FURTHER MECHANICS / CENTRE OF MASS

Centre of Mass of Polygonal Laminae

Mark scheme 10 questions · 92 marks

EXAM BOARDS Edexcel FM2 9FM0/4C · AQA 7367/3M · OCR A Y543 · OCR MEI Mech Minor Y431 OCR MEI Mech Major Y421 · CIE 9231 Paper 3

Questions

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	Total	**92**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last mark of a part whose answer is printed in the question: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	Areas: rectangle $8a^2$, triangle $4a^2$	**B1**	Both areas, or their ratio $2:1$.
	Distances from CD: rectangle a, triangle $2a + \frac{1}{3}(2a) = \frac{8a}{3}$	**B1**	Both, or both from another stated line parallel to CD.
	$(8a^2 + 4a^2)\bar{y} = 8a^2(a) + 4a^2\left(\frac{8a}{3}\right)$	**M1**	Forms a moment equation about CD, or a parallel line, with both pieces' areas and distances and the total area. Condone slips in values.
	$12a^2\bar{y} = \frac{56a^3}{3} \Rightarrow \bar{y} = \frac{14a}{9}$	**A1 AG**	Correct substituted equation leading without error to $\frac{14a}{9}$ from CD; combined steps accepted.
		[4]	
1(b)	$T_E(2a) = Mg\left(\frac{14a}{9}\right)$	**M1**	Equates moments about D of T_E and Mg, using $\frac{14a}{9}$; or moments about another point with resolving vertically. Condone a slip in one distance.
	$T_E = \frac{7Mg}{9}$	**A1**	oe.
		[2]	
	Total for Question 1	**6**	

Question	Scheme	Marks	Notes
2(a)	Lamina uniform and symmetric about the dashed line	**E1**	States that the lamina is symmetric about the dashed line; omitting "uniform" is condoned.
		[1]	
2(b)	$AN = \sqrt{0.5^2 - 0.4^2} = 0.3$ Areas 0.48 and 0.12	**B1**	Both areas, or their ratio $4:1$.
	Distances from CD: 0.3 and 0.7	**B1**	Both, or both from another common line, such as -0.3 and 0.1 from BE.
	$(0.48 + 0.12)\bar{y} = 0.48(0.3) + 0.12(0.7)$	**M1**	Forms a moment equation with each piece's area times its distance and the total area times $\bar{y}$, using their values.
	$0.6\bar{y} = 0.228 \Rightarrow \bar{y} = 0.38$	**A1 AG**	Correct substituted equation leading without error to 0.38; from another line, converted to the distance from CD.
		[4]	
2(c)	G vertically below B: $\tan\theta = \frac{0.4}{0.6-0.38}$ $\left[= \frac{20}{11}\right]$	**M1 A1**	**M1** Takes BG vertical and forms a tangent from the horizontal and vertical distances of G from B, either way up. **A1** $\tan\theta = \frac{0.4}{0.22}$ for the angle with the horizontal, or $\frac{0.22}{0.4}$ for its complement.
	$\theta = 61.2°$	**A1**	awrt 61.2°, or awrt 1.07 radians.
	C is higher than D	**B1**	
		[4]	
	Total for Question 2	**9**	

Question	Scheme	Marks	Notes
3(a)	$5a^2,\ 3a^2,\ a^2\ [\mathcal{A} = 9a^2]$	**B1**	Areas of the pieces of any valid split; a removed piece may be counted negative.
	$\left(\frac{a}{2}, \frac{5a}{2}\right),\ \left(\frac{5a}{2}, \frac{9a}{2}\right),\ \left(\frac{3a}{2}, \frac{5a}{2}\right)$	**B1**	Centres of mass of the pieces of their split, as coordinates or distances from OS and OA.
	(i) $9a^2\bar{x} = 5a^2\left(\frac{a}{2}\right) + 3a^2\left(\frac{5a}{2}\right) + a^2\left(\frac{3a}{2}\right) \Rightarrow \bar{x} = \ldots$	**M1**	Forms a moment equation about OS with an area-times-distance term for every piece and their total area. Condone slips in values.
	$\bar{x} = \frac{23a}{18}$	**A1**	oe, or awrt $1.28a$.
	(ii) $9a^2\bar{y} = 5a^2\left(\frac{5a}{2}\right) + 3a^2\left(\frac{9a}{2}\right) + a^2\left(\frac{5a}{2}\right) \Rightarrow \bar{y} = \ldots$	**M1**	Forms a moment equation about OA with an area-times-distance term for every piece and their total area. Condone slips in values.
	$\bar{y} = \frac{19a}{6}$	**A1**	oe, or awrt $3.17a$.
		[6]	
3(b)	G vertically below O: $\tan\alpha = \dfrac{\bar{x}}{\bar{y}} = \dfrac{\text{“}23a/18\text{”}}{\text{“}19a/6\text{”}}$	**M1**	Takes OG vertical and forms $\tan\alpha$ as their distance from OS over their distance from OA. Condone the ratio inverted.
	$\tan\alpha = \frac{23}{57}$	**A1ft**	Exact, the right way up; ft their $\bar{x}$ and $\bar{y}$.
		[2]	
	Total for Question 3	**8**	

