CORE PURE / MISCELLANEOUS PURE

Standard Summations

Worked solutions 15 questions · 120 marks

EXAM BOARDS Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 · CIE 9231

Questions

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1. Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=5}^{12}(2r-1)(r+3) = 1556 \qquad \textbf{[4]}$$

Solution

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Expanding the general term gives

$$(2r-1)(r+3) = 2r^2 + 6r - r - 3 = 2r^2 + 5r - 3$$

so

$$\sum_{r=1}^{n}(2r-1)(r+3) = 2\sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r - 3n$$

The required sum is the sum from $r = 1$ to 12 minus the sum from $r = 1$ to 4:

$$\sum_{r=5}^{12}(2r-1)(r+3) = \sum_{r=1}^{12}(2r-1)(r+3) - \sum_{r=1}^{4}(2r-1)(r+3)$$

For $n = 12$:

$$\sum_{r=1}^{12} r^2 = \frac{1}{6} \times 12 \times 13 \times 25 = 650, \qquad \sum_{r=1}^{12} r = \frac{1}{2} \times 12 \times 13 = 78$$

$$\sum_{r=1}^{12}(2r-1)(r+3) = 2 \times 650 + 5 \times 78 - 3 \times 12 = 1300 + 390 - 36 = 1654$$

For $n = 4$:

$$\sum_{r=1}^{4} r^2 = \frac{1}{6} \times 4 \times 5 \times 9 = 30, \qquad \sum_{r=1}^{4} r = \frac{1}{2} \times 4 \times 5 = 10$$

$$\sum_{r=1}^{4}(2r-1)(r+3) = 2 \times 30 + 5 \times 10 - 3 \times 4 = 60 + 50 - 12 = 98$$

Therefore

$$\sum_{r=5}^{12}(2r-1)(r+3) = 1654 - 98 = 1556$$

as required.

2. Using standard summation results, find the exact value of

$$\sum_{r=21}^{60}(3r-1)(r+2) \qquad \textbf{[4]}$$

Solution

Expanding the bracket:

$$(3r-1)(r+2) = 3r^2 + 5r - 2$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and also $\sum_{r=1}^{n} 1 = n$. Hence

$$\begin{aligned}\sum_{r=1}^{n}(3r-1)(r+2) &= 3\sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r - 2\sum_{r=1}^{n} 1 \\ &= \frac{1}{2}n(n+1)(2n+1) + \frac{5}{2}n(n+1) - 2n \\ &= \frac{1}{2}n(n+1)[(2n+1)+5] - 2n \\ &= n(n+1)(n+3) - 2n \\ &= n[(n+1)(n+3) - 2] \\ &= n\left(n^2 + 4n + 1\right)\end{aligned}$$

The required sum is the sum from $r = 1$ to 60 minus the sum from $r = 1$ to 20:

$$\begin{aligned}\sum_{r=21}^{60}(3r-1)(r+2) &= \sum_{r=1}^{60}(3r-1)(r+2) - \sum_{r=1}^{20}(3r-1)(r+2) \\ &= 60\left(60^2 + 4\times 60 + 1\right) - 20\left(20^2 + 4\times 20 + 1\right) \\ &= 60\times 3841 - 20\times 481 \\ &= 230\,460 - 9620 \\ &= 220\,840\end{aligned}$$

3. Using standard summation results, find the exact value of

$$\sum_{r=11}^{40} r(r-3)(r+3) \qquad \textbf{[4]}$$

Solution

First expand the general term:

$$r(r-3)(r+3) = r\left(r^2-9\right) = r^3 - 9r$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

so

$$\begin{aligned}\sum_{r=1}^{n}\left(r^3-9r\right) &= \frac{1}{4}n^2(n+1)^2 - \frac{9}{2}n(n+1) \\ &= \frac{1}{4}n(n+1)[n(n+1)-18]\end{aligned}$$

The required sum is the sum from $r=1$ to 40 minus the sum from $r=1$ to 10:

$$\sum_{r=11}^{40}\left(r^3-9r\right) = \sum_{r=1}^{40}\left(r^3-9r\right) - \sum_{r=1}^{10}\left(r^3-9r\right)$$

With $n=40$:

$$\sum_{r=1}^{40}\left(r^3-9r\right) = \frac{1}{4}\times 40\times 41\times(1640-18) = 410\times 1622 = 665\,020$$

With $n=10$:

$$\sum_{r=1}^{10}\left(r^3-9r\right) = \frac{1}{4}\times 10\times 11\times(110-18) = \frac{110}{4}\times 92 = 2530$$

