

FURTHER MECHANICS / CENTRE OF MASS

Centre of Mass of Circular Laminae

Worked solutions

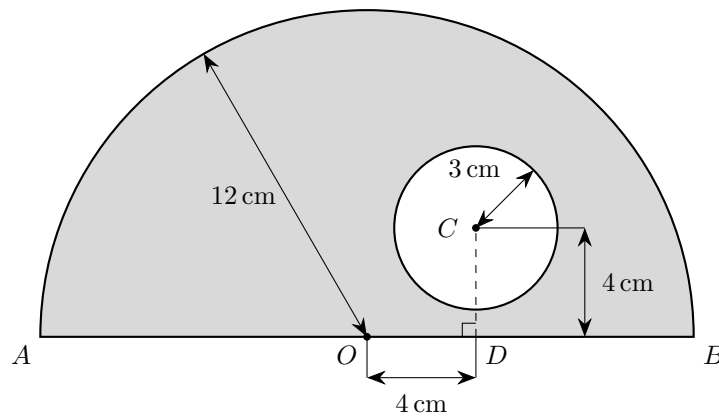
7 questions · 77 marks

Instructions

Unless a question states otherwise, take $g = 9.8 \text{ m s}^{-2}$ wherever a numerical value of g is required. Solutions also give final answers for $g = 10 \text{ m s}^{-2}$ where these differ.

Questions

Question 1	6 marks	2
Question 2	11 marks	4
Question 3	9 marks	6
Question 4	9 marks	8
Question 5	11 marks	10
Question 6	14 marks	12
Question 7	17 marks	15



1. A uniform semicircular lamina has radius 12 cm and diameter AB , with midpoint O . A circular hole of radius 3 cm is cut from the lamina. Its centre C is 4 cm above a point D on OB , where $OD = 4$ cm and CD is perpendicular to AB , as shown in the diagram above.

The remaining lamina is freely suspended from B and hangs in equilibrium.

Determine the acute angle that AB makes with the vertical, giving your answer to the nearest 0.1° .

You may use the result that the centre of mass of a uniform semicircular lamina of radius R is on its axis of symmetry at a distance $\frac{4R}{3\pi}$ from its diameter. [6]

Solution

Take O as the origin, with the x -axis along OB and the y -axis perpendicular to AB , towards the semicircle. Measure coordinates in centimetres, so $B = (12, 0)$ and $C = (4, 4)$.

For a uniform semicircular lamina of radius R , the centre of mass lies on its axis of symmetry at distance $\frac{4R}{3\pi}$ from the diameter. Thus the centre of mass of the complete semicircle has coordinates

$$\left(0, \frac{4 \times 12}{3\pi}\right) = \left(0, \frac{16}{\pi}\right)$$

Let $\mathcal{A}_s$, $\mathcal{A}_h$ and $\mathcal{A}$ denote the areas of the complete semicircle, the hole and the remaining lamina respectively. Then

$$\mathcal{A}_s = \frac{1}{2}\pi(12)^2 = 72\pi \text{ cm}^2$$

$$\mathcal{A}_h = \pi(3)^2 = 9\pi \text{ cm}^2$$

$$\mathcal{A} = 72\pi - 9\pi = 63\pi \text{ cm}^2$$

Let the remaining lamina have centre of mass G , with coordinates $(\bar{x}, \bar{y})$. Since the density is uniform, masses are proportional to areas. The moments of the remaining lamina and the removed circle together equal those of the complete semicircle. Taking moments about the y -axis:

$$63\pi\bar{x} + 9\pi \times 4 = 72\pi \times 0$$

$$63\pi\bar{x} = -36\pi$$

$$\bar{x} = -\frac{4}{7} \text{ cm}$$

Taking moments about the x -axis:

$$63\pi\bar{y} + 9\pi \times 4 = 72\pi \times \frac{16}{\pi}$$

$$63\pi\bar{y} = 1152 - 36\pi$$

$$\bar{y} = \frac{128 - 4\pi}{7\pi} \text{ cm}$$

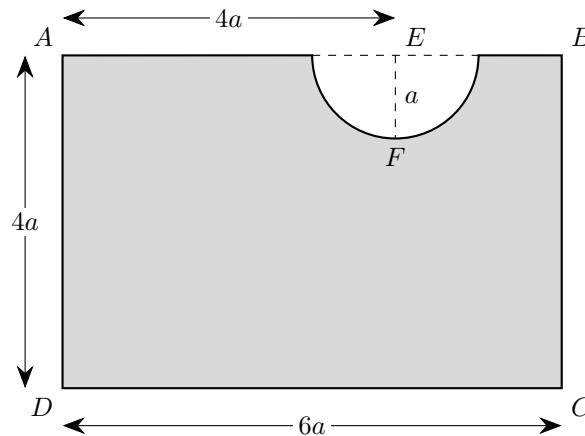
When suspended in equilibrium, G lies vertically below B . Hence the required acute angle θ is the angle between BA and BG . Using the perpendicular distances from B to G :

$$\tan \theta = \frac{\bar{y}}{12 - \bar{x}}$$

$$= \frac{\frac{128 - 4\pi}{7\pi}}{12 + \frac{4}{7}}$$

$$= \frac{32 - \pi}{22\pi}$$

$$\theta = \arctan\left(\frac{32 - \pi}{22\pi}\right) \approx 22.7^\circ \quad (1 \text{ d.p.})$$



2. A uniform rectangular lamina $ABCD$ has $AB = 6a$ and $AD = 4a$, where $a > 0$.

A semicircular notch of radius a is cut from the edge AB .

The centre E of the semicircle lies on AB , with $AE = 4a$, and the semicircle lies inside the rectangle, as shown in the diagram above.

You may use the result that the centre of mass of a uniform semicircular lamina of radius r is a distance $\frac{4r}{3\pi}$ from its diameter.

- (a) (i) Show that the perpendicular distance of the centre of mass of the remaining lamina from AD is

$$\frac{4(36 - \pi)}{48 - \pi}a$$

- (ii) The remaining lamina is freely suspended from A and hangs in equilibrium. Calculate the acute angle that AB makes with the downward vertical, giving your answer to the nearest 0.1° .

[11]

Solution

- (a) (i) Take coordinate axes through D , with the x -axis along DC and the y -axis along DA . Let the centre of mass of the remaining lamina be $G(\bar{x}, \bar{y})$.

The area S of the remaining lamina is

$$S = (6a)(4a) - \frac{1}{2}\pi a^2 = \frac{48 - \pi}{2}a^2$$

The rectangle has its centre of mass at $x = 3a$. By symmetry, the removed semicircle has its centre of mass on the line through E perpendicular to AB , so its x -coordinate is $4a$.

Since the lamina is uniform, masses are proportional to areas. The moments of the remaining lamina and the removed semicircle together equal the moment of the original rectangle. Taking moments about AD :

$$\begin{aligned} S\bar{x} + \frac{\pi a^2}{2}(4a) &= 24a^2(3a) \\ \frac{48 - \pi}{2}a^2\bar{x} &= (72 - 2\pi)a^3 \\ \bar{x} &= \frac{4(36 - \pi)}{48 - \pi}a \end{aligned}$$

- (ii) For a uniform semicircular lamina of radius r , its centre of mass lies on its axis of symmetry, a distance $\frac{4r}{3\pi}$ from its diameter towards the curved boundary.

Thus the removed semicircle has its centre of mass at height $4a - \frac{4a}{3\pi}$ above DC . The original rectangle has its centre of mass at height $2a$. Taking moments about DC :

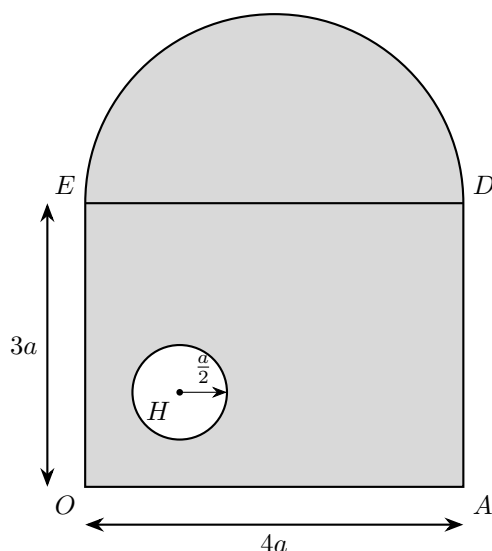
$$\begin{aligned} S\bar{y} + \frac{\pi a^2}{2} \left(4a - \frac{4a}{3\pi} \right) &= 24a^2(2a) \\ \frac{48 - \pi}{2} a^2 \bar{y} &= 48a^3 - 2\pi a^3 + \frac{2}{3} a^3 \\ \bar{y} &= \frac{292 - 12\pi}{3(48 - \pi)} a \end{aligned}$$