Question	Scheme	Marks	Notes
4(a)	Centre of mass of ABG: $\left(\frac{4a+4a+6a}{3}, \frac{0+2a+0}{3}\right)$	**M1**	Uses the mean of the vertices' coordinates, $\frac{1}{3}$ of the perpendicular height from AB, or $\frac{2}{3}$ along the median from G, for the triangle's centre of mass. Condone slips in values.
	$= \left(\frac{14a}{3}, \frac{2a}{3}\right)$	**A1**	Correct x-coordinate $\frac{14a}{3}$, or the height from AB as $AG = 2a$, with AG perpendicular to AB stated or used.
	$\frac{14a}{3} - 4a = \frac{2a}{3}$	**A1 AG**	$\frac{14a}{3} - 4a$ from a correct x-coordinate, or $\frac{1}{3}(2a)$ from a correct height, leading without error to $\frac{2a}{3}$.
		[3]	
4(b)	Areas $8a^2$, $4a^2$, $2a^2$ [total $14a^2$]	**B1**	All three, soi in the moment equations.
	$(2a, a)$, $(a, 3a)$, $\left(\frac{14a}{3}, \frac{2a}{3}\right)$	**B1**	All three centres of mass, soi in the moment equations.
	$14a^2\bar{x} = 8a^2(2a) + 4a^2(a) + 2a^2\left(\frac{14a}{3}\right)$	**M1**	Takes moments about the y-axis or another vertical line, with a term for each piece and their total area. Condone slips in values.
	$\bar{x} = \frac{44a}{21}$	**A1**	oe, or awrt $2.10a$.
	$14a^2\bar{y} = 8a^2(a) + 4a^2(3a) + 2a^2\left(\frac{2a}{3}\right)$	**M1**	Takes moments about the x-axis or another horizontal line, with a term for each piece and their total area. Condone slips in values.
	$\bar{y} = \frac{32a}{21}$	**A1**	oe, or awrt $1.52a$.
		[6]	
4(c)	Centre of mass vertically below $F(0, 4a)$: $\tan\theta = \frac{\text{“}44a/21\text{”}}{4a - \text{“}32a/21\text{”}}$	**M1**	Takes F vertically above their centre of mass and uses a trig ratio of its offsets from F for the angle with FC. Condone the ratio inverted.
	$\tan\theta = \frac{11}{13}$	**A1ft**	ft their $\bar{x}$ and $\bar{y}$ from (b), as $\tan\theta = \frac{\bar{x}}{4a-\bar{y}}$ in any form.
	$\theta = 40.2$	**A1**	awrt 40.2.
		[3]	
	Total for Question 4	**12**	

