

## FURTHER MECHANICS / CENTRE OF MASS

# Centre of Mass of Laminae by Integration

Questions by topic

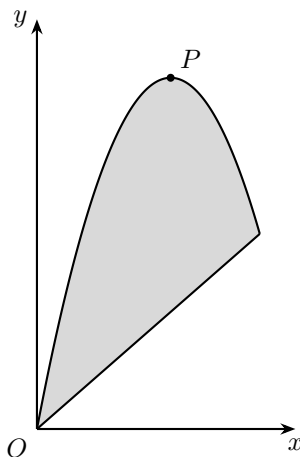
10 questions · 113 marks

**Instructions**

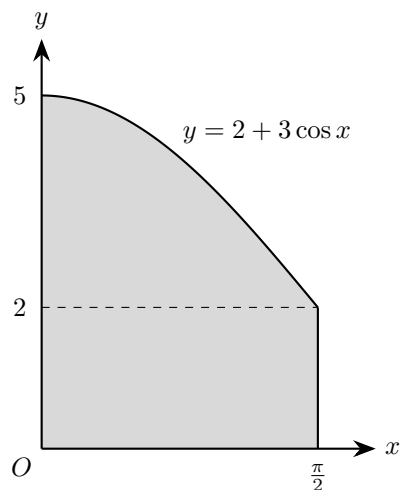
Unless a question states otherwise, take  $g = 9.8 \text{ m s}^{-2}$  wherever a numerical value of  $g$  is required. Solutions also give final answers for  $g = 10 \text{ m s}^{-2}$  where these differ.

## Questions

|             |          |    |
|-------------|----------|----|
| Question 1  | 8 marks  | 2  |
| Question 2  | 5 marks  | 3  |
| Question 3  | 13 marks | 4  |
| Question 4  | 12 marks | 5  |
| Question 5  | 11 marks | 6  |
| Question 6  | 9 marks  | 7  |
| Question 7  | 14 marks | 8  |
| Question 8  | 13 marks | 9  |
| Question 9  | 10 marks | 10 |
| Question 10 | 18 marks | 11 |



1. A uniform lamina occupies the region enclosed by the curve  $y = 6x - x^2$  and the straight line  $y = x$ . The unit of length on each axis is one centimetre. The diagram above shows the lamina and the point  $P(3, 9)$  at the maximum of the curve.
- (a) Find the area of the lamina and use integration to find the coordinates of its centre of mass. Give your answers in exact form. [5]
- (b) The lamina is freely suspended from  $P$  and hangs in equilibrium. Find the acute angle between the vertical and the straight edge of the lamina, giving your answer to the nearest degree. [3]



2. A uniform lamina  $L$  is cut to form a decorative panel.

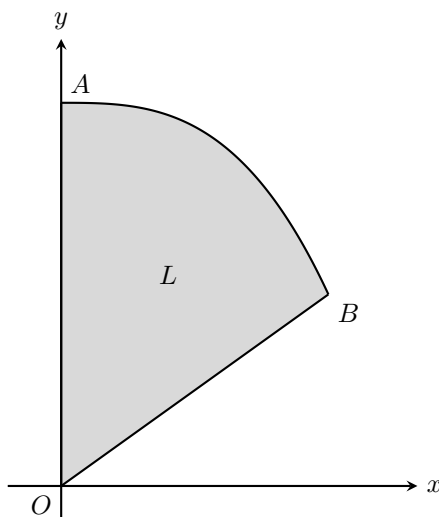
It occupies the region bounded by the curve  $y = 2 + 3 \cos x$ , the  $x$ -axis and the vertical lines  $x = 0$  and  $x = \frac{\pi}{2}$ , as shown in the diagram above.

Its centre of mass is  $(\bar{x}, \bar{y})$ .

- (a) Using integration, show that

$$\bar{x} = \frac{\pi^2 + 6\pi - 12}{4(\pi + 3)} \quad [2]$$

- (b) Find the exact value of  $\bar{y}$ . Hence show, without using decimal approximations, that the centre of mass lies above the line  $y = 2$ . [3]



3. The diagram above shows a region bounded by the  $y$ -axis, the straight line  $y = x$  and the curve

$$y = 6 - \frac{x^3}{9}$$

The curve meets the  $y$ -axis at  $A = (0, 6)$  and meets the line  $y = x$  at  $B = (3, 3)$ .  
The origin is  $O$ .