Let h be the perpendicular distance of G below AB . Then

$$\begin{aligned} h &= 4a - \bar{y} \\ &= \frac{12(48 - \pi) - (292 - 12\pi)}{3(48 - \pi)} a \\ &= \frac{284}{3(48 - \pi)} a \end{aligned}$$

When the lamina hangs in equilibrium, G is vertically below A , so AG gives the downward vertical. Let θ be the acute angle between AB and AG . Using the right-angled triangle with components $\bar{x}$ parallel to AB and h perpendicular to AB , together with part (a)(i):

$$\begin{aligned} \tan \theta &= \frac{h}{\bar{x}} \\ &= \frac{284}{3(48 - \pi)} \times \frac{48 - \pi}{4(36 - \pi)} \\ &= \frac{71}{3(36 - \pi)} \\ \theta &= \arctan \left(\frac{71}{3(36 - \pi)} \right) \approx 35.8^\circ \quad (1 \text{ d.p.}) \end{aligned}$$



3. A uniform lamina is made by joining a rectangle $OADE$ to a semicircle along its diameter ED , and then removing a circular disc from the rectangular portion, as shown in the diagram above.

The rectangle has $OA = 4a$ and $OE = 3a$, where $a > 0$.

The removed disc has radius $\frac{a}{2}$ and centre H , which is a distance a from each of the sides OE and OA .

You may use the result that the centre of mass of a uniform semicircular lamina of radius r is a distance $\frac{4r}{3\pi}$ from its diameter.

- (a) Find, in terms of a and π , the distance of the centre of mass of the remaining lamina from

- (i) OE ,
- (ii) OA .

[7]

The remaining lamina is freely suspended from A and hangs in equilibrium.

The angle between AO and the downward vertical is α .

- (b) Find the exact value of $\tan \alpha$.

[2]

Solution

- (a) (i) Let the centre of mass be G , at distances $\bar{x}$ from OE and $\bar{y}$ from OA . Since the lamina is uniform, masses are proportional to areas, and the removed disc contributes a negative moment.

The semicircle has radius $2a$, so the remaining area S is

$$\begin{aligned} S &= (4a)(3a) + \frac{1}{2}\pi(2a)^2 - \pi\left(\frac{a}{2}\right)^2 \\ &= \frac{a^2(48 + 7\pi)}{4} \end{aligned}$$

The rectangle and semicircle each have their centre of mass at distance $2a$ from OE ; the removed disc has its centre at distance a . Taking moments about OE :

$$\begin{aligned} S\bar{x} &= 12a^2(2a) + 2\pi a^2(2a) - \frac{\pi a^2}{4}(a) \\ &= \frac{a^3(96 + 15\pi)}{4} \\ \bar{x} &= \frac{a(96 + 15\pi)}{48 + 7\pi} \end{aligned}$$

- (ii) The centre of mass of a uniform semicircular lamina of radius r is a distance $\frac{4r}{3\pi}$ from its diameter, on its axis of symmetry. Here the diameter is $3a$ above OA , so the semicircle's centre of mass is at distance

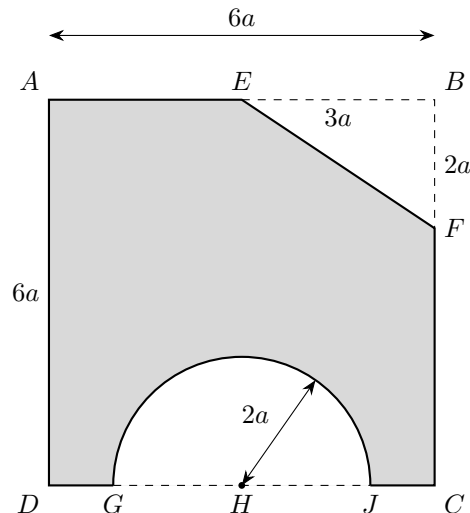
$$3a + \frac{4(2a)}{3\pi} = 3a + \frac{8a}{3\pi}$$

from OA . The rectangle's centre is at distance $\frac{3a}{2}$, and the removed disc's centre is at distance a . Taking moments about OA :

$$\begin{aligned} S\bar{y} &= 12a^2\left(\frac{3a}{2}\right) + 2\pi a^2\left(3a + \frac{8a}{3\pi}\right) - \frac{\pi a^2}{4}(a) \\ &= a^3\left(18 + 6\pi + \frac{16}{3} - \frac{\pi}{4}\right) \\ &= \frac{a^3(280 + 69\pi)}{12} \\ \bar{y} &= \frac{a(280 + 69\pi)}{3(48 + 7\pi)} \end{aligned}$$

