Questions

1. The cubic equation

$$x^3 - 5x^2 + px - q = 0$$

where p and q are positive real constants, has roots α, β and γ.

Given that

$$(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 = 14$$

(a) show that $p = 6$ **[3]**

Given that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 3$$

(b) determine the value of q **[3]**

(c) Without solving the cubic equation, determine the value of $(\alpha+1)(\beta+1)(\gamma+1)$ **[4]**

Solution

(a) For the cubic

$$x^3 - 5x^2 + px - q = 0$$

with roots α, β, γ, the relations between roots and coefficients give

$$\alpha + \beta + \gamma = 5, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = p$$

Now expand the given expression:

$$\begin{aligned}(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 &= (\alpha^2+\beta^2-2\alpha\beta) + (\beta^2+\gamma^2-2\beta\gamma) + (\gamma^2+\alpha^2-2\gamma\alpha)\\ &= 2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\end{aligned}$$

Also,

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

So

$$\begin{aligned}(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 &= 2\big((\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\big) - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\\ &= 2\big((\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\big)\end{aligned}$$

Substituting the known values,

$$\begin{aligned}14 &= 2(5^2 - 3p)\\ 14 &= 2(25-3p)\\ 7 &= 25 - 3p\\ 3p &= 18\\ p &= 6\end{aligned}$$

Hence, $p = 6$.

(b) For this cubic,

$$\alpha\beta\gamma = q$$

Also,

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}\\ &= \frac{p}{q}\end{aligned}$$

Given that this sum is 3,

$$\frac{p}{q} = 3$$

From part (a), $p = 6$, so

$$\begin{aligned} \frac{6}{q} &= 3 \\ 6 &= 3q \\ q &= 2 \end{aligned}$$

Hence, $q = 2$.

(c) Expand:

$$(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + (\alpha\beta+\beta\gamma+\gamma\alpha) + (\alpha+\beta+\gamma) + 1$$

Using the root relations,

$$\alpha\beta\gamma = q, \qquad \alpha\beta+\beta\gamma+\gamma\alpha = p, \qquad \alpha+\beta+\gamma = 5$$

Therefore

$$\begin{aligned} (\alpha+1)(\beta+1)(\gamma+1) &= q + p + 5 + 1 \\ &= 2 + 6 + 5 + 1 \\ &= 14 \end{aligned}$$

Hence,

$$(\alpha+1)(\beta+1)(\gamma+1) = 14$$

2. The quartic equation $x^4 + bx^3 + cx^2 + dx + 3 = 0$ has roots α, β, γ, δ. It is given that

$$\alpha + \beta + \gamma + \delta = -2, \qquad (\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2 = 6, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 2$$

(a) Find the values of b, c and d **[6]**

(b) Given also that

$$(\alpha+1)^3 + (\beta+1)^3 + (\gamma+1)^3 + (\delta+1)^3 = 20$$

find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$ **[2]**

Solution

(a) For the monic quartic

$$x^4 + bx^3 + cx^2 + dx + 3 = 0$$

with roots $\alpha, \beta, \gamma, \delta$, the standard relations between coefficients and roots give

$$\alpha + \beta + \gamma + \delta = -b$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -d$$

$$\alpha\beta\gamma\delta = 3$$

We are given

$$\alpha + \beta + \gamma + \delta = -2$$

so

$$-b = -2$$

hence

$$b = 2$$

Next,

$$\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

Therefore

$$2 = \frac{-d}{3}$$

so

$$d = -6$$

Now use

$$(\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2 = 6$$

Expanding,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha + \beta + \gamma + \delta) + 4 = 6$$

since there are 4 lots of 1^2.

Substituting $\alpha + \beta + \gamma + \delta = -2$,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(-2) + 4 = 6$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 6$$

Also,

$$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

so

$$(-2)^2 = 6 + 2c$$

$$4 = 6 + 2c$$

$$2c = -2$$

$$c = -1$$

Therefore,

$$b = 2, \qquad c = -1, \qquad d = -6$$

(b) Using the extra condition

$$(\alpha + 1)^3 + (\beta + 1)^3 + (\gamma + 1)^3 + (\delta + 1)^3 = 20$$

expand:

$$\sum(\text{root} + 1)^3 = \sum \text{root}^3 + 3\sum \text{root}^2 + 3\sum \text{root} + 4$$

Hence

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + 3(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + 3(\alpha + \beta + \gamma + \delta) + 4 = 20$$

Using $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 6$ and $\alpha + \beta + \gamma + \delta = -2$,

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + 3(6) + 3(-2) + 4 = 20$$

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + 18 - 6 + 4 = 20$$

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 4$$

From part (a), the quartic is

$$x^4 + 2x^3 - x^2 - 6x + 3 = 0$$

so each root r satisfies

$$r^4 + 2r^3 - r^2 - 6r + 3 = 0$$

and therefore

$$r^4 = -2r^3 + r^2 + 6r - 3$$

Summing this for $r = \alpha, \beta, \gamma, \delta$ gives

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -2(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + 6(\alpha + \beta + \gamma + \delta) - 12$$

$$= -2(4) + 6 + 6(-2) - 12$$

$$= -8 + 6 - 12 - 12$$

$$= -26$$

Therefore,

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -26$$

3. The equation $x^3 - 4x^2 - x + 6 = 0$ has roots α, β and γ.

Determine the value of

$$\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$$

[4]

