Questions

Question 1 **2**

Question 2 **4**

Question 3 **7**

Question 4 **9**

Question 5 **12**

Question 6 **15**

Question 7 **18**

Question 8 **20**

Question 9 **23**

Question 10 **27**

Question 11 **29**

Question 12 **33**

Question 13 **35**

Question 14 **38**

1. A curve has equation

$$y = 2 - \frac{3}{x+1}$$

(a) State the equations of the horizontal and vertical asymptotes of this curve. **[2]**

(b) The straight line $y = 3x + c$ meets the curve. Show that the x-coordinates of the points of intersection satisfy

$$3x^2 + (c+1)x + (c+1) = 0$$

[3]

(c) It is given that the line $y = 3x + c$ is a tangent to the curve.

(i) Without using calculus, find the two possible values of c. **[3]**
(ii) Hence find the coordinates of each point of contact. **[4]**

Solution

(a) For

$$y = 2 - \frac{3}{x+1}$$

as $x \to \pm\infty$, the term $\dfrac{3}{x+1} \to 0$, so $y \to 2$.

Hence the horizontal asymptote is

$$y = 2$$

The denominator is zero when

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

Hence the vertical asymptote is

$$x = -1$$

(b) At a point of intersection, the y-values are equal, so

$$3x + c = 2 - \frac{3}{x+1}$$

Rearranging,

$$3x + c - 2 = -\frac{3}{x+1}$$

Multiplying by $(x+1)$,

$$(3x + c - 2)(x + 1) = -3$$

Expanding,

$$\begin{aligned} 3x^2 + 3x + (c-2)x + (c-2) &= -3 \\ 3x^2 + (c+1)x + c - 2 &= -3 \\ 3x^2 + (c+1)x + (c+1) &= 0 \end{aligned}$$

Hence the x-coordinates of the points of intersection satisfy

$$3x^2 + (c+1)x + (c+1) = 0$$

(c) **(i)** If the line is a tangent, the two points of intersection coincide, so the quadratic in part (b) has equal roots.
Therefore its discriminant is zero:

$$(c+1)^2 - 4(3)(c+1) = 0$$

So

$$\begin{aligned}(c+1)^2 - 12(c+1) &= 0\\ (c+1)\big((c+1)-12\big) &= 0\\ (c+1)(c-11) &= 0\end{aligned}$$

Hence

$$c = -1 \quad \text{or} \quad c = 11$$

(ii) From part (b), the tangent case gives a repeated root of

$$3x^2 + (c+1)x + (c+1) = 0$$

so

$$x = \frac{-(c+1)}{2 \cdot 3} = -\frac{c+1}{6}$$

If $c = -1$, then

$$x = -\frac{-1+1}{6} = 0$$

Using $y = 3x + c$,

$$y = 3(0) - 1 = -1$$

So one point of contact is

$$(0, -1)$$

If $c = 11$, then

$$x = -\frac{11+1}{6} = -2$$

Using $y = 3x + c$,

$$y = 3(-2) + 11 = 5$$

So the other point of contact is

$$(-2, 5)$$

Hence the points of contact are

$$(0, -1) \quad \text{and} \quad (-2, 5)$$

2. The curve C has equation $y = \dfrac{x+1}{x^2-4x+3}$.

(a) Find the equations of the asymptotes of C. **[2]**

(b) Find the coordinates of any stationary points on C. **[4]**

(c) Sketch C, stating the coordinates of the intersections with the axes. **[3]**

(d) Sketch the curve with equation $y = \left|\dfrac{x+1}{x^2-4x+3}\right|$ and find in exact form the set of values of x for which $2|x+1| > |x^2-4x+3|$. **[6]**

Solution

(a) Factorising the denominator,

$$x^2-4x+3 = (x-1)(x-3)$$

so the vertical asymptotes are

$$x = 1 \quad \text{and} \quad x = 3$$

Since the degree of the denominator is greater than the degree of the numerator, $y \to 0$ as $x \to \pm\infty$. Hence the horizontal asymptote is

$$y = 0$$

So the asymptotes are $x = 1$, $x = 3$ and $y = 0$.

(b) Using the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2-4x+3)(1)-(x+1)(2x-4)}{(x^2-4x+3)^2} \\ &= \frac{x^2-4x+3-(2x^2-2x-4)}{(x^2-4x+3)^2} \\ &= \frac{7-2x-x^2}{(x^2-4x+3)^2}\end{aligned}$$

Stationary points occur when $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, so

$$7-2x-x^2 = 0$$

$$x^2+2x-7 = 0$$

$$x = \frac{-2 \pm \sqrt{4+28}}{2} = -1 \pm 2\sqrt{2}$$

To find the corresponding y-values, use

$$x^2+2x-7 = 0 \implies x^2 = 7-2x$$

so

$$x^2-4x+3 = (7-2x)-4x+3 = 10-6x$$

Hence

$$y = \frac{x+1}{10-6x}$$

For $x = -1 - 2\sqrt{2}$,

$$\begin{aligned} y &= \frac{-2\sqrt{2}}{10 - 6(-1 - 2\sqrt{2})} \\ &= \frac{-2\sqrt{2}}{16 + 12\sqrt{2}} \\ &= \frac{-2\sqrt{2}(16 - 12\sqrt{2})}{16^2 - (12\sqrt{2})^2} \\ &= \frac{-32\sqrt{2} + 48}{-32} \\ &= \sqrt{2} - \frac{3}{2} \end{aligned}$$

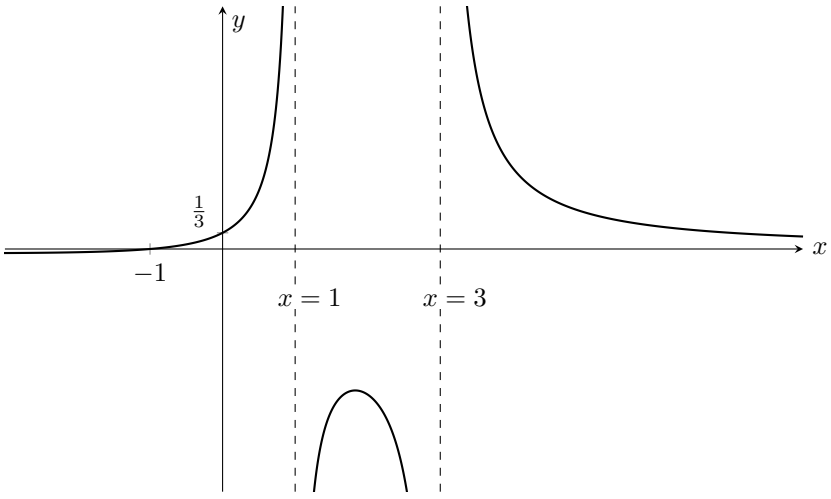
For $x = -1 + 2\sqrt{2}$,

$$\begin{aligned} y &= \frac{2\sqrt{2}}{10 - 6(-1 + 2\sqrt{2})} \\ &= \frac{2\sqrt{2}}{16 - 12\sqrt{2}} \\ &= \frac{2\sqrt{2}(16 + 12\sqrt{2})}{16^2 - (12\sqrt{2})^2} \\ &= \frac{32\sqrt{2} + 48}{-32} \\ &= -\sqrt{2} - \frac{3}{2} \end{aligned}$$