Therefore

$$\sum_{r=11}^{40} r(r-3)(r+3) = 665\,020 - 2530 = 662\,490$$

4. The positive integer k is such that

$$\sum_{r=1}^{k}\left(6r^2-k^2\right)=205$$

Find the value of k, showing clearly that there is only one possible value. **[4]**

Solution

The standard result is

$$\sum_{r=1}^{k} r^2=\frac{1}{6}k(k+1)(2k+1)$$

The term k^2 does not depend on r, so summing it k times gives $k\times k^2=k^3$. Hence

$$\begin{aligned}\sum_{r=1}^{k}\left(6r^2-k^2\right)&=6\times\frac{1}{6}k(k+1)(2k+1)-k^3\\&=k(k+1)(2k+1)-k^3\\&=2k^3+3k^2+k-k^3\\&=k^3+3k^2+k\end{aligned}$$

Setting this equal to 205:

$$k^3+3k^2+k-205=0$$

When $k=5$, $125+75+5-205=0$, so $(k-5)$ is a factor. Factorising:

$$(k-5)\left(k^2+8k+41\right)=0$$

For $k^2+8k+41=0$, the discriminant is

$$b^2-4ac=8^2-4\times1\times41=64-164=-100<0$$

so this quadratic has no real roots. Therefore the only real solution is $k=5$, which is a positive integer, so there is only one possible value:

$$k=5$$

5. **(a)** Prove that, for all positive integers n,

$$\sum_{r=1}^{n}(2r-1)^3 = n^2\left(2n^2-1\right) \qquad \textbf{[5]}$$

(b) Jo evaluated

$$\frac{1}{n^2}\sum_{r=1}^{n}(2r-1)^3$$

for $n=2$, $n=3$ and $n=4$, and obtained a prime number each time. Jo then made the statement:

"For all integers $n \geq 2$,

$$\frac{1}{n^2}\sum_{r=1}^{n}(2r-1)^3$$

is a prime number."

Disprove Jo's statement. **[2]**

Solution

(a) The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

together with $\sum_{r=1}^{n} 1 = n$.

Expanding the cubic:

$$(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$$

So

$$\begin{aligned}
\sum_{r=1}^{n}(2r-1)^3 &= 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\
&= 8 \times \frac{1}{4}n^2(n+1)^2 - 12 \times \frac{1}{6}n(n+1)(2n+1) + 6 \times \frac{1}{2}n(n+1) - n \\
&= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n \\
&= n\left[2n(n+1)^2 - 2(n+1)(2n+1) + 3(n+1) - 1\right] \\
&= n\left[2n^3 + 4n^2 + 2n - 4n^2 - 6n - 2 + 3n + 3 - 1\right] \\
&= n\left(2n^3 - n\right) \\
&= n^2\left(2n^2 - 1\right)
\end{aligned}$$

as required.

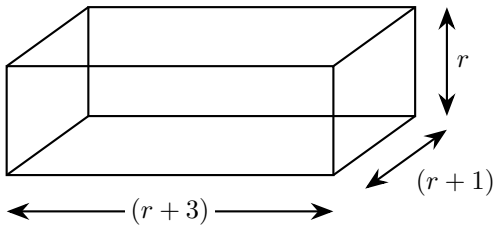
(b) Using part (a), for $n \geq 2$:

$$\frac{1}{n^2}\sum_{r=1}^{n}(2r-1)^3 = 2n^2 - 1$$

When $n = 5$:

$$2 \times 5^2 - 1 = 49 = 7 \times 7$$

Since 49 is not prime, $n = 5$ is a counterexample, so Jo's statement is false.



6. A block in the shape of a cuboid has length $(r+3)$ cm, width $(r+1)$ cm and height r cm, as shown in the diagram above.

A set consists of n such blocks. The first block has a height of 1 cm, the second block has a height of 2 cm, the third block has a height of 3 cm, and so on, so that the rth block has height r cm.

(a) Use the standard results for $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that the **total** volume, V cm^3, of all n blocks in the set is given by

$$V = \frac{n}{12}(n+1)(n+2)(3n+13), \qquad n \geq 1$$ **[5]**

(b) Find the smallest number of blocks in the set for which the total volume of the set exceeds 5000 cm^3. **[2]**

Solution

(a) The rth block has volume

$$r(r+1)(r+3) = r\left(r^2+4r+3\right) = r^3+4r^2+3r$$

so the total volume is

$$V = \sum_{r=1}^{n}\left(r^3+4r^2+3r\right) = \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Substituting these and taking out the common factor $\frac{n(n+1)}{12}$:

$$\begin{aligned} V &= \frac{1}{4}n^2(n+1)^2 + \frac{4}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) \\ &= \frac{n(n+1)}{12}\Big[3n(n+1) + 8(2n+1) + 18\Big] \\ &= \frac{n(n+1)}{12}\left(3n^2+3n+16n+8+18\right) \\ &= \frac{n(n+1)}{12}\left(3n^2+19n+26\right) \end{aligned}$$

Factorising the quadratic, $3n^2+19n+26 = (n+2)(3n+13)$, so

$$V = \frac{n}{12}(n+1)(n+2)(3n+13)$$

as required.