Solution

Let

$$s_1 = \alpha+\beta+\gamma, \qquad s_2 = \alpha\beta+\beta\gamma+\gamma\alpha, \qquad s_3 = \alpha\beta\gamma$$

From

$$x^3 - 4x^2 - x + 6 = 0$$

comparing with $x^3 - s_1x^2 + s_2x - s_3 = 0$, we get

$$s_1 = 4, \qquad s_2 = -1, \qquad s_3 = -6$$

We need

$$\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$$

Taking a common denominator,

$$\begin{aligned}\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha} &= \frac{(\beta+\gamma)+(\gamma+\alpha)+(\alpha+\beta)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)} \\ &= \frac{2(\alpha+\beta+\gamma)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)} \\ &= \frac{2s_1}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}\end{aligned}$$

Now expand the denominator:

$$\begin{aligned}(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) &= (\alpha+\beta)\big(\beta\gamma+\beta\alpha+\gamma^2+\gamma\alpha\big) \\ &= \alpha^2\beta+\alpha^2\gamma+\alpha\beta^2+\beta^2\gamma+\alpha\gamma^2+\beta\gamma^2+2\alpha\beta\gamma\end{aligned}$$

Also,

$$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) = \alpha^2\beta+\alpha^2\gamma+\alpha\beta^2+\beta^2\gamma+\alpha\gamma^2+\beta\gamma^2+3\alpha\beta\gamma$$

So

$$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = s_1s_2 - s_3$$

Hence

$$\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha} = \frac{2s_1}{s_1s_2 - s_3}$$

Substitute $s_1 = 4$, $s_2 = -1$, $s_3 = -6$:

$$\begin{aligned}\frac{2s_1}{s_1s_2 - s_3} &= \frac{2(4)}{4(-1)-(-6)} \\ &= \frac{8}{-4+6} \\ &= \frac{8}{2} \\ &= 4\end{aligned}$$

Therefore, the required value is 4.

4. The quartic equation $3x^4 - x^3 - 10x^2 + 8x + 4 = 0$ has roots α, β, γ and δ.

Without solving the equation, find equations with integer coefficients whose roots are

(a) $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$ **[6]**

(b) $\frac{2}{\alpha-1}$, $\frac{2}{\beta-1}$, $\frac{2}{\gamma-1}$ and $\frac{2}{\delta-1}$ **[6]**

Solution

(a) Let

$$f(x) = 3x^4 - x^3 - 10x^2 + 8x + 4$$

and let $x = \alpha$ be any root, so $f(\alpha) = 0$.

We want an equation whose roots are

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$$

Since the constant term of $f(x)$ is 4, none of the roots is 0, so reciprocals are defined.

Let

$$y = \frac{1}{x}$$

so that

$$x = \frac{1}{y}$$

Substitute $x = \frac{1}{y}$ into $f(x) = 0$:

$$3\left(\frac{1}{y}\right)^4 - \left(\frac{1}{y}\right)^3 - 10\left(\frac{1}{y}\right)^2 + 8\left(\frac{1}{y}\right) + 4 = 0$$

This gives

$$\frac{3}{y^4} - \frac{1}{y^3} - \frac{10}{y^2} + \frac{8}{y} + 4 = 0$$

Multiply through by y^4:

$$3 - y - 10y^2 + 8y^3 + 4y^4 = 0$$

Rearrange in descending powers:

$$4y^4 + 8y^3 - 10y^2 - y + 3 = 0$$

So the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ is

$$4x^4 + 8x^3 - 10x^2 - x + 3 = 0$$

(b) We now want an equation whose roots are

$$\frac{2}{\alpha-1}, \frac{2}{\beta-1}, \frac{2}{\gamma-1}, \frac{2}{\delta-1}$$

First check that 1 is not a root of the original equation:

$$f(1) = 3 - 1 - 10 + 8 + 4 = 4 \neq 0$$

so $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$ are all non-zero.

Let

$$y = \frac{2}{x-1}$$

Then

$$x - 1 = \frac{2}{y} \qquad \Rightarrow \qquad x = 1 + \frac{2}{y}$$

Substitute $x = 1 + \dfrac{2}{y}$ into $f(x) = 0$:

$$3\left(1 + \frac{2}{y}\right)^4 - \left(1 + \frac{2}{y}\right)^3 - 10\left(1 + \frac{2}{y}\right)^2 + 8\left(1 + \frac{2}{y}\right) + 4 = 0$$

Now expand each power:

$$\begin{aligned}
\left(1 + \frac{2}{y}\right)^2 &= 1 + \frac{4}{y} + \frac{4}{y^2} \\
\left(1 + \frac{2}{y}\right)^3 &= 1 + \frac{6}{y} + \frac{12}{y^2} + \frac{8}{y^3} \\
\left(1 + \frac{2}{y}\right)^4 &= 1 + \frac{8}{y} + \frac{24}{y^2} + \frac{32}{y^3} + \frac{16}{y^4}
\end{aligned}$$

So

$$\begin{aligned}
0 = 3\left(1 + \frac{8}{y} + \frac{24}{y^2} + \frac{32}{y^3} + \frac{16}{y^4}\right) - \left(1 + \frac{6}{y} + \frac{12}{y^2} + \frac{8}{y^3}\right) \\
- 10\left(1 + \frac{4}{y} + \frac{4}{y^2}\right) + 8\left(1 + \frac{2}{y}\right) + 4
\end{aligned}$$