Therefore the stationary points are

$$\left(-1 - 2\sqrt{2},\ \sqrt{2} - \frac{3}{2}\right) \quad \text{and} \quad \left(-1 + 2\sqrt{2},\ -\sqrt{2} - \frac{3}{2}\right)$$

(c) The sketch should look like:



For the x-axis, set $y = 0$. Then

$$\frac{x+1}{x^2 - 4x + 3} = 0 \implies x + 1 = 0 \implies x = -1$$

so the x-intercept is

$$(-1, 0)$$

For the y-axis, set $x = 0$. Then

$$y = \frac{0+1}{0-0+3} = \frac{1}{3}$$

so the y-intercept is

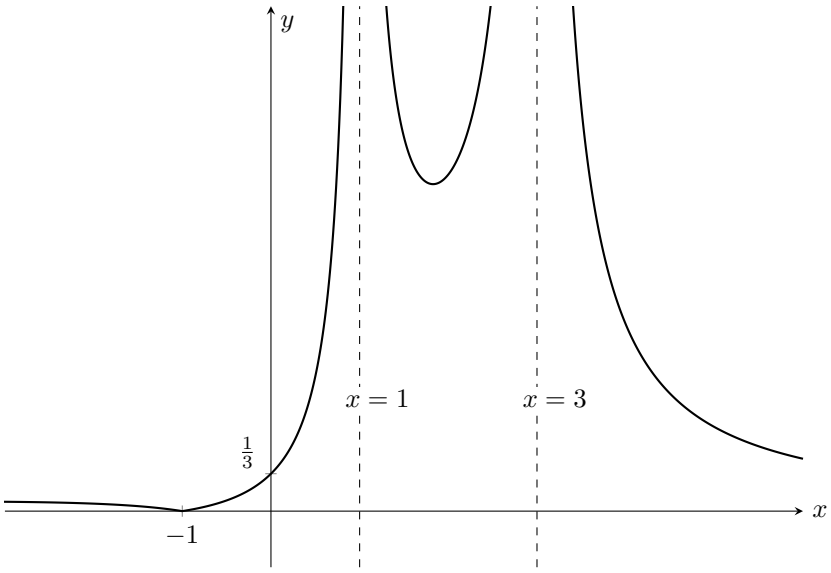
$$\left(0, \frac{1}{3}\right)$$

Also, from

$$y = \frac{x+1}{(x-1)(x-3)}$$

the curve is below the x-axis on $(-\infty, -1)$ and $(1, 3)$, and above the x-axis on $(-1, 1)$ and $(3, \infty)$. Together with the asymptotes and the stationary points from part (b), this gives the sketch shown.

(d) The sketch should look like:



This is the graph from part (c) with the parts below the x-axis reflected in the x-axis. So it has the same asymptotes $x = 1$, $x = 3$, $y = 0$, and there is a cusp at $(-1, 0)$.

Now solve

$$2|x+1| > |x^2 - 4x + 3|$$

Both sides are non-negative, so squaring keeps the inequality in the same direction:

$$\begin{aligned} 4(x+1)^2 &> (x^2 - 4x + 3)^2 \\ 0 &> (x^2 - 4x + 3)^2 - 4(x+1)^2 \\ &= \big(x^2 - 4x + 3 - 2(x+1)\big)\big(x^2 - 4x + 3 + 2(x+1)\big) \\ &= (x^2 - 6x + 1)(x^2 - 2x + 5) \end{aligned}$$

Now

$$x^2 - 2x + 5 = (x-1)^2 + 4 > 0 \quad \text{for all } x$$

so we need

$$x^2 - 6x + 1 < 0$$

Solving $x^2 - 6x + 1 = 0$,

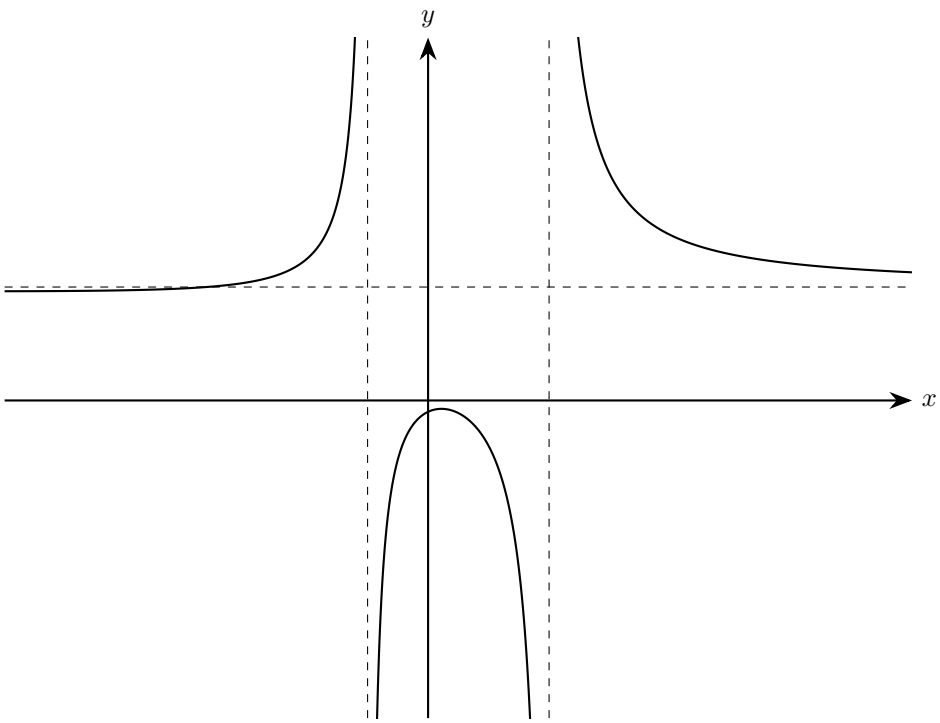
$$x = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$$

Since the quadratic opens upwards, it is negative between its roots. Therefore

$$3 - 2\sqrt{2} < x < 3 + 2\sqrt{2}$$

So the set of values of x is

$$3 - 2\sqrt{2} < x < 3 + 2\sqrt{2}$$



3. The diagram shows the curve C with equation

$$y = \frac{5x^2 + ax - b}{x^2 + bx + a}$$

where a and b are integers.

The vertical asymptotes of C are $x = -1$ and $x = 2$.

(a) State the equation of the asymptote of C that is parallel to the x-axis. **[1]**

(b) Use the vertical asymptotes to find the value of a and the value of b. **[2]**

(c) Hence, or otherwise, find the coordinates of the point where C meets the y-axis. **[1]**

(d) Without using calculus, show that the line $y = 4$ does not intersect C. **[5]**

Solution

(a) The degrees of the numerator and denominator are both 2, so the asymptote parallel to the x-axis is given by the ratio of the leading coefficients:

$$y = \frac{5}{1} = 5$$

So the asymptote is

$$y = 5$$

(b) Vertical asymptotes occur where the denominator is zero.