Question	Scheme	Marks	Notes
5(a)	Areas: 24, $8x$, 16	**B1**	All three areas.
	Distances below SR: -1, x, $2x+1$	**B1**	All three, from SR or any other horizontal line, with consistent signs.
	$(24+8x+16)\bar{y} = 24(-1)+8x(x)+16(2x+1)$	**M1**	Forms a moment equation with all three pieces, their areas and distances, and the total area. Condone sign slips.
	$\bar{y} = \frac{8x^2+32x-8}{8x+40} = \frac{x^2+4x-1}{x+5}$	**A1 AG**	Correct moment equation leading without error to the printed distance below SR, converting from any other reference line.
		[4]	
5(b)	$\bar{y} = \frac{2^2+4(2)-1}{2+5} = \frac{11}{7}$	**B1**	oe, or awrt 1.57
	Centre of mass vertically below P: $\tan\theta = \frac{6}{2+\text{“}\frac{11}{7}\text{”}}$	**M1**	Takes the centre of mass vertically below P and forms a tangent as the ratio of its horizontal and vertical offsets from P. Condone the ratio inverted.
	$\tan\theta = \frac{6}{\frac{25}{7}}$	**A1**	Offsets 6 and $\frac{25}{7}$ in a correct tangent of θ, or of its complement.
	$\tan\theta = \frac{42}{25}$	**A1 AG**	Printed result reached without error; a route through the complement needs its link to θ stated.
		[4]	
Total for Question 5		**8**	

Question	Scheme	Marks	Notes
6(a)	Areas $4a^2$, h^2; remaining $4a^2 - h^2$	**B1**	Removed area subtracted; soi by the moment equations.
	Centres (a, a) and $\left(2a - \frac{h}{2}, \frac{h}{2}\right)$	**B1**	Both coordinates of both centres, measured from AD and AB; soi by the moment equations.
	$(4a^2 - h^2)\bar{x} = 4a^2(a) - h^2\left(2a - \frac{h}{2}\right)$	**M1**	Forms a moment equation about AD with the square's and the removed square's moments, the removed one subtracted. Condone slips in their distances.
	$\bar{x} = a - \frac{h^2}{2(2a+h)}$	**A1**	oe, such as $\frac{4a^3 - 2ah^2 + \frac{1}{2}h^3}{4a^2 - h^2}$; need not be simplified.
	$(4a^2 - h^2)\bar{y} = 4a^2(a) - h^2\left(\frac{h}{2}\right)$	**M1**	Forms a moment equation about AB with the square's and the removed square's moments, the removed one subtracted. Condone slips in their distances.
	$\bar{y} = a + \frac{h^2}{2(2a+h)}$	**A1**	oe, such as $\frac{4a^3 - \frac{1}{2}h^3}{4a^2 - h^2}$; need not be simplified.
		[6]	
6(b)	$AE = 2a - h$ "$a - \frac{h^2}{2(2a+h)}$" $\leq 2a - h$	**M1**	Forms the condition that their distance from AD, in terms of a and h, is at most $AE = 2a - h$. Condone $<$ or $=$.
	$h^2 + 2ah - 4a^2 \leq 0 \Rightarrow h = \ldots$	**dM1**	Clears fractions to a three-term quadratic in h and solves it by formula, completing the square, factorising or calculator, reaching $h = \ldots$
	$0 < h \leq a(\sqrt{5} - 1)$	**A1**	Or $0 < h \leq$ awrt $1.24a$. Needs the lower bound 0 and no negative root; condone $0 \leq h$, and $<$ at the upper end.
		[3]	
	Total for Question 6	**9**	

Question	Scheme	Marks	Notes
7(a)	$AD = \frac{3a}{k},\ AE = \frac{4a}{k}$ $\mathcal{A} = 6a^2 - \frac{6a^2}{k^2}$	**B1**	Areas $6a^2$ and $\frac{6a^2}{k^2}$ oe, the removed one subtracted; soi by the moments equations.
	$\left(\frac{4a}{3}, a\right)$ and $\left(\frac{4a}{3k}, \frac{a}{k}\right)$	**B1**	
	$6a^2\left(1 - \frac{1}{k^2}\right)\bar{x}$ $= 6a^2\left(\frac{4a}{3}\right) - \frac{6a^2}{k^2}\left(\frac{4a}{3k}\right)$	**M1**	Uses their areas and x-coordinates in a moments equation, the removed triangle taken as negative. Condone slips in substitution.
	$\bar{x} = \frac{4a(k^2+k+1)}{3k(k+1)}$	**A1**	Any correct expression in a and k, such as $\frac{4a(k^3-1)}{3k(k^2-1)}$.
	$6a^2\left(1 - \frac{1}{k^2}\right)\bar{y}$ $= 6a^2(a) - \frac{6a^2}{k^2}\left(\frac{a}{k}\right)$	**M1**	Uses their areas and y-coordinates in a moments equation, the removed triangle taken as negative. Condone slips in substitution.
	$\bar{y} = \frac{a(k^2+k+1)}{k(k+1)}$	**A1**	Any correct expression in a and k, such as $\frac{a(k^3-1)}{k(k^2-1)}$.
		[6]	
7(b)	G vertically below C: $\tan\theta = \frac{\bar{y}}{4a-\bar{x}} = \frac{21}{52}$	**M1**	Forms $\tan\theta$ as $\bar{y}$ over the horizontal distance of G from C, oe, with their (a) expressions in a and k, equated to $\frac{21}{52}$.
	$\frac{3(k^2+k+1)}{4(2k^2+2k-1)} = \frac{21}{52}$ $\Rightarrow k^2 + k - 20 = 0 \Rightarrow k = \ldots$	**dM1**	Eliminates a to a three-term quadratic in k and solves it by factorising, formula or completing the square, or by calculator.
	$k = 4\ (k > 1)$	**A1**	$k = 4$ only; A0 if $k = -5$ is also given as an answer.
		[3]	
	Total for Question 7	**9**	