- (b) In equilibrium, G lies vertically below A , so α is the angle between AO and AG . Relative to the original diagram, the distances from A to G parallel and perpendicular to AO are $4a - \bar{x}$ and $\bar{y}$, respectively.

$$\begin{aligned} 4a - \bar{x} &= 4a - \frac{a(96 + 15\pi)}{48 + 7\pi} \\ &= \frac{a(96 + 13\pi)}{48 + 7\pi} \\ \tan \alpha &= \frac{\bar{y}}{4a - \bar{x}} \\ &= \frac{280 + 69\pi}{3(96 + 13\pi)} \end{aligned}$$



4. A uniform square lamina $ABCD$ has side $6a$.

A template T , shown shaded in the diagram above, is made by removing two portions of the lamina.

The first portion is the right-angled triangle EBF , where E lies on AB , F lies on BC , $EB = 3a$ and $BF = 2a$.

The second portion is a semicircle of radius $2a$.

Its centre H is the midpoint of DC , its diameter lies on DC , and the semicircle lies inside the square.

- (a) Find the distances of the centre of mass of T from AD and from DC , giving both answers exactly in terms of a and π . [6]

The template is now suspended freely from A and hangs in equilibrium in a vertical plane.

- (b) Find the acute angle between AD and the vertical, giving your answer in degrees to one decimal place. [3]

Solution

- (a) Take D as the origin, with the positive x -axis along DC and the positive y -axis along DA . Then $\bar{x}$ and $\bar{y}$ are the required distances from AD and DC , respectively.

The centre of mass of a uniform triangular lamina is the average of its three vertex coordinates:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Here $E(3a, 6a)$, $B(6a, 6a)$ and $F(6a, 4a)$, so the triangle's centre of mass is

$$\left(\frac{3a + 6a + 6a}{3}, \frac{6a + 6a + 4a}{3} \right) = \left(5a, \frac{16a}{3} \right)$$

For a uniform semicircular lamina of radius r , the centre of mass lies on its axis of symmetry, at distance $\frac{4r}{3\pi}$ from its diameter. Thus the areas and centre-of-mass coordinates are

Portion	Area	x	y
Square	$36a^2$	$3a$	$3a$
Triangle removed	$3a^2$	$5a$	$\frac{16a}{3}$
Semicircle removed	$2\pi a^2$	$3a$	$\frac{8a}{3\pi}$

Since the lamina is uniform, masses are proportional to areas. The remaining area is

$$\mathcal{A}_T = 36a^2 - 3a^2 - 2\pi a^2 = (33 - 2\pi)a^2$$

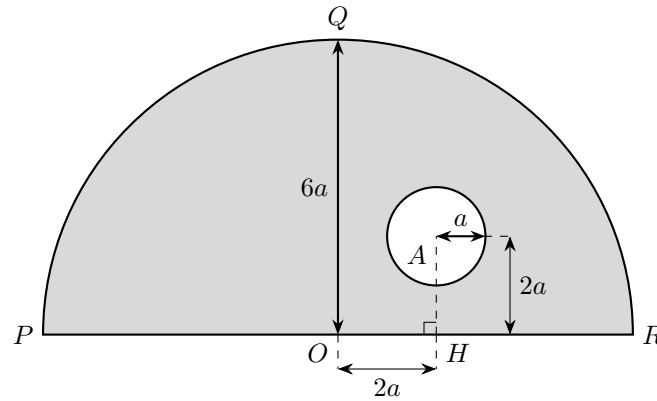
Subtracting the moments of the removed portions from those of the square:

$$\begin{aligned}(33 - 2\pi)a^2\bar{x} &= 36a^2(3a) - 3a^2(5a) - 2\pi a^2(3a) \\ &= (93 - 6\pi)a^3 \\ \bar{x} &= \frac{(93 - 6\pi)a}{33 - 2\pi} \\ (33 - 2\pi)a^2\bar{y} &= 36a^2(3a) - 3a^2\left(\frac{16a}{3}\right) - 2\pi a^2\left(\frac{8a}{3\pi}\right) \\ &= \left(108 - 16 - \frac{16}{3}\right)a^3 = \frac{260a^3}{3} \\ \bar{y} &= \frac{260a}{3(33 - 2\pi)}\end{aligned}$$

- (b) When the template hangs in equilibrium, its centre of mass is vertically below A . Let θ be the acute angle between AD and the vertical.