So the quadratic

$$x^2 + bx + a$$

has roots $x = -1$ and $x = 2$.

Hence

$$x^2 + bx + a = (x + 1)(x - 2)$$

Expanding:

$$(x+1)(x-2) = x^2 - x - 2$$

Comparing with $x^2 + bx + a$, we get

$$b = -1, \qquad a = -2$$

(c) The curve meets the y-axis when $x = 0$.

Using $a = -2$ and $b = -1$:

$$y = \frac{5(0)^2 + a(0) - b}{(0)^2 + b(0) + a} = \frac{-b}{a} = \frac{-(-1)}{-2} = \frac{1}{-2} = -\frac{1}{2}$$

So the point is

$$\left(0, -\frac{1}{2}\right)$$

(d) Using $a = -2$ and $b = -1$, the curve is

$$y = \frac{5x^2 - 2x + 1}{x^2 - x - 2}$$

To find any intersection with the line $y = 4$, solve

$$\frac{5x^2 - 2x + 1}{x^2 - x - 2} = 4$$

So

$$5x^2 - 2x + 1 = 4(x^2 - x - 2)$$

Expanding the right-hand side:

$$5x^2 - 2x + 1 = 4x^2 - 4x - 8$$

Bring all terms to one side:

$$5x^2 - 2x + 1 - 4x^2 + 4x + 8 = 0$$
$$x^2 + 2x + 9 = 0$$

Now consider the discriminant:

$$\Delta = 2^2 - 4(1)(9) = 4 - 36 = -32$$

Since

$$\Delta < 0$$

the equation has no real solutions.

Therefore there are no real values of x for which $y = 4$, so the line $y = 4$ does not intersect C.

4. The function f is defined by

$$f(x) = \frac{-2x^2 + 11x - 20}{2x - 3} \qquad (x \in \mathbb{R},\ x \neq \tfrac{3}{2})$$

(a) Find the range of f **[5]**

(b) Find the coordinates of the two stationary points of the graph of $y = f(x)$. **[2]**

(c) Show that the graph of $y = f(x)$ has an oblique asymptote and find its equation. **[2]**

(d) Sketch the graph of $y = f(x)$, showing clearly the asymptotes and stationary points. **[4]**

Solution

(a) Let $y = f(x)$. Then

$$y = \frac{-2x^2 + 11x - 20}{2x - 3}$$

so

$$y(2x - 3) = -2x^2 + 11x - 20$$
$$2x^2 + (2y - 11)x + (20 - 3y) = 0$$

For a given value of y, this is a quadratic in x. Real values of x exist if and only if the discriminant is non-negative:

$$\begin{aligned}\Delta &= (2y - 11)^2 - 4(2)(20 - 3y)\\ &= 4y^2 - 44y + 121 - 160 + 24y\\ &= 4y^2 - 20y - 39\\ &= (2y - 13)(2y + 3)\end{aligned}$$

So we require

$$(2y - 13)(2y + 3) \geq 0$$

This gives

$$y \leq -\frac{3}{2} \qquad \text{or} \qquad y \geq \frac{13}{2}$$

We must also check that the solution is not the excluded value $x = \frac{3}{2}$. Substituting $x = \frac{3}{2}$ into

$$2x^2 + (2y - 11)x + (20 - 3y)$$

gives

$$2\left(\frac{3}{2}\right)^2 + (2y - 11)\left(\frac{3}{2}\right) + (20 - 3y) = 8 \neq 0$$

so $x = \frac{3}{2}$ is never a root. Hence every y satisfying the inequality is genuinely taken by the function.
Now check the endpoints.
If $y = \frac{13}{2}$, then

$$2x^2 + 2x + \frac{1}{2} = 0$$
$$4x^2 + 4x + 1 = 0$$
$$(2x + 1)^2 = 0$$

so $x = -\frac{1}{2}$.

If $y = -\frac{3}{2}$, then

$$\begin{aligned} 2x^2 - 14x + \frac{49}{2} &= 0 \\ 4x^2 - 28x + 49 &= 0 \\ (2x-7)^2 &= 0 \end{aligned}$$

so $x = \frac{7}{2}$.

Thus the endpoints are included, so the range of f is

$$\left(-\infty, -\tfrac{3}{2}\right] \cup \left[\tfrac{13}{2}, \infty\right)$$

(b) First rewrite $f(x)$ as

$$-2x^2 + 11x - 20 = (-x+4)(2x-3) - 8$$

so

$$f(x) = -x + 4 - \frac{8}{2x-3}$$

Differentiate:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-x + 4 - 8(2x-3)^{-1}\right) \\ &= -1 - 8(-1)(2)(2x-3)^{-2} \\ &= -1 + \frac{16}{(2x-3)^2} \end{aligned}$$

For stationary points, set $f'(x) = 0$:

$$\begin{aligned} -1 + \frac{16}{(2x-3)^2} &= 0 \\ \frac{16}{(2x-3)^2} &= 1 \\ (2x-3)^2 &= 16 \end{aligned}$$

so

$$2x - 3 = \pm 4$$

giving

$$x = -\frac{1}{2} \quad \text{or} \quad x = \frac{7}{2}$$

Now find the corresponding y-values:

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= -\left(-\frac{1}{2}\right) + 4 - \frac{8}{-4} = \frac{1}{2} + 4 + 2 = \frac{13}{2} \\ f\left(\frac{7}{2}\right) &= -\frac{7}{2} + 4 - \frac{8}{4} = \frac{1}{2} - 2 = -\frac{3}{2} \end{aligned}$$

So the stationary points are

$$\left(-\frac{1}{2}, \frac{13}{2}\right) \quad \text{and} \quad \left(\frac{7}{2}, -\frac{3}{2}\right)$$

(c) From part (b),

$$f(x) = -x + 4 - \frac{8}{2x-3}$$

Hence

$$f(x) - (-x+4) = -\frac{8}{2x-3}$$

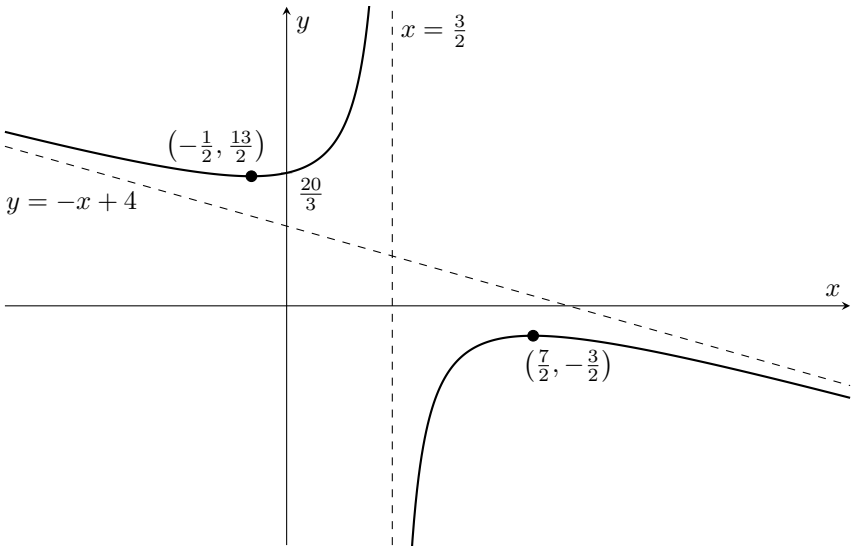
As $x \to \pm\infty$,

$$-\frac{8}{2x-3} \to 0$$

So the graph approaches the line $y = -x + 4$. Therefore the oblique asymptote is

$$y = -x + 4$$

(d) The sketch should look like:



Key features for the sketch are as follows.