(b) Each block has positive volume, so V increases as n increases. Using part (a):

$$n = 10: \quad V = \frac{10}{12} \times 11 \times 12 \times 43 = 4730 < 5000$$

$$n = 11: \quad V = \frac{11}{12} \times 12 \times 13 \times 46 = 6578 > 5000$$

The smallest number of blocks is 11.

7. **(a)** Use the standard results for $\sum r$, $\sum r^2$ and $\sum r^3$ to show that for all positive integers n

$$\sum_{r=1}^{n} r(2r-1)^2 = \frac{1}{6}n(n+1)(an^2+bn+c)$$

where a, b and c are integers to be determined. **[5]**

(b) Hence determine the positive integer k for which

$$\sum_{r=1}^{k} r(2r-1)^2 = 193\sum_{r=1}^{k} r$$ **[4]**

Solution

(a) First expand the general term:

$$r(2r-1)^2 = r\big(4r^2-4r+1\big) = 4r^3-4r^2+r$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

So

$$\begin{aligned}\sum_{r=1}^{n} r(2r-1)^2 &= 4\sum_{r=1}^{n} r^3 - 4\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r \\ &= 4\times\frac{1}{4}n^2(n+1)^2 - 4\times\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= n^2(n+1)^2 - \frac{2}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\end{aligned}$$

Taking out the common factor $\frac{1}{6}n(n+1)$:

$$\begin{aligned}\sum_{r=1}^{n} r(2r-1)^2 &= \frac{1}{6}n(n+1)\big[6n(n+1) - 4(2n+1) + 3\big] \\ &= \frac{1}{6}n(n+1)\big(6n^2+6n-8n-4+3\big) \\ &= \frac{1}{6}n(n+1)\big(6n^2-2n-1\big)\end{aligned}$$

This is in the required form with $a=6$, $b=-2$, $c=-1$.

(b) Using part (a) and the standard result for $\sum r$, with n replaced by k, the equation becomes

$$\frac{1}{6}k(k+1)\big(6k^2-2k-1\big) = 193\times\frac{1}{2}k(k+1)$$

Since k is a positive integer, $k(k+1) \neq 0$, so dividing both sides by $\frac{1}{6}k(k+1)$ gives

$$6k^2 - 2k - 1 = 579$$

$$6k^2 - 2k - 580 = 0$$

$$3k^2 - k - 290 = 0$$

$$(k-10)(3k+29) = 0$$

So $k = 10$ or $k = -\frac{29}{3}$. Since k is a positive integer, $k = -\frac{29}{3}$ is rejected, so

$$k = 10$$

8. **(a)** Show that

$$\sum_{r=1}^{n} r^2(r+3)$$

can be expressed in the form

$$kn(n+1)(an^2+bn+c)$$

where k is a rational number and a, b and c are integers. **[4]**

(b) Show that there is exactly one positive integer n for which

$$2\sum_{r=1}^{n} r^2(r+3) = 19\sum_{r=1}^{n} r(r+1)$$

and state this value of n. **[6]**

Solution

(a) Expanding the general term gives $r^2(r+3) = r^3 + 3r^2$, so

$$\sum_{r=1}^{n} r^2(r+3) = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2$$

The standard results are

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

so

$$\begin{aligned}\sum_{r=1}^{n} r^2(r+3) &= \frac{1}{4}n^2(n+1)^2 + 3\times\frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1)\\ &= \frac{1}{4}n(n+1)[n(n+1)+2(2n+1)]\\ &= \frac{1}{4}n(n+1)\left(n^2+n+4n+2\right)\\ &= \frac{1}{4}n(n+1)\left(n^2+5n+2\right)\end{aligned}$$

This is in the form $kn(n+1)(an^2+bn+c)$ with $k=\frac{1}{4}$, $a=1$, $b=5$, $c=2$.