Expand fully:

$$\begin{aligned}
0 = \left(3 + \frac{24}{y} + \frac{72}{y^2} + \frac{96}{y^3} + \frac{48}{y^4}\right) - \left(1 + \frac{6}{y} + \frac{12}{y^2} + \frac{8}{y^3}\right) \\
- \left(10 + \frac{40}{y} + \frac{40}{y^2}\right) + \left(8 + \frac{16}{y}\right) + 4
\end{aligned}$$

Collect like terms:

$$\begin{aligned}
0 = (3 - 1 - 10 + 8 + 4) + \left(\frac{24}{y} - \frac{6}{y} - \frac{40}{y} + \frac{16}{y}\right) \\
+ \left(\frac{72}{y^2} - \frac{12}{y^2} - \frac{40}{y^2}\right) + \left(\frac{96}{y^3} - \frac{8}{y^3}\right) + \frac{48}{y^4}
\end{aligned}$$

Hence

$$0 = 4 - \frac{6}{y} + \frac{20}{y^2} + \frac{88}{y^3} + \frac{48}{y^4}$$

Multiply through by y^4:

$$4y^4 - 6y^3 + 20y^2 + 88y + 48 = 0$$

All coefficients are even, so divide by 2:

$$2y^4 - 3y^3 + 10y^2 + 44y + 24 = 0$$

Therefore the equation whose roots are $\dfrac{2}{\alpha - 1}, \dfrac{2}{\beta - 1}, \dfrac{2}{\gamma - 1}, \dfrac{2}{\delta - 1}$ is

$$2x^4 - 3x^3 + 10x^2 + 44x + 24 = 0$$

5. The cubic equation $x^3 + x^2 - 2x - 1 = 0$ has roots α, β and γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$ **[3]**

(b) Find the value of

$$\left(\frac{1}{\alpha+1}\right)^2 + \left(\frac{1}{\beta+1}\right)^2 + \left(\frac{1}{\gamma+1}\right)^2$$

[2]

(c) Find the value of

$$\left(\frac{1}{\alpha+1}\right)^3 + \left(\frac{1}{\beta+1}\right)^3 + \left(\frac{1}{\gamma+1}\right)^3$$

[2]

Solution

(a) Since $x = -1$ is not a root of $x^3 + x^2 - 2x - 1 = 0$, the quantities

$$\frac{1}{\alpha+1}, \quad \frac{1}{\beta+1}, \quad \frac{1}{\gamma+1}$$

are all defined.

Let

$$y = \frac{1}{x+1}$$

so

$$x = \frac{1}{y} - 1 = \frac{1-y}{y}$$

Substitute into the given cubic:

$$\left(\frac{1-y}{y}\right)^3 + \left(\frac{1-y}{y}\right)^2 - 2\left(\frac{1-y}{y}\right) - 1 = 0$$

Multiply through by y^3:

$$(1-y)^3 + y(1-y)^2 - 2y^2(1-y) - y^3 = 0$$

Expanding:

$$(1 - 3y + 3y^2 - y^3) + (y - 2y^2 + y^3) - 2y^2 + 2y^3 - y^3 = 0$$

Collecting terms:

$$1 - 2y - y^2 + y^3 = 0$$

So

$$y^3 - y^2 - 2y + 1 = 0$$

When $x = \alpha, \beta, \gamma$, the corresponding values of y are

$$\frac{1}{\alpha+1}, \quad \frac{1}{\beta+1}, \quad \frac{1}{\gamma+1}$$

so the required cubic equation is

$$y^3 - y^2 - 2y + 1 = 0$$

(b) Let the roots of

$$y^3 - y^2 - 2y + 1 = 0$$

be p, q, r, where

$$p = \frac{1}{\alpha + 1}, \quad q = \frac{1}{\beta + 1}, \quad r = \frac{1}{\gamma + 1}$$

From the cubic,

$$p + q + r = 1, \qquad pq + pr + qr = -2$$

Now

$$\begin{aligned} p^2 + q^2 + r^2 &= (p + q + r)^2 - 2(pq + pr + qr) \\ &= 1^2 - 2(-2) \\ &= 5 \end{aligned}$$

Therefore

$$\left(\frac{1}{\alpha + 1}\right)^2 + \left(\frac{1}{\beta + 1}\right)^2 + \left(\frac{1}{\gamma + 1}\right)^2 = 5$$

(c) Using the same roots p, q, r, we also have

$$pqr = -1$$

Use

$$p^3 + q^3 + r^3 = (p + q + r)^3 - 3(p + q + r)(pq + pr + qr) + 3pqr$$

Hence

$$\begin{aligned} p^3 + q^3 + r^3 &= 1^3 - 3(1)(-2) + 3(-1) \\ &= 1 + 6 - 3 \\ &= 4 \end{aligned}$$

Therefore

$$\left(\frac{1}{\alpha + 1}\right)^3 + \left(\frac{1}{\beta + 1}\right)^3 + \left(\frac{1}{\gamma + 1}\right)^3 = 4$$