From part (c), the oblique asymptote is

$$y = -x + 4$$

Also, from

$$f(x) = -x + 4 - \frac{8}{2x - 3}$$

as $x \to \left(\frac{3}{2}\right)^-$, $2x - 3 \to 0^-$, so

$$f(x) \to +\infty$$

and as $x \to \left(\frac{3}{2}\right)^+$, $2x - 3 \to 0^+$, so

$$f(x) \to -\infty$$

Hence there is a vertical asymptote at

$$x = \frac{3}{2}$$

Also

$$f(x) - (-x + 4) = -\frac{8}{2x - 3}$$

so for $x < \frac{3}{2}$ this is positive, meaning the left-hand branch lies above the oblique asymptote, and for $x > \frac{3}{2}$ it is negative, meaning the right-hand branch lies below it.

From part (b), the stationary points are

$$\left(-\frac{1}{2}, \frac{13}{2}\right) \quad \text{and} \quad \left(\frac{7}{2}, -\frac{3}{2}\right)$$

The y-intercept is

$$f(0) = \frac{-20}{-3} = \frac{20}{3}$$

so the graph passes through $\left(0, \frac{20}{3}\right)$.

For x-intercepts, solve

$$-2x^2 + 11x - 20 = 0$$

Its discriminant is

$$11^2 - 4(-2)(-20) = 121 - 160 = -39 < 0$$

so there are no x-intercepts.

Finally, from part (a), the graph has no points with

$$-\frac{3}{2} < y < \frac{13}{2}$$

This gives the required sketch.

5. The curve C has equation $y = \dfrac{x^2 + ax - 6}{x + 1}$, where $a > 0$.

(a) Find the equations of the asymptotes of C. **[3]**

(b) Show that C has no stationary points. **[4]**

(c) Sketch C, stating the coordinates of the point of intersection with the y-axis and labelling the asymptotes. **[3]**

(d) (i) Sketch the curve with equation $y = \left|\dfrac{x^2 + ax - 6}{x + 1}\right|$. **[2]**

(ii) On your sketch in part (i), draw the line $y = a + 1$. **[1]**

(iii) It is given that $\left|\dfrac{x^2 + ax - 6}{x + 1}\right| < a + 1$ for $-11 < x < -3$ and $0 < x < 4$.
Find the value of a. **[2]**

Solution

(a) Divide the numerator by $x + 1$:

$$x^2 + ax - 6 = (x + 1)(x + a - 1) - (a + 5)$$

so

$$y = \frac{x^2 + ax - 6}{x + 1} = x + a - 1 - \frac{a + 5}{x + 1}$$

Hence the asymptotes are

$$x = -1 \qquad \text{and} \qquad y = x + a - 1$$

(b) From part (a),

$$y = x + a - 1 - (a + 5)(x + 1)^{-1}$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + (a + 5)(x + 1)^{-2} = 1 + \frac{a + 5}{(x + 1)^2}$$

Since $a > 0$, we have $a + 5 > 0$, and for $x \neq -1$,

$$\frac{a + 5}{(x + 1)^2} > 0$$

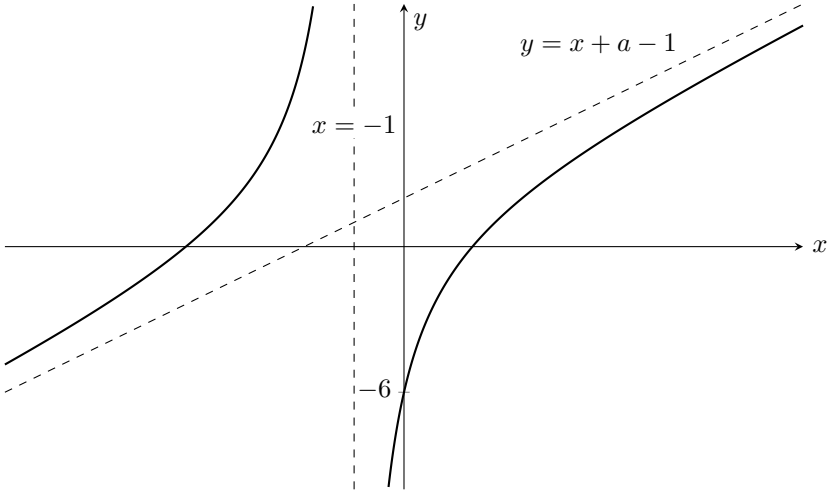
Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} > 0 \quad \text{for all } x \neq -1$$

A stationary point would require $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, which never happens.

Therefore C has no stationary points.

(c) The sketch should look like:



The point where the curve meets the y-axis is found by putting $x = 0$:

$$y = \frac{0^2 + a(0) - 6}{0 + 1} = -6$$

so the point is $(0, -6)$.

Also,

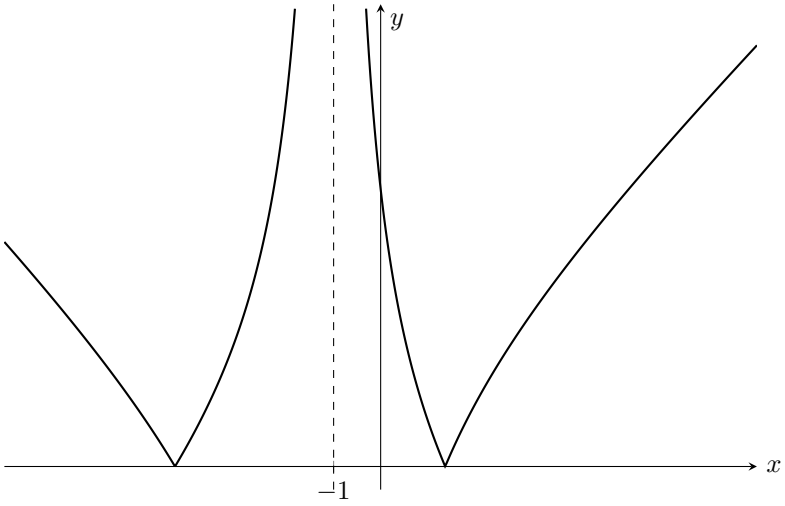
$$y - (x + a - 1) = -\frac{a + 5}{x + 1}$$

so for $x < -1$ the curve is above the oblique asymptote, and for $x > -1$ it is below it.