(b) First find $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$. The standard result for $\sum r$ is

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

and together with the result for $\sum r^2$ this gives

$$\begin{aligned}\sum_{r=1}^{n} r(r+1) &= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\ &= \frac{1}{6}n(n+1)[(2n+1)+3] \\ &= \frac{1}{6}n(n+1)(2n+4) \\ &= \frac{1}{3}n(n+1)(n+2)\end{aligned}$$

Using part (a), the equation becomes

$$2 \times \frac{1}{4}n(n+1)\left(n^2+5n+2\right) = 19 \times \frac{1}{3}n(n+1)(n+2)$$

For a positive integer n, $n(n+1) \neq 0$, so dividing by $n(n+1)$ gives

$$\frac{1}{2}\left(n^2+5n+2\right) = \frac{19}{3}(n+2)$$

Multiplying by 6:

$$\begin{aligned}3n^2+15n+6 &= 38n+76 \\ 3n^2-23n-70 &= 0 \\ (3n+7)(n-10) &= 0\end{aligned}$$

So $n = 10$ or $n = -\frac{7}{3}$. Since $-\frac{7}{3}$ is not a positive integer, the only solution that is a positive integer is $n = 10$. Hence there is exactly one positive integer satisfying the equation, namely $n = 10$.

9. **(a)** Find an expression for

$$\sum_{r=1}^{2m}(r-1)(r+3)$$

in the form $\frac{1}{3}m(am^2+bm+c)$, where a, b, c are integers whose values are to be determined. **[4]**

(b) Hence show that

$$\sum_{r=2m+1}^{4m}(r-1)(r+3)=\frac{1}{3}m\left(56m^2+54m-11\right)$$

and use this result to find the value of

$$\sum_{r=21}^{40}(r-1)(r+3)$$ **[5]**

Solution

(a) Expand the general term:

$$(r-1)(r+3)=r^2+2r-3$$

The standard results are

$$\sum_{r=1}^{m}r=\frac{1}{2}m(m+1),\qquad \sum_{r=1}^{m}r^2=\frac{1}{6}m(m+1)(2m+1)$$

Using them with m replaced by $2m$, together with $\sum_{r=1}^{2m}3=3\times 2m=6m$, gives

$$\begin{aligned}\sum_{r=1}^{2m}(r-1)(r+3)&=\sum_{r=1}^{2m}r^2+2\sum_{r=1}^{2m}r-\sum_{r=1}^{2m}3\\&=\frac{1}{6}(2m)(2m+1)(4m+1)+2\times\frac{1}{2}(2m)(2m+1)-6m\\&=\frac{1}{3}m(2m+1)(4m+1)+2m(2m+1)-6m\\&=\frac{1}{3}m\left[(8m^2+6m+1)+6(2m+1)-18\right]\\&=\frac{1}{3}m\left(8m^2+6m+1+12m+6-18\right)\\&=\frac{1}{3}m\left(8m^2+18m-11\right)\end{aligned}$$

This is in the form $\frac{1}{3}m(am^2+bm+c)$ with $a=8$, $b=18$, $c=-11$.

(b) Replacing m by $2m$ in the result of part (a):

$$\begin{aligned}\sum_{r=1}^{4m}(r-1)(r+3)&=\frac{1}{3}(2m)\left(8(2m)^2+18(2m)-11\right)\\&=\frac{1}{3}(2m)\left(32m^2+36m-11\right)\\&=\frac{1}{3}m\left(64m^2+72m-22\right)\end{aligned}$$

The required sum is the sum from 1 to $4m$ minus the sum from 1 to $2m$, so

$$\begin{aligned}\sum_{r=2m+1}^{4m}(r-1)(r+3) &= \sum_{r=1}^{4m}(r-1)(r+3) - \sum_{r=1}^{2m}(r-1)(r+3)\\ &= \frac{1}{3}m\big(64m^2+72m-22\big) - \frac{1}{3}m\big(8m^2+18m-11\big)\\ &= \frac{1}{3}m\big(56m^2+54m-11\big)\end{aligned}$$

as required.

For $\sum_{r=21}^{40}(r-1)(r+3)$, take $2m+1=21$, so $m=10$, and then $4m=40$ as needed. Substituting $m=10$:

$$\begin{aligned}\sum_{r=21}^{40}(r-1)(r+3) &= \frac{1}{3}\times 10(56\times 100+54\times 10-11)\\ &= \frac{10}{3}\times 6129\\ &= 20\,430\end{aligned}$$