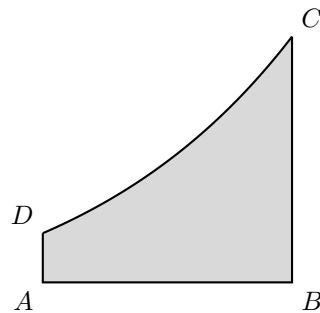
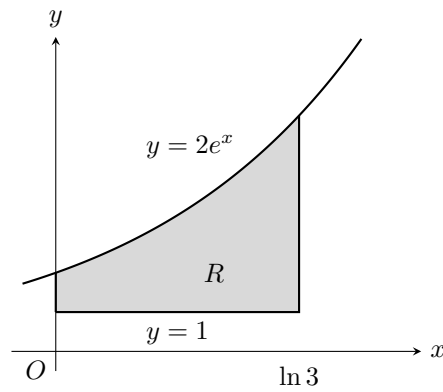
A uniform lamina  $L$  occupies the shaded region  $OAB$ .

The unit of length on each axis is one centimetre, and its centre of mass has coordinates  $(\bar{x}, \bar{y})$ .

- (a) Use integration to find the area of  $L$  and show that

$$\bar{y} = \frac{118}{35} \quad [5]$$

- (b) The lamina is freely suspended from  $B$  and hangs in equilibrium. Find the acute angle between the straight edge  $OB$  and the downward vertical, giving your answer in degrees to one decimal place. [8]



4. The shaded region  $R$  is bounded by the curve  $y = 2e^x$ , the horizontal line  $y = 1$ , the  $y$ -axis and the line  $x = \ln 3$ .

The unit of length on each axis is one metre.

A uniform lamina occupies  $R$ , and its centre of mass has coordinates  $(\bar{x}, \bar{y})$ .

- (a) Use integration to find the area of  $R$  and show that

$$\bar{y} = \frac{16 - \ln 3}{8 - 2 \ln 3} \quad [5]$$

The second diagram above shows the same lamina, with vertices  $A = (0, 1)$ ,  $B = (\ln 3, 1)$ ,  $C = (\ln 3, 6)$  and  $D = (0, 2)$ .

The lamina is freely suspended from  $D$  and hangs in equilibrium.

Let  $\theta$  be the angle between the directed segment from  $D$  to  $A$  and the downward vertical, where  $0^\circ \leq \theta \leq 180^\circ$ .

- (b) Use integration to find  $\bar{x}$  in exact form. Hence find  $\theta$ , giving your answer to one decimal place. [7]

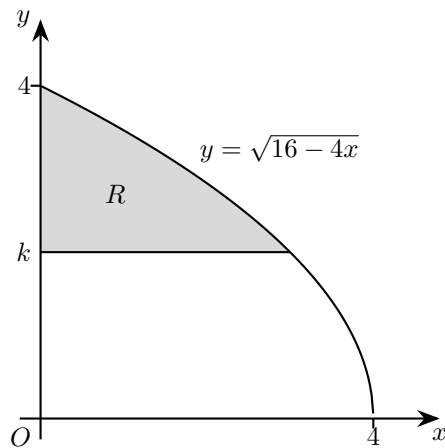
5. A uniform lamina occupies the region bounded by the curve

$$y = \frac{k}{x+1}$$

the  $x$ -axis and the vertical lines  $x = 0$  and  $x = 3$ , where  $k$  is a positive constant.

- (i) Find the exact  $x$ -coordinate of the centre of mass of the lamina. [6]

- (ii) The centre of mass lies on the line  $y = x$ .  
Find the exact value of  $k$  and hence the exact area of the lamina. [5]



6. The shaded region  $R$  is bounded by the curve  $y = \sqrt{16 - 4x}$ , the  $y$ -axis and the horizontal line  $y = k$ , where  $0 < k < 4$ .