From part (b), the curve is increasing on both sides of $x = -1$, with

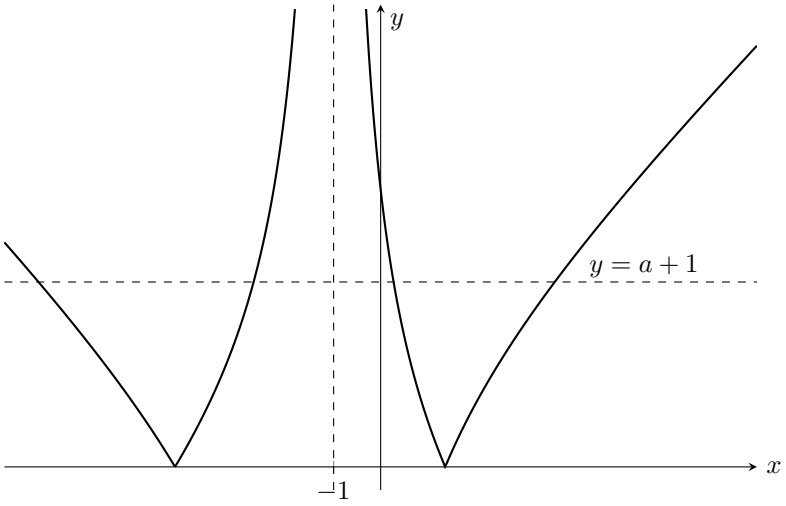
$$y \to +\infty \text{ as } x \to -1^-, \qquad y \to -\infty \text{ as } x \to -1^+$$

(d) (i) The sketch should look like:



This is obtained by reflecting the parts of C below the x-axis in the x-axis.

(ii) Draw the horizontal line $y = a + 1$ on the sketch:



(iii) The endpoints of the given intervals are where

$$\left|\frac{x^2 + ax - 6}{x + 1}\right| = a + 1$$

From the intervals

$$-11 < x < -3 \quad \text{and} \quad 0 < x < 4$$

we can see that $x = 0$ is one boundary point. So

$$\left|\frac{0^2 + a(0) - 6}{0 + 1}\right| = a + 1$$

$$6 = a + 1$$

$$a = 5$$

6. The function f is defined by

$$f(x) = \frac{x^2 + 2x - 3}{x^2 + px + 4} \qquad x \in \mathbb{R}$$

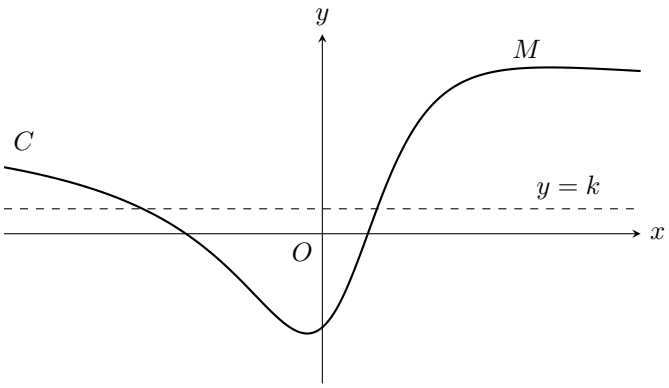
where p is a constant.

The graph of $y = f(x)$ has only one asymptote.

(a) Write down the equation of the asymptote. **[1]**

(b) Find the set of possible values of p. **[4]**

(c) Find the coordinates of the points at which the graph of $y = f(x)$ intersects the coordinate axes. **[3]**



(d) A curve C has equation

$$y = \frac{x^2 + 2x - 3}{x^2 - x + 4}$$

The curve C has a local maximum at the point M as shown in the diagram.
The line $y = k$ intersects curve C.

(i) Show that

$$15k^2 - 8k - 16 \leq 0$$ **[5]**

(ii) Hence, find the y-coordinate of point M. **[2]**

Solution

(a) The numerator and denominator have the same degree, and both have leading coefficient 1.

So the horizontal asymptote is

$$y = 1$$

(b) From part (a), one asymptote is already $y = 1$.

For the graph to have *only one* asymptote, there must be no vertical asymptotes. Vertical asymptotes would occur if the denominator

$$x^2 + px + 4$$

had any real roots.

Therefore we require the quadratic $x^2 + px + 4 = 0$ to have no real solutions, so its discriminant must be negative:

$$p^2 - 4(1)(4) < 0$$

$$p^2 - 16 < 0$$

$$p^2 < 16$$

$$-4 < p < 4$$

(c) To find the x-intercepts, set $y = 0$. Then the numerator must be zero:

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3 \quad \text{or} \quad x = 1$$

So the possible points are $(-3, 0)$ and $(1, 0)$.

We must also check the denominator is not zero at these values. Since from part (b), $-4 < p < 4$,

$$(-3)^2 + p(-3) + 4 = 13 - 3p > 1$$

and

$$1^2 + p(1) + 4 = p + 5 > 1$$

so both are non-zero.

Hence the graph intersects the x-axis at

$$(-3, 0) \quad \text{and} \quad (1, 0)$$

To find the y-intercept, set $x = 0$:

$$y = \frac{0^2 + 2(0) - 3}{0^2 + p(0) + 4} = \frac{-3}{4}$$

So the graph intersects the y-axis at

$$\left(0, -\frac{3}{4}\right)$$

(d) (i) The line $y = k$ intersects the curve

$$y = \frac{x^2 + 2x - 3}{x^2 - x + 4}$$

when

$$k = \frac{x^2 + 2x - 3}{x^2 - x + 4}$$

First note that

$$x^2 - x + 4 = \left(x - \frac{1}{2}\right)^2 + \frac{15}{4} > 0$$

for all real x, so we may multiply through safely.

Hence

$$x^2 + 2x - 3 = k(x^2 - x + 4)$$

Rearranging,

$$\begin{aligned} x^2 + 2x - 3 &= kx^2 - kx + 4k \\ 0 &= kx^2 - kx + 4k - x^2 - 2x + 3 \\ 0 &= (k-1)x^2 - (k+2)x + (4k+3) \end{aligned}$$

Equivalently,

$$(1-k)x^2 + (2+k)x - (3+4k) = 0$$

For the line to intersect the curve, this quadratic in x must have at least one real root. So its discriminant is non-negative:

$$(2+k)^2 - 4(1-k)\big(-(3+4k)\big) \geq 0$$

$$(2+k)^2 + 4(1-k)(3+4k) \geq 0$$

Now expand:

$$\begin{aligned}(2+k)^2+4(1-k)(3+4k) &\geq 0\\ (k^2+4k+4)+4(3+4k-3k-4k^2) &\geq 0\\ k^2+4k+4+4(3+k-4k^2) &\geq 0\\ k^2+4k+4+12+4k-16k^2 &\geq 0\\ -15k^2+8k+16 &\geq 0\end{aligned}$$

Therefore

$$15k^2-8k-16\leq 0$$

as required.