10. **(i)** Use standard series to show that

$$\sum_{r=1}^{n} r(r+1)(4r-k) = \frac{1}{3}n(n+1)(n+2)(3n+1-k)$$

where k is a constant. **[5]**

(ii) Using your result from part (i) with $k = 1$, find

$$\sum_{r=n+1}^{2n} r(r+1)(4r-1)$$

giving your answer in fully factorised form. **[3]**

Solution

(i) Expanding the general term:

$$r(r+1)(4r-k) = (r^2+r)(4r-k) = 4r^3 + (4-k)r^2 - kr$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

so

$$\begin{aligned}
\sum_{r=1}^{n} r(r+1)(4r-k) &= 4\sum_{r=1}^{n} r^3 + (4-k)\sum_{r=1}^{n} r^2 - k\sum_{r=1}^{n} r \\
&= n^2(n+1)^2 + \frac{1}{6}(4-k)n(n+1)(2n+1) - \frac{1}{2}kn(n+1) \\
&= \frac{1}{6}n(n+1)\big[6n(n+1) + (4-k)(2n+1) - 3k\big] \\
&= \frac{1}{6}n(n+1)\big[6n^2 + 6n + 8n + 4 - 2kn - k - 3k\big] \\
&= \frac{1}{6}n(n+1)\big[6n^2 + 14n - 2kn + 4 - 4k\big] \\
&= \frac{1}{3}n(n+1)\big[3n^2 + 7n - kn + 2 - 2k\big]
\end{aligned}$$

Expanding gives $(n+2)(3n+1-k) = 3n^2 + n - kn + 6n + 2 - 2k = 3n^2 + 7n - kn + 2 - 2k$, so

$$\sum_{r=1}^{n} r(r+1)(4r-k) = \frac{1}{3}n(n+1)(n+2)(3n+1-k)$$

as required.

(ii) Using part (i) with $k = 1$:

$$\sum_{r=1}^{n} r(r+1)(4r-1) = \frac{1}{3}n(n+1)(n+2)(3n) = n^2(n+1)(n+2)$$

Replacing n by $2n$:

$$\sum_{r=1}^{2n} r(r+1)(4r-1) = (2n)^2(2n+1)(2n+2) = 8n^2(n+1)(2n+1)$$

Therefore

$$\begin{aligned}\sum_{r=n+1}^{2n} r(r+1)(4r-1) &= \sum_{r=1}^{2n} r(r+1)(4r-1) - \sum_{r=1}^{n} r(r+1)(4r-1) \\ &= 8n^2(n+1)(2n+1) - n^2(n+1)(n+2) \\ &= n^2(n+1)\big[16n+8-n-2\big] \\ &= n^2(n+1)(15n+6) \\ &= 3n^2(n+1)(5n+2)\end{aligned}$$

11. **(a)** Use the standard results for $\sum_{r=1}^{n} r^3$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}(2r-1)^3 = n^2\left(2n^2-1\right)$$

for all positive integers n. **[5]**

(b) Hence find any positive integer values of n for which

$$\sum_{r=n+1}^{2n}(2r-1)^3 = 29n^4 - 34n^2 + \sum_{r=1}^{\infty} 240\left(\frac{3}{5}\right)^r$$ **[5]**

Solution

(a) Expanding the cube:

$$(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

and $\sum_{r=1}^{n} 1 = n$. Therefore

$$\begin{aligned}
\sum_{r=1}^{n}(2r-1)^3 &= 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\
&= 8 \times \frac{1}{4}n^2(n+1)^2 - 12 \times \frac{1}{6}n(n+1)(2n+1) + 6 \times \frac{1}{2}n(n+1) - n \\
&= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + 3n(n+1) - n \\
&= \left(2n^4+4n^3+2n^2\right) - \left(4n^3+6n^2+2n\right) + \left(3n^2+3n\right) - n \\
&= 2n^4 - n^2 \\
&= n^2\left(2n^2-1\right)
\end{aligned}$$

as required.

(b) Using part (a) with n replaced by $2n$:

$$\sum_{r=1}^{2n}(2r-1)^3 = (2n)^2\left(2(2n)^2-1\right) = 4n^2\left(8n^2-1\right) = 32n^4 - 4n^2$$

So the left-hand side is

$$\begin{aligned}
\sum_{r=n+1}^{2n}(2r-1)^3 &= \sum_{r=1}^{2n}(2r-1)^3 - \sum_{r=1}^{n}(2r-1)^3 \\
&= \left(32n^4-4n^2\right) - \left(2n^4-n^2\right) \\
&= 30n^4 - 3n^2
\end{aligned}$$

The infinite series is geometric with first term $240 \times \frac{3}{5} = 144$ and common ratio $\frac{3}{5}$. Since $\left|\frac{3}{5}\right| < 1$, using

$$S_\infty = \frac{a}{1-r} \qquad (|r| < 1)$$

gives

$$\sum_{r=1}^{\infty} 240\left(\frac{3}{5}\right)^r = \frac{144}{1-\frac{3}{5}} = \frac{144}{\frac{2}{5}} = 360$$