The centre of mass is $\bar{x}$ perpendicular to AD , and $6a - \bar{y}$ along AD from A . Using part (a):

$$\begin{aligned}\tan \theta &= \frac{\bar{x}}{6a - \bar{y}} \\ &= \frac{93 - 6\pi}{6(33 - 2\pi) - \frac{260}{3}} \\ &= \frac{279 - 18\pi}{334 - 36\pi} \\ \theta &= \arctan\left(\frac{279 - 18\pi}{334 - 36\pi}\right) \approx 45.2^\circ \quad (1 \text{ d.p.})\end{aligned}$$



5. A uniform semicircular lamina has centre O , radius $6a$ and diameter PR .
 A circular disc of radius a is removed to form the lamina L shown shaded in the diagram above.
 The centre of the removed disc is A .
 The point H lies on OR , with $OH = 2a$, $HA = 2a$ and HA perpendicular to PR .

Take O as the origin, with the positive x -axis directed towards R and the positive y -axis directed towards Q .

- (a) Find the exact coordinates of the centre of mass of L . [5]

The mass of L is M .

A particle of mass kM is attached at P .

The system is suspended from R and hangs freely in equilibrium, with P below R and the diameter PR making an acute angle of $\arctan\left(\frac{1}{6}\right)$ with the downward vertical through R .

- (b) Find the exact value of k and the magnitude of the supporting force at R in terms of M and g . [6]

Solution

- (a) For a uniform semicircular lamina of radius r , the centre of mass lies on its axis of symmetry, a distance $\frac{4r}{3\pi}$ from the centre of the circle.

Thus the original semicircle has area and centre of mass

$$\frac{1}{2}\pi(6a)^2 = 18\pi a^2, \quad \left(0, \frac{4(6a)}{3\pi}\right) = \left(0, \frac{8a}{\pi}\right)$$

The removed disc has area πa^2 and centre of mass $A(2a, 2a)$. Therefore the area of L is

$$18\pi a^2 - \pi a^2 = 17\pi a^2$$

Let the centre of mass of L be $(\bar{x}, \bar{y})$. Since the lamina is uniform, masses are proportional to areas. Subtracting the moments of the removed disc:

$$17\pi a^2 \bar{x} = 18\pi a^2(0) - \pi a^2(2a) = -2\pi a^3$$

$$17\pi a^2 \bar{y} = 18\pi a^2 \left(\frac{8a}{\pi}\right) - \pi a^2(2a) = (144 - 2\pi)a^3$$

Dividing by $17\pi a^2$, the exact coordinates are

$$\left(-\frac{2a}{17}, \frac{(144 - 2\pi)a}{17\pi}\right)$$

(b) Write $\theta = \arctan\left(\frac{1}{6}\right)$, so $\tan \theta = \frac{1}{6}$.

Relative to R , the centre of mass of L is a distance $6a - \bar{x}$ towards P and a distance $\bar{y}$ perpendicular to PR , on the Q side. Also, $RP = 12a$.

Resolve the weights parallel and perpendicular to PR . The parallel component of the lamina's weight has magnitude $Mg \cos \theta$, while the perpendicular components have magnitudes $Mg \sin \theta$ and $kMg \sin \theta$. Taking moments about R :

$$Mg\bar{y} \cos \theta = Mg(6a - \bar{x}) \sin \theta + kMg(12a) \sin \theta$$

Dividing by $Mg \sin \theta$ gives

$$\bar{y} \cot \theta = 6a - \bar{x} + 12ak$$

Substituting the coordinates from part (a) and $\cot \theta = 6$:

$$6\left(\frac{(144 - 2\pi)a}{17\pi}\right) = 6a + \frac{2a}{17} + 12ak$$

$$\frac{(864 - 12\pi)a}{17\pi} = \frac{104a}{17} + 12ak$$

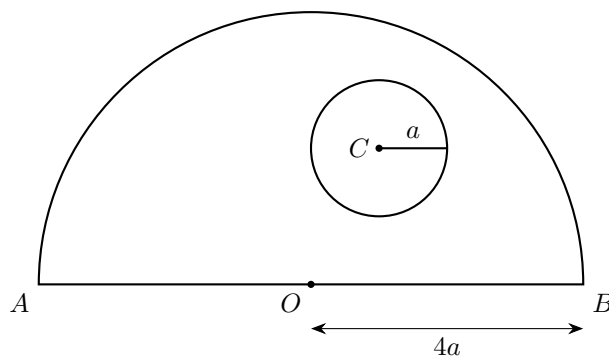
$$864 - 12\pi = 104\pi + 204\pi k$$

$$204\pi k = 864 - 116\pi$$

$$k = \frac{216 - 29\pi}{51\pi}$$

The supporting force F balances the total weight of the system, so

$$\begin{aligned} F &= (M + kM)g \\ &= \left(1 + \frac{216 - 29\pi}{51\pi}\right)Mg \\ &= \frac{216 + 22\pi}{51\pi}Mg \end{aligned}$$



6. A uniform semicircular lamina has radius $4a$ and diameter AB , whose midpoint is O .

A circular hole of radius a is cut from the lamina.

In the orientation shown above, take O as the origin, the positive x -direction along OB and the positive y -direction perpendicular to AB towards the semicircular arc.

The centre C of the hole has coordinates $(a, 2a)$.

- (i) Show that the centre of mass G of the remaining lamina has coordinates

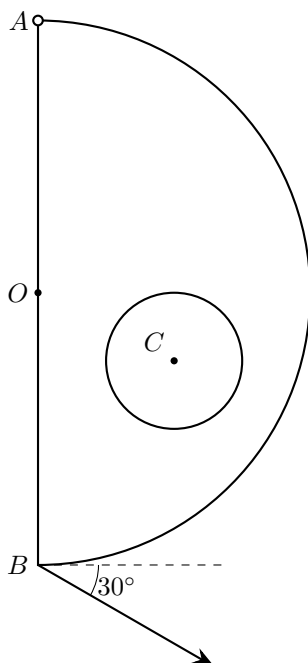
$$\left(-\frac{a}{7}, \frac{(128 - 6\pi)a}{21\pi} \right) \quad [6]$$

The remaining lamina has mass 7 kg .

It is freely pivoted at A and turned so that AB is vertical, with A above B and the semicircular arc to the right of AB , as shown in the diagram below.

A light string attached to B holds the lamina in equilibrium.

The string lies in the plane of the lamina and makes an angle of 30° below the horizontal.



- (ii) Calculate the tension in the string. [3]

- (iii) Find the magnitude of the force exerted on the lamina by the pivot at A and the angle this force makes with the upward vertical, stating on which side of the vertical it acts. [5]

Solution

- (i) For a uniform semicircular lamina of radius r , the centre of mass lies on its axis of symmetry, at distance $\frac{4r}{3\pi}$ from the centre of the circle towards the arc. Thus the original lamina has centre of mass

$$\left(0, \frac{4(4a)}{3\pi}\right) = \left(0, \frac{16a}{3\pi}\right)$$

The areas of the original semicircle and the removed circle are

$$\frac{1}{2}\pi(4a)^2 = 8\pi a^2, \quad \pi a^2$$

Since the lamina is uniform, mass is proportional to area. Let the removed circle have mass m . The original semicircle then has mass $8m$, and the remaining lamina has mass $7m$.

Let the centre of mass of the remaining lamina be $G(\bar{x}, \bar{y})$. Combining the remaining lamina with the removed circle restores the original semicircle. Taking moments about the y -axis gives

$$7m\bar{x} + ma = 8m(0)$$

$$7\bar{x} = -a$$

$$\bar{x} = -\frac{a}{7}$$

Similarly, taking moments about the x -axis gives

$$7m\bar{y} + m(2a) = 8m\left(\frac{16a}{3\pi}\right)$$

$$7\bar{y} = \frac{128a}{3\pi} - 2a$$

$$\bar{y} = \frac{(128 - 6\pi)a}{21\pi}$$

Hence G has coordinates

$$\left(-\frac{a}{7}, \frac{(128 - 6\pi)a}{21\pi}\right)$$

- (ii) The lamina has been turned through 90° clockwise, so the original y -coordinate of G is now its horizontal distance to the right of AB .