6. The equation $x^3 - 14x^2 + 56x + c = 0$, where c is a constant, has three roots in geometric progression.

(a) Determine the roots of the equation **[6]**

(b) Find c **[1]**

Solution

(a) Let the three roots be

$$\frac{a}{r}, \quad a, \quad ar$$

where $r \neq 0$.
For the cubic

$$x^3 - 14x^2 + 56x + c = 0$$

the sum of the roots is 14, and the sum of the products of the roots taken two at a time is 56.
So

$$\frac{a}{r} + a + ar = 14$$
$$a\left(\frac{1}{r} + 1 + r\right) = 14$$

Also,

$$\left(\frac{a}{r}\right)(a) + a(ar) + \left(\frac{a}{r}\right)(ar) = 56$$
$$\frac{a^2}{r} + a^2r + a^2 = 56$$
$$a^2\left(\frac{1}{r} + 1 + r\right) = 56$$

Now divide the second equation by the first:

$$\frac{a^2\left(\frac{1}{r} + 1 + r\right)}{a\left(\frac{1}{r} + 1 + r\right)} = \frac{56}{14}$$

so

$$a = 4$$

Substitute into

$$a\left(\frac{1}{r} + 1 + r\right) = 14$$

to get

$$4\left(\frac{1}{r} + 1 + r\right) = 14$$
$$\frac{1}{r} + 1 + r = \frac{7}{2}$$

Multiply through by $2r$:

$$2 + 2r + 2r^2 = 7r$$
$$2r^2 - 5r + 2 = 0$$

Factorising,

$$(2r - 1)(r - 2) = 0$$

so

$$r = \frac{1}{2} \quad \text{or} \quad r = 2$$

Hence the roots are

$$\frac{4}{2},\ 4,\ 4(2)$$

that is

$$2,\ 4,\ 8$$

So the roots are $\boxed{2,\ 4,\ 8}$.

(b) The product of the roots is

$$2 \cdot 4 \cdot 8 = 64$$

For

$$x^3 - 14x^2 + 56x + c = 0$$

the product of the roots is $-c$.

So

$$-c = 64$$

and therefore

$$c = -64$$

So $\boxed{c = -64}$.

7. The quadratic equation

$$x^2 - kx + 1 = 0$$

where k is an integer, has roots α and β.

(a) Write down, in terms of k where appropriate, the value of $\alpha + \beta$ and the value of $\alpha\beta$ **[2]**

(b) Determine, in simplest form in terms of k, the value of

$$(\alpha^2 + \beta) + (\beta^2 + \alpha)$$ **[3]**

(c) Determine a quadratic equation which has roots

$$\alpha^2 + \beta \text{ and } \beta^2 + \alpha$$

giving your answer in the form $px^2 + qx + r = 0$, where p, q and r are integer values in terms of k **[4]**

Solution

(a) For the quadratic $x^2 - kx + 1 = 0$, we use

$$\alpha + \beta = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}, \qquad \alpha\beta = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

So

$$\alpha + \beta = -\frac{-k}{1} = k, \qquad \alpha\beta = \frac{1}{1} = 1$$

Hence

$$\alpha + \beta = k, \qquad \alpha\beta = 1$$

(b)

$$\begin{aligned}(\alpha^2 + \beta) + (\beta^2 + \alpha) &= \alpha^2 + \beta^2 + \alpha + \beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta + (\alpha + \beta) \\ &= k^2 - 2(1) + k \\ &= k^2 + k - 2\end{aligned}$$

Hence the value is

$$k^2 + k - 2$$

(c) Let the new roots be

$$\alpha^2 + \beta \quad \text{and} \quad \beta^2 + \alpha$$

Their sum is, from part (b),

$$(\alpha^2 + \beta) + (\beta^2 + \alpha) = k^2 + k - 2$$

Now find their product:

$$\begin{aligned}(\alpha^2 + \beta)(\beta^2 + \alpha) &= \alpha^2\beta^2 + \alpha^3 + \beta^3 + \alpha\beta \\ &= (\alpha\beta)^2 + \alpha^3 + \beta^3 + \alpha\beta \\ &= 1 + \alpha^3 + \beta^3 + 1 \\ &= \alpha^3 + \beta^3 + 2\end{aligned}$$

Next,

$$\begin{aligned}\alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= k^3 - 3(1)(k) \\ &= k^3 - 3k\end{aligned}$$

So the product is

$$(\alpha^2 + \beta)(\beta^2 + \alpha) = k^3 - 3k + 2$$

A quadratic with roots r_1 and r_2 is

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

Therefore the required equation is

$$x^2 - (k^2 + k - 2)x + (k^3 - 3k + 2) = 0$$

So, in the form $px^2 + qx + r = 0$,

$$x^2 - (k^2 + k - 2)x + (k^3 - 3k + 2) = 0$$

with $p = 1$, $q = -(k^2 + k - 2)$, $r = k^3 - 3k + 2$.

8. The roots of the equation

$$x^3 + 5x^2 - 4x - 24 = 0$$

are α, β and γ.

Without solving the equation,

(a) write down the value of each of

$$\alpha + \beta + \gamma \qquad \alpha\beta + \beta\gamma + \gamma\alpha \qquad \alpha\beta\gamma$$ **[1]**

(b) Hence determine the value of

(i)

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$$ **[2]**

(ii)

$$(\alpha + 2)(\beta + 2)(\gamma + 2)$$ **[2]**

(iii)

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$$ **[3]**

Solution

(a) For a monic cubic

$$x^3 + bx^2 + cx + d = 0$$

with roots α, β, γ,

$$\alpha + \beta + \gamma = -b, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = c, \qquad \alpha\beta\gamma = -d$$

Here

$$x^3 + 5x^2 - 4x - 24 = 0$$

so

$$\alpha + \beta + \gamma = -5, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -4, \qquad \alpha\beta\gamma = 24$$

Therefore the values are -5, -4 and 24.