(ii) At the local maximum, the horizontal line $y=k$ just touches the curve, so there is exactly one value of x. Therefore the discriminant is zero, so

$$15k^2-8k-16=0$$

Solving,

$$k=\frac{8\pm\sqrt{(-8)^2-4(15)(-16)}}{2(15)}=\frac{8\pm\sqrt{64+960}}{30}=\frac{8\pm 32}{30}$$

So

$$k=\frac{40}{30}=\frac{4}{3}\qquad\text{or}\qquad k=\frac{-24}{30}=-\frac{4}{5}$$

The local maximum has the larger y-value, so the y-coordinate of M is

$$\frac{4}{3}$$

7. A curve has equation

$$y = \frac{x^2 - 9}{x^2 - 4}$$

(a) Write down the equations of the asymptotes to the curve. **[3]**

(b) Sketch the curve, indicating the coordinates of the points where the curve intersects the coordinate axes. **[5]**

(c) Hence, or otherwise, solve the inequality

$$0 < \frac{x^2 - 9}{x^2 - 4} < 1$$ **[2]**

Solution

(a) Vertical asymptotes occur where the denominator is zero:

$$x^2 - 4 = 0$$

so

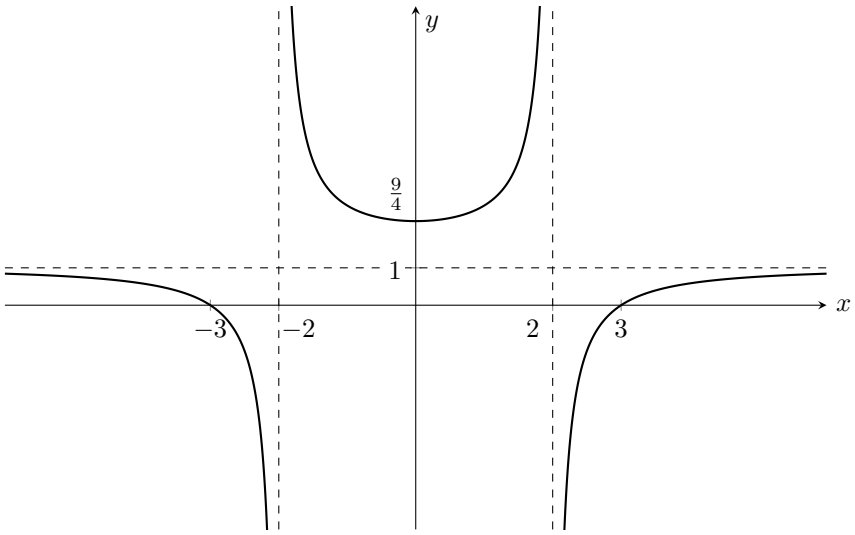
$$x = \pm 2$$

For large $|x|$, the numerator and denominator have the same degree, so the horizontal asymptote is the ratio of the leading coefficients:

$$y = 1$$

Hence the asymptotes are $x = -2$, $x = 2$ and $y = 1$.

(b) The sketch should look like:



First write

$$y = \frac{x^2 - 9}{x^2 - 4} = 1 - \frac{5}{x^2 - 4}$$

Since the equation depends on x^2, the curve is symmetric about the y-axis.

The x-intercepts are found by setting $y = 0$:

$$\frac{x^2 - 9}{x^2 - 4} = 0 \quad \Rightarrow \quad x^2 - 9 = 0$$

so

$$x = \pm 3$$

giving intercepts $(-3, 0)$ and $(3, 0)$.

The y-intercept is found by setting $x = 0$:

$$y = \frac{0-9}{0-4} = \frac{9}{4}$$

so the point is

$$\left(0, \frac{9}{4}\right)$$

Also,

$$y = 1 - \frac{5}{x^2 - 4}$$

so the curve never meets the line $y = 1$.

For $|x| < 2$, we have $x^2 - 4 < 0$, so $-\dfrac{5}{x^2-4} > 0$ and hence $y > 1$.

For $2 < |x| < 3$, the numerator is negative and the denominator is positive, so $y < 0$.

For $|x| > 3$, both numerator and denominator are positive, so $0 < y < 1$.

So the sketch has the correct intercepts and asymptotes as shown.

(c) Let

$$f(x) = \frac{x^2 - 9}{x^2 - 4}$$

From the sketch in part **(b)**, the inequality

$$0 < f(x) < 1$$

means that the curve must lie *above* the x-axis and *below* the horizontal line $y = 1$.

Looking at the graph, this happens only on the two outer branches, that is for

$$x < -3 \quad \text{or} \quad x > 3.$$

Hence the solution to

$$0 < \frac{x^2 - 9}{x^2 - 4} < 1$$

is

$$\boxed{x < -3 \quad \text{or} \quad x > 3.}$$

8. The curve C has equation

$$y = \frac{x^2 + 3}{x + 1}$$

(a) Find the equations of the asymptotes of C. **[3]**

(b) Show that there is no point on C for which $-6 < y < 2$. **[4]**

(c) Sketch C. **[2]**

(d) (i) Sketch the graphs of

$$y = \left|\frac{x^2 + 3}{x + 1}\right| \text{ and } y = |x - 1|$$

on the same axes, stating the coordinates of any intersections with the axes. **[4]**

(ii) Use your sketch to find the set of values of c for which

$$\left|\frac{x^2 + 3}{x + 1}\right| \leq |x - 1| + c$$

has no solution. **[1]**

Solution

(a) Divide the numerator by the denominator:

$$x^2 + 3 = (x + 1)(x - 1) + 4$$

so

$$y = \frac{x^2 + 3}{x + 1} = x - 1 + \frac{4}{x + 1}$$

Hence:

- as $x \to -1$, $\dfrac{4}{x + 1} \to \pm\infty$, so there is a vertical asymptote

$$x = -1$$

- as $x \to \pm\infty$, $\dfrac{4}{x + 1} \to 0$, so the curve approaches

$$y = x - 1$$

Therefore the asymptotes are $x = -1$ and $y = x - 1$.