Question	Scheme	Marks	Notes
8(a)	$60a^2$, $3ah$ $\mathcal{A} = 60a^2 - 3ah$	**B1**	Both areas, with the removed area subtracted; soi in a moment equation.
	$(5a, 3a)$, $\left(10a - \frac{h}{2}, \frac{9a}{2}\right)$	**B1**	Both centres of mass relative to AB and AD; soi in moment equations.
	$(60a^2 - 3ah)\bar{y}$ $= 60a^2(3a) - 3ah\left(\frac{9a}{2}\right)$	**M1**	Forms a moment equation about AB with the remaining area and the difference of the two pieces' moments. Condone slips in values.
	$\bar{y} = \frac{3a(40a-3h)}{2(20a-h)}$	**A1**	oe; need not be simplified.
	$(60a^2 - 3ah)\bar{x}$ $= 60a^2(5a) - 3ah\left(10a - \frac{h}{2}\right)$	**M1**	Forms the corresponding moment equation about AD. Condone slips in values.
	$\bar{x} = \frac{300a^3 - 30a^2h + \frac{3}{2}ah^2}{3a(20a-h)}$ $= \frac{h^2 - 20ah + 200a^2}{2(20a-h)}$	**A1 AG**	Correct moment equation about AD leading without error to the printed result; combined steps accepted.
		[6]	
8(b)	$\tan\theta = \frac{\text{“}\bar{x}\text{”}}{6a - \text{“}\bar{y}\text{”}} = \frac{13}{12}$ $\Rightarrow \frac{h^2 - 20ah + 200a^2}{3a(40a-h)} = \frac{13}{12}$	**M1**	Equates $\frac{13}{12}$ to the centre of mass's horizontal distance from D over its vertical distance below D, using their (a) expressions in h.
	$12h^2 - 201ah + 840a^2 = 0 \Rightarrow h = \ldots$	**dM1**	Clears fractions to a three-term quadratic in h and solves it by formula, factorising, completing the square or calculator.
	$h = 8a$, $h = \frac{35a}{4}$	**A1**	Both, oe, and no others.
		[3]	
	Total for Question 8	**9**	