(b) Using

$$\alpha + \beta + \gamma = -5, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -4, \qquad \alpha\beta\gamma = 24$$

(i)

$$\begin{aligned}\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} &= \frac{\gamma}{\alpha\beta\gamma} + \frac{\alpha}{\alpha\beta\gamma} + \frac{\beta}{\alpha\beta\gamma} \\ &= \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} \\ &= \frac{-5}{24}\end{aligned}$$

Hence

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = -\frac{5}{24}$$

(ii)

$$\begin{aligned}(\alpha + 2)(\beta + 2)(\gamma + 2) &= (\alpha\beta + 2\alpha + 2\beta + 4)(\gamma + 2) \\ &= \alpha\beta\gamma + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma + 4\alpha + 4\beta + 4\gamma + 8 \\ &= \alpha\beta\gamma + 2(\alpha\beta + \beta\gamma + \gamma\alpha) + 4(\alpha + \beta + \gamma) + 8\end{aligned}$$

Substituting the values from part (a),

$$\begin{aligned}(\alpha+2)(\beta+2)(\gamma+2) &= 24+2(-4)+4(-5)+8\\ &= 24-8-20+8\\ &= 4\end{aligned}$$

Therefore

$$(\alpha+2)(\beta+2)(\gamma+2) = 4$$

(iii) Start with

$$(\alpha\beta+\beta\gamma+\gamma\alpha)^2$$

Expanding,

$$\begin{aligned}(\alpha\beta+\beta\gamma+\gamma\alpha)^2 &= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\cdot\beta\gamma+2\beta\gamma\cdot\gamma\alpha+2\gamma\alpha\cdot\alpha\beta\\ &= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta^2\gamma+2\alpha\beta\gamma^2+2\alpha^2\beta\gamma\\ &= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)\end{aligned}$$

So

$$\begin{aligned}\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 &= (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)\\ &= (-4)^2-2(24)(-5)\\ &= 16+240\\ &= 256\end{aligned}$$

Hence

$$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = 256$$

9. The roots of the equation $3x^3 + px^2 - 12x - 8 = 0$ are α, β and γ.

(a) Given that $\alpha + \beta + \gamma = 2$, write down the value of p **[1]**

(b) Write down values for $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$ **[1]**

(c) Hence find the value of $\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)\left(1 + \frac{1}{\gamma}\right)$ **[3]**

Solution

(a) For $3x^3 + px^2 - 12x - 8 = 0$, the sum of the roots is

$$\alpha + \beta + \gamma = -\frac{p}{3}$$

Given $\alpha + \beta + \gamma = 2$,

$$-\frac{p}{3} = 2$$

so

$$p = -6$$

Therefore, $p = -6$.

(b) For $3x^3 + px^2 - 12x - 8 = 0$,

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-12}{3} = -4$$

and

$$\alpha\beta\gamma = -\frac{-8}{3} = \frac{8}{3}$$

Therefore,

$$\alpha\beta + \beta\gamma + \gamma\alpha = -4, \qquad \alpha\beta\gamma = \frac{8}{3}$$

(c)

$$\begin{aligned}\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)\left(1 + \frac{1}{\gamma}\right) &= \frac{(\alpha+1)(\beta+1)(\gamma+1)}{\alpha\beta\gamma} \\ &= \frac{\alpha\beta\gamma + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha + \beta + \gamma + 1}{\alpha\beta\gamma}\end{aligned}$$

Using the results from parts (a) and (b),

$$\alpha + \beta + \gamma = 2, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -4, \qquad \alpha\beta\gamma = \frac{8}{3}$$

so

$$\begin{aligned}\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)\left(1 + \frac{1}{\gamma}\right) &= \frac{\frac{8}{3} - 4 + 2 + 1}{\frac{8}{3}} \\ &= \frac{\frac{8}{3} - 1}{\frac{8}{3}} \\ &= \frac{\frac{5}{3}}{\frac{8}{3}} \\ &= \frac{5}{8}\end{aligned}$$

Therefore,

$$\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)\left(1 + \frac{1}{\gamma}\right) = \frac{5}{8}$$

10. The roots of the equation $x^3 + 2x^2 - 5x - 3 = 0$ are α, β and γ.

(a) Find $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ **[4]**

(b) Find an equation with integer coefficients whose roots are α^2, β^2 and γ^2 **[4]**

Solution

(a) Since α, β and γ are roots of

$$x^3 + 2x^2 - 5x - 3 = 0$$

the standard relations between roots and coefficients give

$$\alpha + \beta + \gamma = -2, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -5, \qquad \alpha\beta\gamma = 3$$

Now expand

$$\begin{aligned}(\alpha\beta + \beta\gamma + \gamma\alpha)^2 &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2(\alpha\beta)(\beta\gamma) + 2(\beta\gamma)(\gamma\alpha) + 2(\gamma\alpha)(\alpha\beta) \\ &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta^2\gamma + 2\beta\gamma^2\alpha + 2\gamma\alpha^2\beta \\ &= \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)\end{aligned}$$