(b) Start from

$$y = \frac{x^2 + 3}{x + 1}$$

and rearrange:

$$y(x + 1) = x^2 + 3$$
$$x^2 - yx + (3 - y) = 0$$

This is a quadratic in x. For real values of x, its discriminant must satisfy

$$\Delta \geq 0$$

So

$$\begin{aligned}\Delta &= (-y)^2 - 4(1)(3 - y) \\ &= y^2 - 12 + 4y \\ &= y^2 + 4y - 12 \\ &= (y + 6)(y - 2)\end{aligned}$$

Hence

$$(y+6)(y-2) \geq 0$$

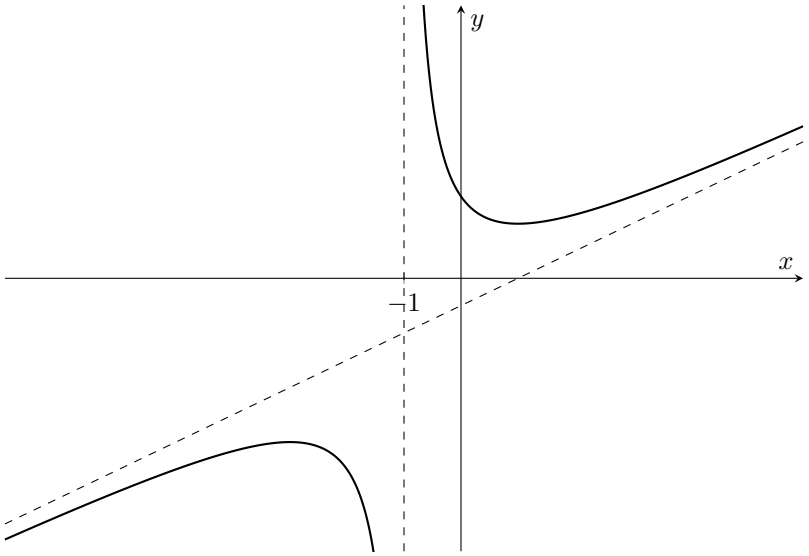
This gives

$$y \leq -6 \quad \text{or} \quad y \geq 2$$

So there is no point on C for which

$$-6 < y < 2$$

(c) The sketch should look like:



Finding the y-intercept:

$$y(0) = 3$$

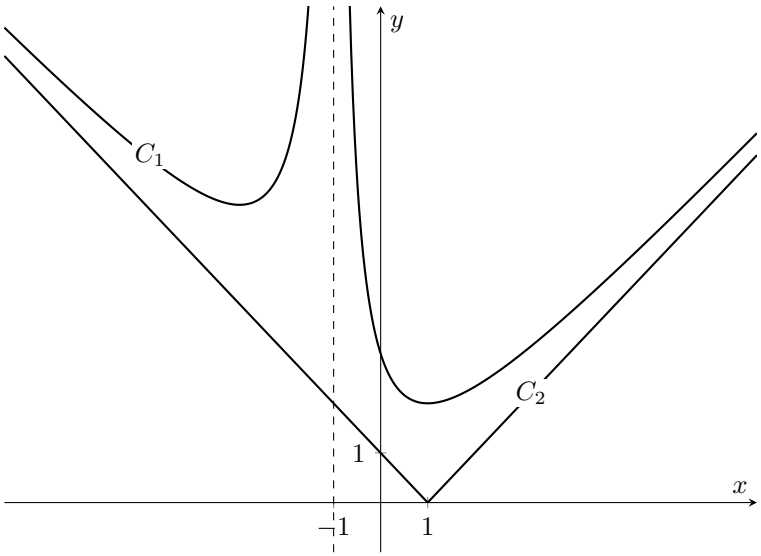
so the y-intercept is $(0, 3)$.

There is no x-intercept because

$$\frac{x^2+3}{x+1} = 0 \;\Rightarrow\; x^2+3 = 0$$

which has no real solution.

(d) (i) The two sketches should look like:



Here

$$C_1 :\; y = \left|\frac{x^2+3}{x+1}\right|, \qquad C_2 :\; y = |x-1|$$

For C_1, the part of the original curve with $x < -1$ is reflected in the x-axis.The right-hand branch is unchanged.

Its intersections with the axes are:

$$y = \left|\frac{0^2+3}{0+1}\right| = 3$$

so the y-intercept is $(0, 3)$.
There is no x-intercept, since $x^2 + 3 > 0$ for all real x.
Also, C_1 has vertical asymptote $x = -1$, and its oblique asymptotes are

$$y = 1 - x \quad \text{as } x \to -\infty, \qquad y = x - 1 \quad \text{as } x \to \infty$$

For $C_2 : y = |x - 1|$:

$$|x - 1| = 0 \Rightarrow x = 1$$

so the x-intercept is $(1, 0)$, and

$$y = |0 - 1| = 1$$

so the y-intercept is $(0, 1)$.

(ii) From the sketch, $y = \left|\frac{x^2+3}{x+1}\right|$ lies above $y = |x - 1|$ for all x, and the gap tends to 0 as $x \to \pm\infty$.
For the equality case $c = 0$,

$$\begin{aligned}\left|\frac{x^2+3}{x+1}\right| = |x-1| &\Rightarrow \frac{(x^2+3)^2}{(x+1)^2} = (x-1)^2 \\ &\Rightarrow (x^2+3)^2 = (x-1)^2(x+1)^2 \\ &\Rightarrow (x^2+3)^2 = (x^2-1)^2 \\ &\Rightarrow x^4 + 6x^2 + 9 = x^4 - 2x^2 + 1 \\ &\Rightarrow 8x^2 + 8 = 0\end{aligned}$$

which is impossible for real x.
So there is no solution when $c = 0$, and therefore no solution for any $c < 0$ either. Since the graphs get arbitrarily close for large positive or negative x, any $c > 0$ gives a solution.
Hence the inequality has no solution for

$$c \leq 0$$

9. Let a be a positive constant.

(a) The curve C_1 has equation

$$y = \frac{x^2 - a^2}{x^2 - 4a^2}$$

Sketch C_1. **[2]**

The curve C_2 has equation

$$y = \left(\frac{x^2 - a^2}{x^2 - 4a^2}\right)^2$$

The curve C_3 has equation

$$y = \left|\frac{x^2 - a^2}{x^2 - 4a^2}\right|$$

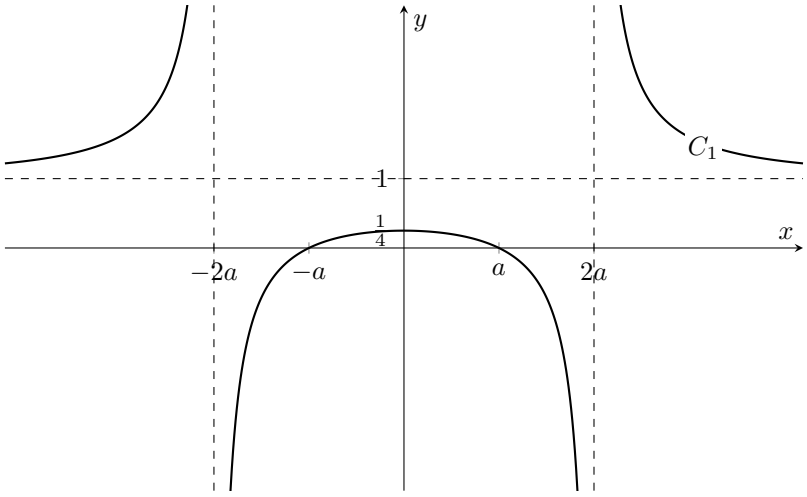
(b) (i) Find the coordinates of all stationary points of C_2. **[3]**
(ii) Find also the coordinates of all points of intersection of C_2 and C_3. **[3]**

(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which

$$\left(\frac{x^2 - a^2}{x^2 - 4a^2}\right)^2 \leq \left|\frac{x^2 - a^2}{x^2 - 4a^2}\right|$$ **[5]**

Solution

(a) The sketch should look like:



Since the equation involves only x^2, C_1 is even, so it is symmetric about the y-axis.