The equation becomes

$$30n^4 - 3n^2 = 29n^4 - 34n^2 + 360$$

$$n^4 + 31n^2 - 360 = 0$$

$$\left(n^2 + 40\right)\left(n^2 - 9\right) = 0$$

Since $n^2 = -40$ has no real solutions, $n^2 = 9$. As n is a positive integer, this gives

$$n = 3$$

12. **(a)** Use the standard results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+a)^2(n+b)$$

where a and b are integers to be found. **[5]**

(b) Given that

$$\sum_{r=11}^{n} r(r+1)(2r+1) = 17\,940$$

use your answer to part (a) to find the value of the positive integer n, showing clear algebraic working. **[4]**

Solution

(a) Expanding the general term:

$$r(r+1)(2r+1) = (r^2+r)(2r+1) = 2r^3 + 3r^2 + r$$

The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

so

$$\begin{aligned}
\sum_{r=1}^{n} r(r+1)(2r+1) &= 2\sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r \\
&= 2 \times \frac{1}{4}n^2(n+1)^2 + 3 \times \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\
&= \frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1) \\
&= \frac{1}{2}n(n+1)[n(n+1) + (2n+1) + 1] \\
&= \frac{1}{2}n(n+1)\left(n^2+3n+2\right) \\
&= \frac{1}{2}n(n+1)(n+1)(n+2) \\
&= \frac{1}{2}n(n+1)^2(n+2)
\end{aligned}$$

This is in the form $\frac{1}{2}n(n+a)^2(n+b)$ with $a = 1$, $b = 2$.

(b) Write $S_n = \sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$. Then

$$\sum_{r=11}^{n} r(r+1)(2r+1) = S_n - S_{10}$$

where

$$S_{10} = \frac{1}{2} \times 10 \times 11^2 \times 12 = 7260$$

Hence

$$\frac{1}{2}n(n+1)^2(n+2) - 7260 = 17\,940$$

$$\frac{1}{2}n(n+1)^2(n+2) = 25\,200$$

$$n(n+2)(n+1)^2 = 50\,400$$

Let $m = n + 1$, so that $n(n+2) = (m-1)(m+1) = m^2 - 1$. Then

$$(m^2 - 1)m^2 = 50\,400$$

$$m^4 - m^2 - 50\,400 = 0$$

$$(m^2 - 225)(m^2 + 224) = 0$$

Since $m^2 > 0$, $m^2 = 225$, and as n is a positive integer, $m = n + 1 > 0$, so $m = 15$. Therefore

$$n = 14$$

13. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3, \quad \sum_{r=1}^{n} r^2 \quad \text{and} \quad \sum_{r=1}^{n} r$$

(a) Show that

$$1^3 - 2^3 + 3^3 - 4^3 + \cdots + (2n-1)^3 - (2n)^3 = -n^2(4n+3) \qquad \textbf{[5]}$$

Ten consecutive positive integers are written down, the smallest of which is odd. Starting with the cube of the smallest integer, the cubes of these ten integers are alternately added and subtracted, so that the cube of the largest integer is subtracted. The result is -3725.

(b) Use the answer to part (a) to determine the smallest of these ten integers. **[4]**

Solution

(a) The standard results are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

The terms pair up as an odd cube minus the next even cube, so the left-hand side is

$$\sum_{r=1}^{n} \left[(2r-1)^3 - (2r)^3\right]$$

Expanding the general term:

$$\begin{aligned}(2r-1)^3 - (2r)^3 &= \left(8r^3 - 12r^2 + 6r - 1\right) - 8r^3 \\ &= -12r^2 + 6r - 1\end{aligned}$$

Summing from $r = 1$ to n, with $\sum_{r=1}^{n} 1 = n$:

$$\begin{aligned}\sum_{r=1}^{n}\left(-12r^2 + 6r - 1\right) &= -12\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= -12 \times \frac{1}{6}n(n+1)(2n+1) + 6 \times \frac{1}{2}n(n+1) - n \\ &= -2n(n+1)(2n+1) + 3n(n+1) - n \\ &= n\left[-2\left(2n^2 + 3n + 1\right) + 3n + 3 - 1\right] \\ &= n\left(-4n^2 - 6n - 2 + 3n + 2\right) \\ &= n\left(-4n^2 - 3n\right) \\ &= -n^2(4n+3)\end{aligned}$$

as required.