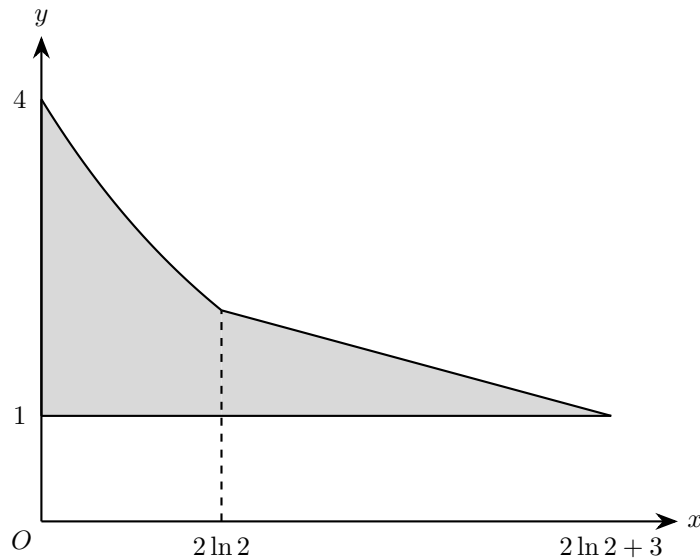
A uniform lamina occupies  $R$ .

- (a) Using integration, determine the  $y$ -coordinate of the centre of mass of the lamina in terms of  $k$ . Simplify your expression fully. [6]

- (b) The centre of mass lies on the line  $y = 3$ . Find the exact value of  $k$ , explaining why any other root of your equation must be rejected. [3]

7. A uniform lamina occupies the region between the curve  $y = 4e^{-x/2}$  and the line  $y = 1$ , for  $0 \leq x \leq 2 \ln 2$ .

- (i) Using integration, find the exact area of this lamina and the exact  $y$ -coordinate of its centre of mass. [7]



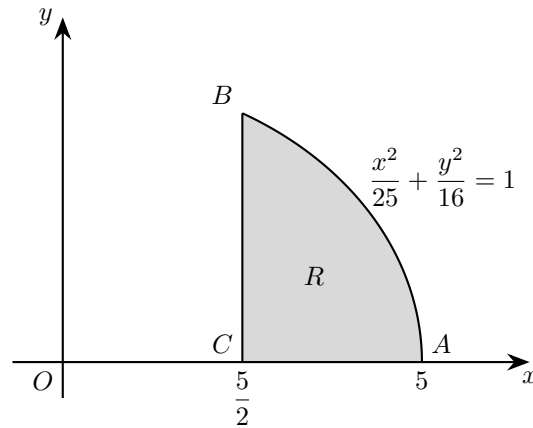
A triangular piece of lamina with the same uniform mass per unit area is attached to its right-hand edge. The resulting lamina is shown in the diagram above.

Its lower boundary is  $y = 1$  and its upper boundary is  $y = f(x)$ , where

$$f(x) = \begin{cases} 4e^{-x/2}, & 0 \leq x \leq 2 \ln 2 \\ 2 - \frac{x - 2 \ln 2}{3}, & 2 \ln 2 \leq x \leq 2 \ln 2 + 3 \end{cases}$$

Its left-hand boundary lies on the  $y$ -axis.

- (ii) Using your area from part (i), find the exact  $x$ -coordinate of the centre of mass of the resulting lamina. [7]



8. A uniform lamina occupies the shaded region  $R$ , bounded by the upper arc of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

the line  $x = \frac{5}{2}$  and the  $x$ -axis.

The points  $C$  and  $A$  have coordinates  $(\frac{5}{2}, 0)$  and  $(5, 0)$  respectively.

The point  $B$  is the upper intersection of the ellipse with  $x = \frac{5}{2}$ .

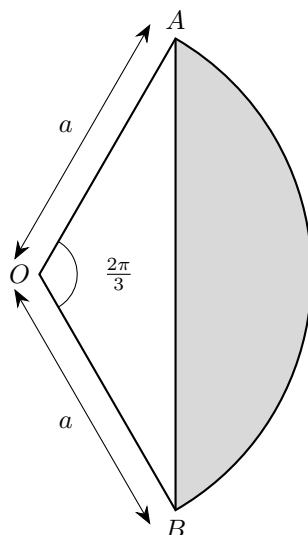
The unit of length on each axis is one metre.