Question	Scheme	Marks	Notes
9(a)	$\bar{x} = 0$	**B1**	
	Rectangle: $\mathcal{A}_1 = 240 \times 100 = 24\,000$ Each triangle: $\mathcal{A}_2 = \frac{1}{2} \times 100 \times 60 = 3000$	**B1**	Correct areas of the pieces of any valid decomposition.
	$y_1 = 50$, $y_2 = 100 - \frac{60}{3} = 80$	**B1**	Correct heights of the centres of mass of the pieces of any valid decomposition.
	$18\,000\bar{y} = 24\,000(50) - 2(3000)(80)$	**M1**	Forms a moment equation about the x-axis with a term for every piece and the total area, removed pieces negative. Condone slips in values.
	$\bar{y} = \frac{720\,000}{18\,000} = 40$	**A1 AG**	Correct substituted equation leading without error to $\bar{y} = 40$.
		[5]	
9(b)	Needs $(0, 40)$ on or inside triangle DEG: DG: $\frac{x-p}{20-p} = \frac{40}{100} \Rightarrow x = \ldots$	**M1**	Uses that $(0, 40)$ must lie in triangle DEG, by where DG or EG meets $y = 40$, a line from D or E through $(0, 40)$, or $R_D, R_E \geq 0$ by moments.
	DG: $x = 8 + \frac{3p}{5}$ EG: $x = -8 + \frac{3p}{5}$	**A1**	Both, oe, or both limiting values $p = \pm\frac{40}{3}$ from lines through $(0, 40)$, or $R_D = W(0.2 - 0.015p)$ and $R_E = W(0.2 + 0.015p)$, oe.
	$-8 + \frac{3p}{5} \leq 0 \leq 8 + \frac{3p}{5}$ $\Rightarrow -\frac{40}{3} \leq p \leq \frac{40}{3}$	**A1**	oe, such as $\lvert p \rvert \leq \frac{40}{3}$, or with awrt 13.3. Condone strict inequalities.
		[3]	
9(c)	$R_D + R_E + R_G = 100$ Moments about AB: $100R_D + 100R_E = 100 \times 40$ $\Rightarrow R_G = \ldots$	**M1**	Obtains an equation in R_G alone, by resolving vertically with moments about AB, or by moments about DE. Condone sign slips.
	$R_G = 60\,\text{N}$	**A1**	
	Moments about the y-axis: $20R_D - 20R_E + 8\text{“}60\text{”} = 0$	**M1**	Takes moments about another axis, such as the y-axis, with all nonzero terms. Condone sign slips.
	$R_D - R_E = -24$, $R_D + R_E = 40$ $\Rightarrow R_D = 8\,\text{N}$, $R_E = 32\,\text{N}$	**A1**	
		[4]	
	Total for Question 9	**12**	

Question	Scheme	Marks	Notes
10	$A = (k+4)(k+3)a^2 - k(k+1)a^2$ $[= 6(k+2)a^2]$	**B1**	Both areas, the hole's subtracted. Need not be simplified.
	$\left(\frac{(k+4)a}{2}, \frac{(k+3)a}{2}\right)$, $\left(\frac{(k+3)a}{2}, \frac{(k+2)a}{2}\right)$	**B1**	Both centres of mass, in any form.
	$6(k+2)a^2\,\bar{x} = (k+4)(k+3)a^2 \cdot \frac{(k+4)a}{2}$ $- k(k+1)a^2 \cdot \frac{(k+3)a}{2}$	**M1**	Forms a moment equation for $\bar{x}$ with their two areas and x-coordinates, the hole's term subtracted. Condone slips in the values.
	$\bar{x} = \frac{a(k+3)(7k+16)}{12(k+2)}$	**A1**	oe, need not be simplified.
	$6(k+2)a^2\,\bar{y} = (k+4)(k+3)a^2 \cdot \frac{(k+3)a}{2}$ $- k(k+1)a^2 \cdot \frac{(k+2)a}{2}$	**M1**	Forms the $\bar{y}$ equation as for $\bar{x}$, with their y-coordinates.
	$\bar{y} = \frac{a(7k^2+31k+36)}{12(k+2)}$	**A1**	oe, need not be simplified.
	$a < \bar{x} < (k+2)a$, $a < \bar{y} < (k+1)a$	**M1**	States all four bounds from the hole's edges, applied to their $\bar{x}$ and $\bar{y}$.
	$\bar{x} > a \Rightarrow 7k^2 + 25k + 24 > 0$ $\bar{x} < (k+2)a \Rightarrow 5k^2 + 11k > 0$ $\bar{y} > a \Rightarrow 7k^2 + 19k + 12 > 0$ True for all $k > 0$	**A1**	All three reduced correctly to a form evidently true for $k > 0$, such as all coefficients positive, with a conclusion.
	$7k^2 + 31k + 36 < 12(k+2)(k+1)$ $\Rightarrow 5k^2 + 5k - 12 > 0 \Rightarrow k = \ldots$	**dM1**	Rearranges $\bar{y} < (k+1)a$ with their $\bar{y}$ into a three-term quadratic inequality and finds its critical values by any means. Depends on the $\bar{y}$ equation's M1, not on the bounds.
	$k > \frac{\sqrt{265}-5}{10}$	**A1**	oe, or $k >$ awrt 1.13, with the negative region rejected. Condone $\geq$.
	Total for Question 10	**10**	