(b) Let the smallest integer be $2m+1$, where m is a non-negative integer. Then the largest is $2m+10$, and the result is

$$\begin{aligned}&(2m+1)^3-(2m+2)^3+\cdots+(2m+9)^3-(2m+10)^3\\&=\left[1^3-2^3+\cdots-(2m+10)^3\right]-\left[1^3-2^3+\cdots-(2m)^3\right]\end{aligned}$$

Using part (a) with $n=m+5$ and with $n=m$:

$$\begin{aligned}-(m+5)^2(4m+23)+m^2(4m+3)&=-3725\\-\left(4m^3+63m^2+330m+575\right)+4m^3+3m^2&=-3725\\-60m^2-330m-575&=-3725\\60m^2+330m-3150&=0\\2m^2+11m-105&=0\\(m-5)(2m+21)&=0\end{aligned}$$

Since m must be a non-negative integer, $m=-\frac{21}{2}$ is rejected, so $m=5$. The smallest integer is $2\times5+1=11$.

14. **(a)** Use the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n} \left(6r^2 + 2r - 8\right) = an(n-b)(n+c)$$

where a, b and c are positive integers to be determined. **[4]**

(b) Explain why, for $k \in \mathbb{N}$,

$$\sum_{r=1}^{4k+1} r \sin\left(90r\right)^\circ = 2k + 1$$ **[3]**

Using the results from part (a) and part (b) and showing all your working,

(c) find the value of the integer n, where $n \geq 3$, that satisfies

$$\sum_{r=3}^{n} \left(6r^2 + 2r - 8\right) = 3\left[\sum_{r=3}^{4n+1} r \sin\left(90r\right)^\circ\right]^2$$

and show that there are no other such values of n. **[6]**

Solution

(a) The standard summation formulae are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Also $\sum_{r=1}^{n} 8 = 8n$. Splitting the sum:

$$\begin{aligned}\sum_{r=1}^{n}\left(6r^2 + 2r - 8\right) &= 6\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 8 \\ &= n(n+1)(2n+1) + n(n+1) - 8n \\ &= n[(n+1)(2n+1) + (n+1) - 8] \\ &= n[(n+1)(2n+2) - 8] \\ &= 2n\left[(n+1)^2 - 4\right]\end{aligned}$$

Using the difference of two squares, $(n+1)^2 - 4 = (n+1-2)(n+1+2) = (n-1)(n+3)$, so

$$\sum_{r=1}^{n}\left(6r^2 + 2r - 8\right) = 2n(n-1)(n+3)$$

This is in the form $an(n-b)(n+c)$ with $a = 2$, $b = 1$, $c = 3$.

(b) The values of $\sin(90r)^\circ$ for $r = 1, 2, 3, 4$ are $1, 0, -1, 0$, and since sin has period 360°, this pattern repeats every four terms. So the sum is

$$1 + 0 - 3 + 0 + 5 + 0 - 7 + 0 + \cdots$$

Group the first $4k$ terms into k blocks of four, $r = 4m+1, \ldots, 4m+4$ for $m = 0, 1, \ldots, k-1$. Each block contributes

$$(4m+1)(1) + (4m+2)(0) + (4m+3)(-1) + (4m+4)(0) = -2$$

so the sum of the first $4k$ terms is $k \times (-2) = -2k$.

The final term has $r = 4k + 1$, and $\sin(360k + 90)^\circ = \sin 90^\circ = 1$, so it contributes $4k + 1$. Therefore

$$\sum_{r=1}^{4k+1} r \sin(90r)^\circ = -2k + (4k + 1) = 2k + 1$$

as required.

(c) For the left-hand side, remove the $r = 1$ and $r = 2$ terms from the result of part (a). When $r = 1$ the term is $6 + 2 - 8 = 0$, and when $r = 2$ it is $24 + 4 - 8 = 20$. So

$$\sum_{r=3}^{n} \big(6r^2 + 2r - 8\big) = 2n(n - 1)(n + 3) - 20$$

For the right-hand side, use part (b) with $k = n$ and remove the $r = 1$ and $r = 2$ terms, which are $1 \times \sin 90^\circ = 1$ and $2 \times \sin 180^\circ = 0$:

$$\sum_{r=3}^{4n+1} r \sin(90r)^\circ = (2n + 1) - 1 - 0 = 2n$$

so the right-hand side is $3(2n)^2 = 12n^2$. The equation becomes

$$2n(n - 1)(n + 3) - 20 = 12n^2$$

$$2n\big(n^2 + 2n - 3\big) - 20 = 12n^2$$

$$2n^3 + 4n^2 - 6n - 20 = 12n^2$$

$$2n^3 - 8n^2 - 6n - 20 = 0$$

$$n^3 - 4n^2 - 3n - 10 = 0$$

Trying $n = 5$: $125 - 100 - 15 - 10 = 0$, so $(n - 5)$ is a factor. Factorising:

$$(n - 5)\big(n^2 + n + 2\big) = 0$$

For $n^2 + n + 2 = 0$, the discriminant is

$$b^2 - 4ac = 1^2 - 4 \times 1 \times 2 = -7 < 0$$

so this quadratic has no real roots. Hence the only solution is $n = 5$, which satisfies $n \geq 3$, and there are no other such values of n.