So

$$\begin{aligned}\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \\ &= (-5)^2 - 2(3)(-2) \\ &= 25 + 12 \\ &= 37\end{aligned}$$

Hence

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = 37$$

(b) Let the required equation have roots α^2, β^2 and γ^2.
First find their sum:

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \\ &= (-2)^2 - 2(-5) \\ &= 4 + 10 \\ &= 14\end{aligned}$$

From part (a),

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = 37$$

Also,

$$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 3^2 = 9$$

So for roots $\alpha^2, \beta^2, \gamma^2$, the monic cubic is

$$x^3 - (\text{sum of roots})x^2 + (\text{sum of pairwise products})x - (\text{product of roots}) = 0$$

Therefore

$$x^3 - 14x^2 + 37x - 9 = 0$$

So the required equation is

$$x^3 - 14x^2 + 37x - 9 = 0$$

11. The graph of $y = x^3 + 3x^2 + px - 20$, where p is a constant, crosses the x-axis at three points whose x-coordinates are equally spaced.

Find the roots of

$$x^3 + 3x^2 + px - 20 = 0$$

[6]

Solution

Since the three roots are equally spaced, let them be

$$a - d, \quad a, \quad a + d$$

For

$$x^3 + 3x^2 + px - 20 = 0$$

the sum of the roots is -3, so

$$(a - d) + a + (a + d) = -3$$
$$3a = -3$$
$$a = -1$$

Also, the product of the roots is 20, because for a monic cubic $x^3 + \cdots - 20$, the product of the roots is $-(-20) = 20$.
So

$$(a - d)a(a + d) = 20$$

Now

$$(a - d)(a + d) = a^2 - d^2$$

so

$$a(a^2 - d^2) = 20$$

Substitute $a = -1$:

$$(-1)\left(1 - d^2\right) = 20$$
$$-1 + d^2 = 20$$
$$d^2 = 21$$
$$d = \sqrt{21}$$

Therefore the roots are

$$a - d = -1 - \sqrt{21}, \qquad a = -1, \qquad a + d = -1 + \sqrt{21}$$

Hence the roots of the equation are

$$-1 - \sqrt{21}, \quad -1, \quad -1 + \sqrt{21}$$

(Checking: the pairwise sum is

$$(-1 - \sqrt{21})(-1) + (-1)(-1 + \sqrt{21}) + (-1 - \sqrt{21})(-1 + \sqrt{21}) = -18$$

so $p = -18$, which is consistent.)

12. The cubic equation $x^3 - 3x + 1 = 0$ has roots α, β and γ.

(a) Find a cubic equation whose roots are $\dfrac{1}{\alpha - 1}, \dfrac{1}{\beta - 1}, \dfrac{1}{\gamma - 1}$. **[3]**

(b) Hence find the value of $\dfrac{1}{(\alpha-1)^2} + \dfrac{1}{(\beta-1)^2} + \dfrac{1}{(\gamma-1)^2}$. **[2]**

(c) Find also the value of $\dfrac{1}{(\alpha-1)^5} + \dfrac{1}{(\beta-1)^5} + \dfrac{1}{(\gamma-1)^5}$. **[3]**

Solution

(a) Let

$$t = \frac{1}{x-1}$$

so that

$$x = 1 + \frac{1}{t}$$

We substitute this into the given equation

$$x^3 - 3x + 1 = 0$$

Then

$$\begin{aligned} \left(1 + \frac{1}{t}\right)^3 - 3\left(1 + \frac{1}{t}\right) + 1 &= 0 \\ \left(1 + \frac{3}{t} + \frac{3}{t^2} + \frac{1}{t^3}\right) - 3 - \frac{3}{t} + 1 &= 0 \\ \frac{1}{t^3} + \frac{3}{t^2} - 1 &= 0 \end{aligned}$$

Multiply through by t^3:

$$\begin{aligned} 1 + 3t - t^3 &= 0 \\ t^3 - 3t - 1 &= 0 \end{aligned}$$

So the cubic equation whose roots are

$$\frac{1}{\alpha - 1}, \quad \frac{1}{\beta - 1}, \quad \frac{1}{\gamma - 1}$$

is

$$t^3 - 3t - 1 = 0$$

(b) Let the roots of

$$t^3 - 3t - 1 = 0$$

be

$$r_1 = \frac{1}{\alpha - 1}, \quad r_2 = \frac{1}{\beta - 1}, \quad r_3 = \frac{1}{\gamma - 1}$$

For the cubic

$$t^3 + 0t^2 - 3t - 1 = 0$$

we have

$$r_1 + r_2 + r_3 = 0, \qquad r_1 r_2 + r_2 r_3 + r_3 r_1 = -3$$

Now

$$(r_1 + r_2 + r_3)^2 = r_1^2 + r_2^2 + r_3^2 + 2(r_1 r_2 + r_2 r_3 + r_3 r_1)$$

So

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_2 r_3 + r_3 r_1) \\ &= 0^2 - 2(-3) \\ &= 6 \end{aligned}$$

Hence

$$\frac{1}{(\alpha - 1)^2} + \frac{1}{(\beta - 1)^2} + \frac{1}{(\gamma - 1)^2} = 6$$