The tension T has horizontal component $T \cos 30^\circ$, acting at B , where $AB = 8a$. Its vertical component acts along AB , so has no moment about A .

Taking moments about A , and using part (i):

$$8aT \cos 30^\circ = 7g \left(\frac{(128 - 6\pi)a}{21\pi} \right)$$

$$T = \frac{7g}{8 \cos 30^\circ} \cdot \frac{128 - 6\pi}{21\pi}$$

Using $g = 9.8 \text{ m s}^{-2}$, the tension is

$$T \approx 16.4 \text{ N}$$

- (iii) Let the pivot force have components R_x horizontally to the right and R_y vertically upwards. Resolving horizontally and vertically for equilibrium:

$$R_x + T \cos 30^\circ = 0$$

$$R_y - T \sin 30^\circ - 7g = 0$$

Thus the pivot force acts to the left and upwards, with components

$$R_x = -T \cos 30^\circ, \quad R_y = 7g + T \sin 30^\circ$$

Using the unrounded value of T , its magnitude is

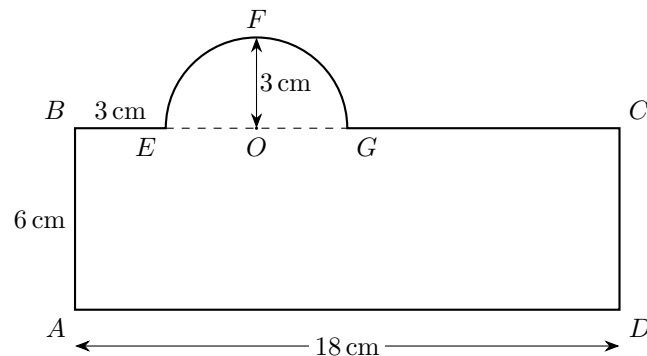
$$R = \sqrt{(T \cos 30^\circ)^2 + (7g + T \sin 30^\circ)^2} \\ \approx 78.1 \text{ N}$$

Let θ be the angle the force makes with the upward vertical. Then

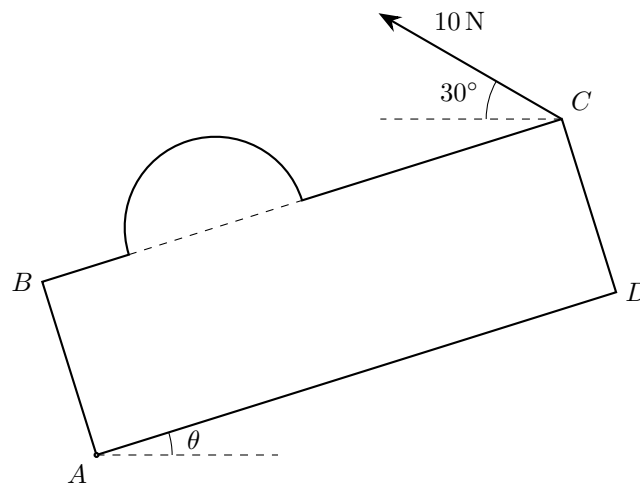
$$\tan \theta = \frac{|R_x|}{R_y} = \frac{T \cos 30^\circ}{7g + T \sin 30^\circ} \\ \theta \approx 10.5^\circ \quad \text{to the left of the upward vertical}$$

Final answers with $g = 10$: (ii) 16.7 N; (iii) magnitude 79.7 N

7. A uniform lamina consists of a rectangle $ABCD$ together with a semicircular tab EFG attached externally along its diameter EG , as shown in the diagram below. The two components have the same mass per unit area. The points E and G lie on BC , and O is the midpoint of EG . The dimensions are $AB = 6$ cm, $BC = 18$ cm and $BE = 3$ cm. The semicircular tab has radius 3 cm.



- (i) Consider the semicircular tab EFG on its own. Using the standard centre-of-mass result for a uniform semicircular lamina, show that its centre of mass is 1.27 cm from the diameter EG , correct to 3 significant figures. State the exact distance. [2]
- (ii) For the complete lamina, find
- the distance from AB to its centre of mass, giving your answer exactly, [5]
 - the distance from AD to its centre of mass, giving your answer exactly. [4]
- (iii) The complete lamina has weight 24 N and is smoothly pivoted at A . A wire attached at C supports it in equilibrium. The wire pulls upwards and to the left at 30° above the horizontal, and its tension is 10 N.
- The side AD makes an angle θ above the horizontal, where $0^\circ < \theta < 45^\circ$, as shown in the diagram below.



