = \left(\frac{3ad}{2(16a + 3d)}, \frac{24a^2 - 3d^2}{2(16a + 3d)} \right)$$

- (b) The downhill rim of the cylinder has x -coordinate $\frac{a}{2} + a = \frac{3a}{2}$. Both component centres lie downhill of the uphill rim, so only downhill toppling is possible.

Resolve each weight into a component down the plane and a component perpendicular to the plane. About the downhill rim, the downhill components give overturning moments and the perpendicular components give restoring moments.

The hemisphere's centre is $d + \frac{3a}{4}$ above the plane and $\frac{3a}{2}$ uphill of the rim. The corresponding distances for the cylinder's centre are $\frac{d}{2}$ and a . For no toppling, therefore,

$$m_H g \sin \theta \left(d + \frac{3a}{4} \right) + m_C g \sin \theta \frac{d}{2} \leq m_H g \cos \theta \frac{3a}{2} + m_C g \cos \theta a$$

Substitute the masses from part (a), divide by $\rho\pi a^2 g \cos \theta$, and use $\tan \theta = \frac{1}{2}$:

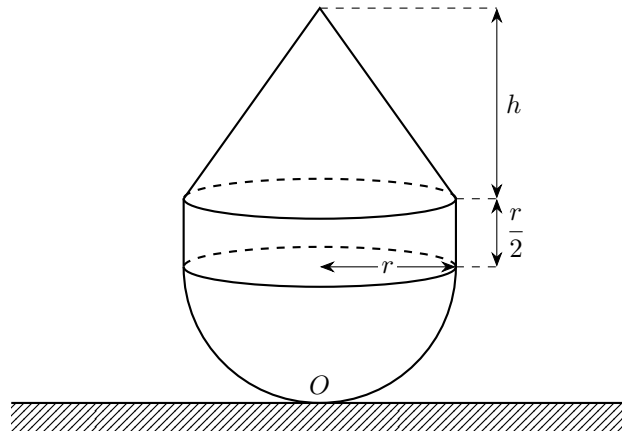
$$\frac{1}{2} \left[\frac{16a}{3} \left(d + \frac{3a}{4} \right) + \frac{d^2}{2} \right] \leq \frac{16a}{3} \cdot \frac{3a}{2} + ad$$

$$\frac{8ad}{3} + 2a^2 + \frac{d^2}{4} \leq 8a^2 + ad$$

$$3d^2 + 20ad - 72a^2 \leq 0$$

Since $a > 0$ and $d > 0$, the greatest permitted height is the positive root, where toppling is just about to occur:

$$d_{\max} = \frac{-20a + \sqrt{(20a)^2 + 4(3)(72a^2)}}{6} = \frac{2\sqrt{79} - 10}{3} a$$



6. A self-righting ornament is formed by joining a solid hemisphere, a solid cylinder and a solid cone, as shown in the diagram above. All three components are made from the same material of uniform density and have a common axis of symmetry.

The hemisphere has radius r . The cylinder has radius r and height $\frac{r}{2}$, and its lower face coincides with the plane face of the hemisphere. The cone has base radius r and height h , and its base coincides with the upper face of the cylinder.

The ornament rests on a horizontal plane at the lowest point O of the hemisphere, with its axis vertical and the cone uppermost.

You may use the following results for uniform solids:

- The centre of mass of a hemisphere of radius r is $\frac{3r}{8}$ from its plane face towards its curved surface.
- The centre of mass of a cone of height h is $\frac{h}{4}$ from its base towards its vertex.

- (a) Find, in terms of r and h , the distance d of the centre of mass of the ornament from O . Simplify your answer. [8]

- (b) Find the exact range of values of $\frac{h}{r}$ for which the ornament is in stable equilibrium in this position, so that it tends to return towards the vertical after a small tilt. [4]

Solution

- (a) By symmetry, the centre of mass lies on the common axis. Let ρ be the common density, and measure heights vertically upwards from O .

The centre of mass of a uniform solid hemisphere of radius R lies on its axis, a distance $\frac{3R}{8}$ from its plane face towards the curved surface. The centre of mass of a uniform solid cylinder lies halfway along its axis. The centre of mass of a uniform solid cone of height H lies on its axis, a distance $\frac{H}{4}$ from its base towards its vertex. Using mass = density $\times$ volume:

Component	Mass m_i	Height z_i above O
Hemisphere	$\rho\left(\frac{2}{3}\pi r^3\right)$	$r - \frac{3r}{8} = \frac{5r}{8}$
Cylinder	$\rho\left(\pi r^2 \times \frac{r}{2}\right)$	$r + \frac{r}{4} = \frac{5r}{4}$
Cone	$\rho\left(\frac{1}{3}\pi r^2 h\right)$	$r + \frac{r}{2} + \frac{h}{4} = \frac{3r}{2} + \frac{h}{4}$

For a composite solid, the centre-of-mass height is

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

Here $\bar{z} = d$.

Substituting and cancelling the common factor $\rho\pi r^2$:

$$\begin{aligned}
 d &= \frac{\frac{2r}{3} \times \frac{5r}{8} + \frac{r}{2} \times \frac{5r}{4} + \frac{h}{3} \left(\frac{3r}{2} + \frac{h}{4} \right)}{\frac{2r}{3} + \frac{r}{2} + \frac{h}{3}} \\
 &= \frac{\frac{5r^2}{12} + \frac{5r^2}{8} + \frac{rh}{2} + \frac{h^2}{12}}{\frac{7r+2h}{6}} \\
 &= \frac{25r^2 + 12rh + 2h^2}{\frac{24}{7r+2h}} \\
 &= \frac{25r^2 + 12rh + 2h^2}{4(7r+2h)}
 \end{aligned}$$

- (b) Let C be the centre of the hemisphere, so $OC = r$. After a small tilt, the weight produces a restoring moment about the point of contact precisely when the centre of mass is below C . If it coincides with C , there is no restoring moment, so stable equilibrium requires the strict inequality $d < r$.

Using part (a), with $r > 0$ and $h > 0$:

$$\begin{aligned}
 \frac{25r^2 + 12rh + 2h^2}{4(7r+2h)} &< r \\
 25r^2 + 12rh + 2h^2 &< 28r^2 + 8rh \\
 2h^2 + 4rh - 3r^2 &< 0 \\
 2\left(\frac{h}{r}\right)^2 + 4\left(\frac{h}{r}\right) - 3 &< 0 \\
 \left(\frac{h}{r} + 1\right)^2 &< \frac{5}{2}
 \end{aligned}$$

Since $\frac{h}{r} > 0$, the required exact range is

$$0 < \frac{h}{r} < \sqrt{\frac{5}{2}} - 1 = \frac{\sqrt{10} - 2}{2}$$