(c) Using the same roots r_1, r_2, r_3, each root satisfies

$$r^3 - 3r - 1 = 0$$

so

$$r^3 = 3r + 1$$

Multiply by r^2:

$$r^5 = r^2 r^3 = r^2(3r + 1) = 3r^3 + r^2$$

Now substitute again for r^3:

$$\begin{aligned} r^5 &= 3(3r + 1) + r^2 \\ &= 9r + 3 + r^2 \end{aligned}$$

Summing for r_1, r_2, r_3 gives

$$\begin{aligned} r_1^5 + r_2^5 + r_3^5 &= 9(r_1 + r_2 + r_3) + 3 \cdot 3 + (r_1^2 + r_2^2 + r_3^2) \\ &= 9(0) + 9 + 6 \\ &= 15 \end{aligned}$$

Therefore

$$\frac{1}{(\alpha - 1)^5} + \frac{1}{(\beta - 1)^5} + \frac{1}{(\gamma - 1)^5} = 15$$

13. The quartic equation $x^4 - 2x^2 + 2x + 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^2, β^2, γ^2, δ^2 and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. **[5]**

(b) Find the value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. **[3]**

(c) Find the value of $\alpha^8 + \beta^8 + \gamma^8 + \delta^8$. **[2]**

Solution

(a) Let

$$f(x) = x^4 - 2x^2 + 2x + 1$$

If $f(\alpha) = 0$, then $f(\alpha)f(-\alpha) = 0$. So α^2 is a root of the equation obtained from $f(x)f(-x) = 0$ after writing everything in terms of x^2. The same will then be true for $\beta^2, \gamma^2, \delta^2$.

Now

$$f(x) = (x^2 - 1)^2 + 2x, \qquad f(-x) = (x^2 - 1)^2 - 2x$$

So

$$\begin{aligned} f(x)f(-x) &= \big((x^2-1)^2 + 2x\big)\big((x^2-1)^2 - 2x\big) \\ &= (x^2-1)^4 - 4x^2 \end{aligned}$$

Let $y = x^2$. Then

$$\begin{aligned} (x^2-1)^4 - 4x^2 &= (y-1)^4 - 4y \\ &= y^4 - 4y^3 + 6y^2 - 4y + 1 - 4y \\ &= y^4 - 4y^3 + 6y^2 - 8y + 1 \end{aligned}$$

Hence the required quartic equation is

$$y^4 - 4y^3 + 6y^2 - 8y + 1 = 0$$

Its roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$.

If these roots are y_1, y_2, y_3, y_4, then for a monic quartic,

$$y_1 + y_2 + y_3 + y_4 = -(\text{coefficient of } y^3) = 4$$

Therefore

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 4$$

(b) Let the roots of

$$h(y) = y^4 - 4y^3 + 6y^2 - 8y + 1$$

be y_1, y_2, y_3, y_4, where

$$y_1 = \alpha^2, \quad y_2 = \beta^2, \quad y_3 = \gamma^2, \quad y_4 = \delta^2$$

Define

$$q_r = y_1^r + y_2^r + y_3^r + y_4^r$$

Then

$$\alpha^6 + \beta^6 + \gamma^6 + \delta^6 = q_3$$

For a monic quartic $y^4 + ay^3 + by^2 + cy + d$, the power sums satisfy

$$\begin{aligned} q_1 + a &= 0 \\ q_2 + aq_1 + 2b &= 0 \\ q_3 + aq_2 + bq_1 + 3c &= 0 \end{aligned}$$

Here $a = -4,\ b = 6,\ c = -8$, so

$$\begin{aligned} q_1 - 4 &= 0 \\ q_2 - 4q_1 + 12 &= 0 \\ q_3 - 4q_2 + 6q_1 - 24 &= 0 \end{aligned}$$

From the first equation,

$$q_1 = 4$$

Then

$$q_2 - 4(4) + 12 = 0$$

so

$$q_2 = 4$$

Now

$$q_3 - 4(4) + 6(4) - 24 = 0$$

which gives

$$q_3 = 16$$

Therefore

$$\alpha^6 + \beta^6 + \gamma^6 + \delta^6 = 16$$

(c) Here

$$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 = q_4$$

Using the next power-sum relation for the same quartic,

$$q_4 - 4q_3 + 6q_2 - 8q_1 + 4 = 0$$

Substitute $q_1 = 4,\ q_2 = 4,\ q_3 = 16$:

$$\begin{aligned} q_4 - 4(16) + 6(4) - 8(4) + 4 &= 0 \\ q_4 - 64 + 24 - 32 + 4 &= 0 \\ q_4 - 68 &= 0 \end{aligned}$$

So

$$q_4 = 68$$

Hence

$$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 = 68$$

14. The equation $z^4 - 2z^3 - z^2 - 2z + 1 = 0$ has roots α, β, γ and δ.

(a) Show that a quartic equation whose roots are $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$, $\gamma + \frac{1}{\gamma}$ and $\delta + \frac{1}{\delta}$ is

$$w^4 - 4w^3 - 2w^2 + 12w + 9 = 0$$

[3]

(b) Hence determine the exact roots of the equation $z^4 - 2z^3 - z^2 - 2z + 1 = 0$. **[3]**