- (a) Using the substitution  $x = 5 \sin t$ , show by integration that the area of  $R$  is

$$\frac{5}{6} (4\pi - 3\sqrt{3}) \text{ m}^2 \quad [4]$$

- (b) Use integration to find the exact coordinates  $(\bar{x}, \bar{y})$  of the centre of mass of the lamina. [6]

- (c) The lamina is freely suspended from  $A$  and hangs in equilibrium. Find the acute angle between  $AC$  and the downward vertical, giving your answer in degrees to one decimal place. [3]



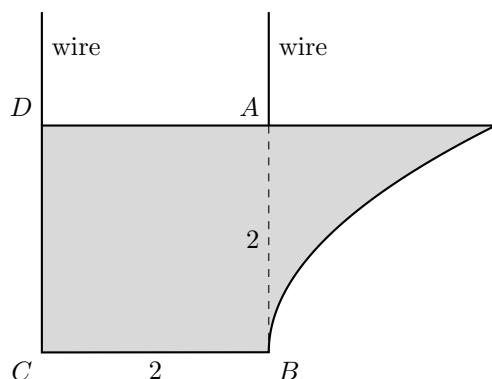
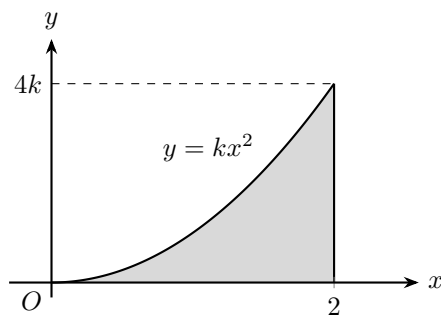
9. A thin uniform metal plate has the shape of the shaded circular segment in the diagram above. It is bounded by the chord  $AB$  and the minor arc  $AB$  of a circle with centre  $O$  and radius  $a$ , where  $a > 0$ . The angle  $AOB$  is  $\frac{2\pi}{3}$ .

The plate is modelled as a uniform lamina with constant mass per unit area  $\sigma$ , where  $\sigma > 0$ .

Take  $O$  as the origin and the positive  $x$ -axis along the bisector of angle  $AOB$ , directed towards the minor arc.

Thus the chord  $AB$  lies on  $x = \frac{a}{2}$ .

- (a) Using integration, find the mass of the plate in terms of  $\sigma$  and  $a$ . Give your answer in exact form. [5]
- (b) Using integration to calculate the first moment of the plate about the  $y$ -axis, find the exact distance of its centre of mass from  $O$ . Explain why the centre of mass lies on the  $x$ -axis. [5]



10. A uniform lamina occupies the region bounded by the curve  $y = kx^2$ , the  $x$ -axis and the line  $x = 2$ , where  $k > 0$ , as shown in the first diagram above.

(i) Use integration to find both coordinates of the centre of mass of this lamina in terms of  $k$ .

[8]

A sign is made from a uniform sheet by joining a square  $ABCD$  of side length 2 to a curved piece having the shape of the lamina in part (i).

The sheet has the same thickness and density throughout.

For the sign, take  $C$  as the origin, with  $CB$  along the positive  $x$ -axis and  $CD$  along the positive  $y$ -axis. Thus  $A = (2, 2)$ ,  $B = (2, 0)$  and  $D = (0, 2)$ .

The curved piece is obtained by reflecting the lamina in part (i) in the line  $y = x$  and then translating it 2 units to the right.

Its curved edge therefore has equation

$$x = 2 + ky^2, \quad 0 \leq y \leq 2$$

The complete sign is shown in the second diagram above.

(ii) Find both coordinates of the centre of mass of the sign in terms of  $k$ .

[5]

The sign is held in equilibrium in a vertical plane by two vertical wires attached at  $A$  and  $D$ . The tension in the wire at  $A$  is twice the tension in the wire at  $D$ .

(iii) Find the exact value of  $k$ .

[5]