15. **(a)** Explain why, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{2n}(-1)^r f(r) = \sum_{r=1}^{n}(f(2r) - f(2r-1))$$

for any function $f(r)$. **[2]**

(b) Use the result of part (a) and the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{2n}(-1)^r r^2(r+1) = n(n+1)(4n+1)$$ **[4]**

(c) Hence show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{2n} r(r+1)\left((-1)^r r + 3\right) = n(n+1)(12n+5)$$ **[3]**

(d) Hence evaluate

$$\sum_{r=11}^{41} r(r+1)\left((-1)^r r + 3\right)$$ **[4]**

Solution

(a) When r is even, $(-1)^r = 1$, and when r is odd, $(-1)^r = -1$. The values $r = 1, 2, \ldots, 2n$ split into n consecutive pairs $r = 2k-1$ (odd) and $r = 2k$ (even), for $k = 1, 2, \ldots, n$. Each pair contributes

$$(-1)^{2k-1} f(2k-1) + (-1)^{2k} f(2k) = f(2k) - f(2k-1)$$

Adding the contributions of all n pairs, and renaming the index k as r, gives

$$\sum_{r=1}^{2n}(-1)^r f(r) = \sum_{r=1}^{n}(f(2r) - f(2r-1))$$

(b) Take $f(r) = r^2(r+1) = r^3 + r^2$. Then

$$f(2r) = (2r)^3 + (2r)^2 = 8r^3 + 4r^2$$

$$f(2r-1) = (2r-1)^2(2r) = 2r\left(4r^2 - 4r + 1\right) = 8r^3 - 8r^2 + 2r$$

so

$$f(2r) - f(2r-1) = 12r^2 - 2r$$

The standard summation formulae are

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Using part (a):

$$\sum_{r=1}^{2n}(-1)^r r^2(r+1) = \sum_{r=1}^{n}\big(12r^2 - 2r\big)$$
$$= 12 \times \frac{1}{6}n(n+1)(2n+1) - 2 \times \frac{1}{2}n(n+1)$$
$$= 2n(n+1)(2n+1) - n(n+1)$$
$$= n(n+1)(4n+2-1)$$
$$= n(n+1)(4n+1)$$

as required.

(c) Expanding the general term:

$$r(r+1)((-1)^r r + 3) = (-1)^r r^2(r+1) + 3\big(r^2 + r\big)$$

Using the standard formulae with n replaced by $2n$:

$$3\sum_{r=1}^{2n}\big(r^2 + r\big) = 3\left[\frac{1}{6}(2n)(2n+1)(4n+1) + \frac{1}{2}(2n)(2n+1)\right]$$
$$= n(2n+1)(4n+1) + 3n(2n+1)$$
$$= n(2n+1)(4n+4)$$
$$= 4n(n+1)(2n+1)$$

Hence, using part (b):

$$\sum_{r=1}^{2n} r(r+1)((-1)^r r + 3) = n(n+1)(4n+1) + 4n(n+1)(2n+1)$$
$$= n(n+1)(4n+1+8n+4)$$
$$= n(n+1)(12n+5)$$

as required.

(d) Let $u_r = r(r+1)((-1)^r r + 3)$ and $S_m = \sum_{r=1}^{m} u_r$. By part (c), $S_{2n} = n(n+1)(12n+5)$. Since the result of part (c) applies only to an even upper limit, write

$$\sum_{r=11}^{41} u_r = S_{41} - S_{10} = S_{40} + u_{41} - S_{10}$$

Using part (c) with $n = 20$ and $n = 5$:

$$S_{40} = 20 \times 21 \times 245 = 102\,900$$
$$S_{10} = 5 \times 6 \times 65 = 1950$$

Since 41 is odd, $(-1)^{41} = -1$, so

$$u_{41} = 41 \times 42 \times (-41 + 3) = 41 \times 42 \times (-38) = -65\,436$$

Therefore

$$\sum_{r=11}^{41} r(r+1)((-1)^r r+3) = 102\,900 - 65\,436 - 1950 = 35\,514$$