Solution

(a) Let

$$f(z) = z^4 - 2z^3 - z^2 - 2z + 1$$

and let z be any root of $f(z) = 0$.
Since the constant term is 1, no root is 0, so we may divide by z^2:

$$z^2 - 2z - 1 - \frac{2}{z} + \frac{1}{z^2} = 0$$
$$\left(z^2 + \frac{1}{z^2}\right) - 2\left(z + \frac{1}{z}\right) - 1 = 0$$

Now let

$$w = z + \frac{1}{z}$$

Then

$$z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2 = w^2 - 2$$

so

$$(w^2 - 2) - 2w - 1 = 0$$
$$w^2 - 2w - 3 = 0$$
$$(w - 3)(w + 1) = 0$$

So the possible values of w are 3 and -1.
Also, the polynomial is unchanged when its coefficients are reversed, so if z is a root then $\frac{1}{z}$ is also a root. Therefore the four roots $\alpha, \beta, \gamma, \delta$ occur in reciprocal pairs, and the four values

$$\alpha + \frac{1}{\alpha}, \quad \beta + \frac{1}{\beta}, \quad \gamma + \frac{1}{\gamma}, \quad \delta + \frac{1}{\delta}$$

are

$$3, 3, -1, -1$$

Hence the required quartic is

$$(w - 3)^2(w + 1)^2 = (w^2 - 2w - 3)^2$$
$$= w^4 - 4w^3 - 2w^2 + 12w + 9$$

So the quartic equation is

$$w^4 - 4w^3 - 2w^2 + 12w + 9 = 0$$

(b) From part (a), the corresponding values of

$$w = z + \frac{1}{z}$$

are 3 and -1.

If $w = 3$, then

$$\begin{aligned} z + \frac{1}{z} &= 3 \\ z^2 - 3z + 1 &= 0 \end{aligned}$$

So

$$z = \frac{3 \pm \sqrt{5}}{2}$$

If $w = -1$, then

$$\begin{aligned} z + \frac{1}{z} &= -1 \\ z^2 + z + 1 &= 0 \end{aligned}$$

So

$$z = \frac{-1 \pm \mathrm{i}\sqrt{3}}{2}$$

Hence the exact roots of

$$z^4 - 2z^3 - z^2 - 2z + 1 = 0$$

are

$$\frac{3 + \sqrt{5}}{2}, \quad \frac{3 - \sqrt{5}}{2}, \quad \frac{-1 + \mathrm{i}\sqrt{3}}{2}, \quad \frac{-1 - \mathrm{i}\sqrt{3}}{2}$$

15. The quartic equation $x^4 - 2x^3 + 3x^2 - 4x + 1 = 0$ has roots α, β, γ, δ.

(a) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$. **[2]**

(b) Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. **[2]**

(c) Find the value of

$$\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 + \left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2 + \left(\frac{1}{\gamma} + \frac{1}{\delta} + \frac{1}{\alpha}\right)^2 + \left(\frac{1}{\delta} + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2$$

[5]

Solution

(a) For the monic quartic

$$x^4 - 2x^3 + 3x^2 - 4x + 1 = 0$$

with roots $\alpha, \beta, \gamma, \delta$, the standard root relations give

$$\alpha + \beta + \gamma + \delta = 2$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 3$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 4$$

$$\alpha\beta\gamma\delta = 1$$

Hence

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} &= \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta} \\ &= \frac{4}{1} \\ &= 4\end{aligned}$$

So

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = 4$$

(b) Let

$$r_1 = \frac{1}{\alpha}, \quad r_2 = \frac{1}{\beta}, \quad r_3 = \frac{1}{\gamma}, \quad r_4 = \frac{1}{\delta}$$

Then from part (a),

$$r_1 + r_2 + r_3 + r_4 = 4$$

Also

$$\begin{aligned}r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta} \\ &= \frac{\gamma\delta + \beta\delta + \beta\gamma + \alpha\delta + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma\delta} \\ &= \frac{\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta}{\alpha\beta\gamma\delta} \\ &= \frac{3}{1} \\ &= 3\end{aligned}$$

Now use

$$(r_1 + r_2 + r_3 + r_4)^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2 + 2(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)$$

So

$$\begin{aligned}\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}&=4^2-2(3)\\&=16-6\\&=10\end{aligned}$$

So

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}=10$$

(c) Using the same notation, let

$$S=r_1+r_2+r_3+r_4=4$$

Then each bracket is the sum of all four reciprocals except one:

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=S-r_4$$

$$\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=S-r_1$$

$$\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha}=S-r_2$$

$$\frac{1}{\delta}+\frac{1}{\alpha}+\frac{1}{\beta}=S-r_3$$

So the required expression is

$$(S-r_1)^2+(S-r_2)^2+(S-r_3)^2+(S-r_4)^2$$

Expanding,

$$\begin{aligned}\sum_{i=1}^{4}(S-r_i)^2&=\sum_{i=1}^{4}\left(S^2-2Sr_i+r_i^2\right)\\&=4S^2-2S\sum_{i=1}^{4}r_i+\sum_{i=1}^{4}r_i^2\end{aligned}$$

From parts (a) and (b),

$$\sum_{i=1}^{4}r_i=S=4\qquad\text{and}\qquad\sum_{i=1}^{4}r_i^2=10$$

Therefore

$$\begin{aligned}\sum_{i=1}^{4}(S-r_i)^2&=4(4^2)-2(4)(4)+10\\&=64-32+10\\&=42\end{aligned}$$

So the value of the expression is

42