Questions

1. **(i)** In each of the following cases, find a 2×2 matrix that represents

(a) a reflection in the line $y = x$ **[1]**

(b) a rotation of $60°$ clockwise about $(0, 0)$ **[1]**

(c) a rotation of $60°$ clockwise about $(0, 0)$ followed by a reflection in the line $y = x$ **[2]**

(ii) The triangle T has vertices at the points $(1, 2)$, $(5, 5)$ and $(k, -1)$, where $k > -3$ is a constant.

Triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}$$

Given that the area of triangle T' is 84 square units, find the value of k. **[6]**

2.

$$\mathbf{P} = \begin{pmatrix} p+1 & 2 \\ 2 & p+1 \end{pmatrix}$$

where p is a positive constant.

The matrix $\mathbf{P}$ represents a linear transformation U.
The parallelogram S has vertices at the points with coordinates $(1, 1)$, $(4, 2)$, $(6, 5)$ and $(3, 4)$.
The area of the image of S under the linear transformation U is 35.

(a) Determine the value of p. **[4]**

The transformation V consists of a reflection in the line $y = x$ followed by a stretch of scale factor 2 parallel to the x-axis with the y-axis invariant. The transformation V is represented by the matrix $\mathbf{Q}$.

(b) Write down the matrix $\mathbf{Q}$. **[2]**

Given that U followed by V is the transformation W, which is represented by the matrix $\mathbf{R}$

(c) find the matrix $\mathbf{R}$. **[2]**

3. (a) Specify fully the transformation $\mathbf{T}$ of the plane associated with the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and $0 < \theta < 2\pi$. **[2]**

(b) (i) Find $\det \mathbf{M}$. **[1]**

(ii) Deduce **two properties** of the transformation $\mathbf{T}$ from the value of $\det \mathbf{M}$. **[2]**

(c) Prove that

$$\mathbf{M}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

where n is a positive integer. **[4]**

(d) Hence specify fully a **single** transformation which is equivalent to n applications of the transformation $\mathbf{T}$. **[1]**

4. Consider the matrix

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(i) Calculate A^2. **[1]**

You are told that the matrix A represents a reflection in a line through the origin.

(ii) Explain why your answer to part (i) is consistent with this information. **[1]**

(iii) By finding the invariant points of the transformation, determine the equation of the mirror line. **[3]**

(iv) Describe fully the transformation represented by

$$B = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

[2]

(v) A composite transformation is formed by the transformation represented by A followed by the transformation represented by B. Find the single matrix that represents this composite transformation. **[2]**

(vi) The composite transformation described in part (v) is equivalent to a single reflection. Find the equation of the mirror line of this reflection. **[1]**

5. The transformation U, represented by the 2×2 matrix P, is a reflection in the x-axis.

(a) Write down the matrix P. **[1]**

The transformation V, represented by the 2×2 matrix Q, is a reflection in the line $y = x$.

(b) Write down the matrix Q. **[1]**

Given that U followed by V is transformation T, which is represented by the matrix R

(c) express R in terms of P and Q, **[1]**

(d) find the matrix R, **[2]**

(e) give a full geometrical description of T as a single transformation. **[2]**

6. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

where $k > 0$ and $k \neq 1$.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations. State the type of each transformation, and make clear the order in which they are applied. **[2]**

The unit square in the x-y plane is transformed by $\mathbf{M}$ onto rectangle $OABC$.

(b) Find, in terms of k, the area of rectangle $OABC$ and the matrix which transforms $OABC$ onto the unit square. **[3]**

(c) Show that the line through the origin with gradient $\frac{1}{\sqrt{k}}$ is invariant under the transformation in the x-y plane represented by $\mathbf{M}$. **[3]**

7. You are given the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(a) Find $\mathbf{A}^5$. **[1]**

(b) Describe the transformation that $\mathbf{A}$ represents. **[2]**

The matrix $\mathbf{B}$ represents a reflection in the plane $x = y$.

(c) Write down the matrix $\mathbf{B}$. **[1]**

The point Q has coordinates $(4, -1, 2)$. The point Q' is the image of Q under the transformation represented by $\mathbf{B}$.

(d) Find the coordinates of Q'. **[1]**

8. A right-angled triangle T has vertices $A(-2,1)$, $B(1,1)$ and $C(1,5)$. When T is transformed by the matrix

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the image is T'.

(a) Find the coordinates of the vertices of T'. **[2]**

(b) Describe fully the transformation represented by P. **[2]**

The matrices

$$Q = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

represent two transformations. When T is transformed by the matrix QR, the image is T''.

(c) Find QR. **[2]**

(d) Find the determinant of QR. **[2]**

(e) Using your answer to part (d), find the area of T''. **[3]**

9. Given that the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

represents a rotation through $\alpha°$ about the y-axis in the positive sense given by the right-hand rule.

(a) determine the value of α. **[1]**

(b) Hence determine the smallest possible positive integer value of k for which $\mathbf{M}^k = \mathbf{I}$. **[2]**

The 3×3 matrix $\mathbf{N}$ represents a reflection in the plane with equation $x = 0$.

(c) Write down the matrix $\mathbf{N}$. **[1]**

The point A has coordinates $(1, 2, 3)$.

The point B is the image of the point A under the transformation represented by matrix $\mathbf{M}$ followed by the transformation represented by matrix $\mathbf{N}$.

(d) Show that the coordinates of B are $(2\sqrt{2}, 2, -\sqrt{2})$. **[2]**

Given that O is the origin

(e) show that, to 3 significant figures, the size of angle AOB is $79.4°$. **[2]**

(f) Hence determine the area of triangle AOB, giving your answer to 3 significant figures. **[2]**

10. Bilal has been asked to describe the transformation given by the matrix

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

He writes his answer as follows:

> The transformation is a rotation about the y-axis through an angle of θ, where
>
> $$\cos\theta = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = \frac{1}{2}$$
>
> $$\theta = 30^\circ$$

Identify and correct the error in Bilal's work. **[2]**

11. Three non-singular 3×3 matrices, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{R}$, are such that

$$\mathbf{RA} = \mathbf{B}$$

The matrix $\mathbf{R}$ represents a rotation about the x-axis through an angle θ and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & \cos\theta & \sin\theta \end{bmatrix}$$

(a) Show that $\mathbf{A}$ is independent of the value of θ. **[3]**

(b) Give a full description of the single transformation represented by the matrix $\mathbf{A}$. **[1]**