Calculate θ , giving your answer to 3 significant figures.

[6]

Solution

- (i) Consider only the semicircular tab EFG . For a uniform semicircular lamina of radius r , the centre of mass lies on its axis of symmetry at distance $\frac{4r}{3\pi}$ from the diameter. Here $r = 3$ cm, so the distance from EG is

$$\frac{4 \times 3}{3\pi} = \frac{4}{\pi} \text{ cm} \approx 1.27 \text{ cm} \quad (3 \text{ s.f.})$$

(ii) Since both components have the same mass per unit area, their masses are proportional to their areas.

- (a) Let $\bar{x}$ be the distance of the complete lamina's centre of mass from AB . The component areas and centre-of-mass distances are

Component	Area (cm ²)	Distance from AB (cm)
Rectangle	$18 \times 6 = 108$	$\frac{18}{2} = 9$
Semicircle	$\frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$	$BE + EO = 3 + 3 = 6$

Taking moments of mass about AB , with the common mass per unit area cancelling:

$$\left(108 + \frac{9\pi}{2}\right)\bar{x} = 108 \times 9 + \frac{9\pi}{2} \times 6$$

$$\begin{aligned}\bar{x} &= \frac{972 + 27\pi}{108 + \frac{9\pi}{2}} \\ &= \frac{216 + 6\pi}{24 + \pi} \text{ cm}\end{aligned}$$

- (b) Let $\bar{y}$ be the distance of the complete lamina's centre of mass from AD . The rectangle's centre of mass is 3 cm from AD . Using part (i), the semicircle's centre of mass is at distance

$$\left(6 + \frac{4}{\pi}\right) \text{ cm}$$

Taking moments of mass about AD :

$$\left(108 + \frac{9\pi}{2}\right)\bar{y} = 108 \times 3 + \frac{9\pi}{2} \left(6 + \frac{4}{\pi}\right)$$

$$= 324 + 27\pi + 18$$

$$\bar{y} = \frac{342 + 27\pi}{108 + \frac{9\pi}{2}}$$

$$= \frac{76 + 6\pi}{24 + \pi} \text{ cm}$$

- (iii) Resolve the tension parallel and perpendicular to AD . The angle between the wire and the direction DA is $\theta + 30^\circ$, so the components are

$$T_{\parallel} = 10 \cos(\theta + 30^\circ) \text{ N} \quad \text{in the direction } DA$$

$$T_{\perp} = 10 \sin(\theta + 30^\circ) \text{ N} \quad \text{in the direction } AB$$

These components both produce anticlockwise moments about A , with perpendicular distances 6 cm and 18 cm, respectively. The reaction at the pivot has zero moment about A .

After rotation through θ , the horizontal distance of the centre of mass from A is $\bar{x} \cos \theta - \bar{y} \sin \theta$. Taking moments about A , in N cm:

$$10[18 \sin(\theta + 30^\circ) + 6 \cos(\theta + 30^\circ)] = 24(\bar{x} \cos \theta - \bar{y} \sin \theta)$$

$$18 \sin(\theta + 30^\circ) + 6 \cos(\theta + 30^\circ) = \frac{12}{5}(\bar{x} \cos \theta - \bar{y} \sin \theta)$$

Expanding the compound angles and collecting terms:

$$\begin{aligned}18\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right) + 6\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) &= \frac{12}{5}(\bar{x}\cos\theta - \bar{y}\sin\theta) \\ \left(9\sqrt{3} - 3 + \frac{12\bar{y}}{5}\right)\sin\theta &= \left(\frac{12\bar{x}}{5} - 9 - 3\sqrt{3}\right)\cos\theta \\ \tan\theta &= \frac{\frac{12\bar{x}}{5} - 9 - 3\sqrt{3}}{9\sqrt{3} - 3 + \frac{12\bar{y}}{5}}\end{aligned}$$

Using the exact values from parts (ii)(a) and (ii)(b), and $0^\circ < \theta < 45^\circ$, gives

$$\theta \approx 17.4^\circ \quad (3 \text{ s.f.})$$