The vertical asymptotes are where

$$x^2 - 4a^2 = 0 \quad \Rightarrow \quad x = \pm 2a$$

and as $x \to \pm\infty$,

$$\frac{x^2 - a^2}{x^2 - 4a^2} \to 1$$

so the horizontal asymptote is $y = 1$.

The intercepts are:

$$y = 0 \Rightarrow x^2 - a^2 = 0 \Rightarrow x = \pm a$$

so $(\pm a, 0)$, and

$$x = 0 \Rightarrow y = \frac{-a^2}{-4a^2} = \frac{1}{4}$$

so $(0, \frac{1}{4})$.

Also, $y > 1$ for $|x| > 2a$, $y < 0$ for $a < |x| < 2a$, and $0 \leq y < 1$ for $|x| < a$.

(b) **(i)** Let

$$g(x) = \left(\frac{x^2 - a^2}{x^2 - 4a^2}\right)^2$$

Write

$$f(x) = \frac{x^2 - a^2}{x^2 - 4a^2}$$

so that $g(x) = \big(f(x)\big)^2$.

Then

$$g'(x) = 2f(x)f'(x)$$

and, using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(2x)(x^2 - 4a^2) - (x^2 - a^2)(2x)}{(x^2 - 4a^2)^2} \\ &= \frac{2x\big((x^2 - 4a^2) - (x^2 - a^2)\big)}{(x^2 - 4a^2)^2} \\ &= \frac{2x(-3a^2)}{(x^2 - 4a^2)^2} \\ &= -\frac{6a^2x}{(x^2 - 4a^2)^2} \end{aligned}$$

So

$$\begin{aligned} g'(x) &= 2\left(\frac{x^2 - a^2}{x^2 - 4a^2}\right)\left(-\frac{6a^2x}{(x^2 - 4a^2)^2}\right) \\ &= -\frac{12a^2x(x^2 - a^2)}{(x^2 - 4a^2)^3} \end{aligned}$$

Stationary points occur when $g'(x) = 0$, so

$$-12a^2x(x^2 - a^2) = 0$$

Hence

$$x = 0 \quad \text{or} \quad x^2 = a^2 \Rightarrow x = \pm a$$

Now find the corresponding y-coordinates:

$$g(0) = \left(\frac{-a^2}{-4a^2}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

and

$$g(\pm a) = 0$$

Therefore the stationary points are

$$(0, \tfrac{1}{16}), \quad (-a, 0), \quad (a, 0)$$

(ii) The curves C_2 and C_3 have equations

$$y = \left(\frac{x^2 - a^2}{x^2 - 4a^2}\right)^2 \qquad \text{and} \qquad y = \left|\frac{x^2 - a^2}{x^2 - 4a^2}\right|$$

Let

$$u = \left|\frac{x^2 - a^2}{x^2 - 4a^2}\right|$$

Then intersections satisfy

$$u^2 = u$$

so

$$u(u-1)=0$$

Hence

$$u=0 \quad \text{or} \quad u=1$$

If $u=0$, then

$$\left|\frac{x^2-a^2}{x^2-4a^2}\right|=0$$

so

$$x^2-a^2=0 \Rightarrow x=\pm a$$

and then $y=0$.
So two intersection points are

$$(\pm a, 0)$$

If $u=1$, then

$$\left|\frac{x^2-a^2}{x^2-4a^2}\right|=1$$

so

$$\frac{x^2-a^2}{x^2-4a^2}=\pm 1$$

First, if

$$\frac{x^2-a^2}{x^2-4a^2}=1$$

then

$$x^2-a^2=x^2-4a^2$$

which gives

$$-a^2=-4a^2$$

impossible.
Next, if

$$\frac{x^2-a^2}{x^2-4a^2}=-1$$

then

$$\begin{aligned} x^2-a^2&=-(x^2-4a^2)\\ x^2-a^2&=-x^2+4a^2\\ 2x^2&=5a^2\\ x^2&=\frac{5a^2}{2} \end{aligned}$$

so

$$x=\pm a\sqrt{\frac{5}{2}}$$

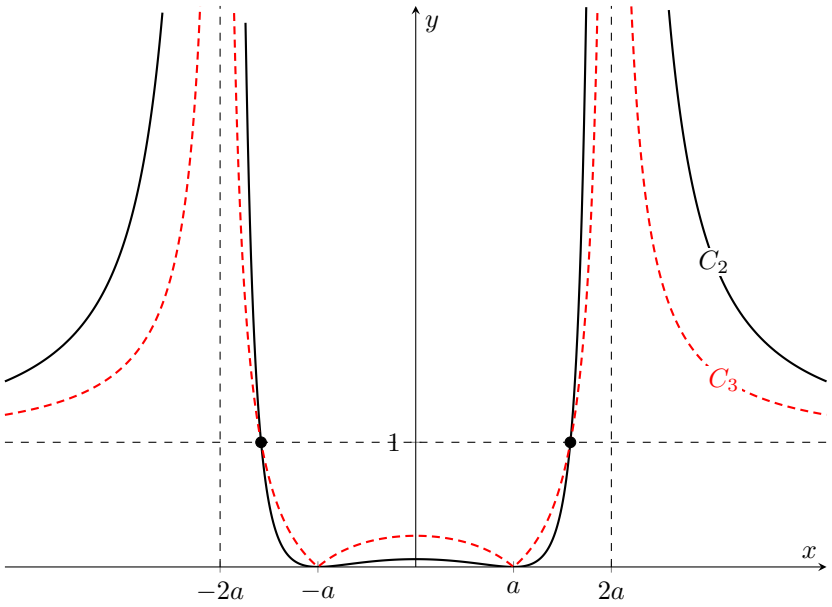
and since $u=1$,

$$y=1$$

Therefore the points of intersection are

$$(\pm a, 0), \quad \left(\pm a\sqrt{\frac{5}{2}}, 1\right)$$

(c) The sketch should look like:



Both curves are even, so both are symmetric about the y-axis.

They have the same vertical asymptotes $x = \pm 2a$ and the same horizontal asymptote $y = 1$.

From part (b)(i), C_2 has stationary points

$$(0, \tfrac{1}{16}), \quad (\pm a, 0)$$

and from part (b)(ii), the curves meet at

$$(\pm a, 0), \quad \left(\pm a\sqrt{\frac{5}{2}}, 1\right)$$

Also, $C_3 = |C_1|$, so it passes through $(0, \frac{1}{4})$ and has cusps at $(\pm a, 0)$.

From the sketch, C_2 lies below or on C_3 between the outer intersection points. Therefore

$$\left(\frac{x^2 - a^2}{x^2 - 4a^2}\right)^2 \leq \left|\frac{x^2 - a^2}{x^2 - 4a^2}\right|$$

for

$$-a\sqrt{\frac{5}{2}} \leq x \leq a\sqrt{\frac{5}{2}}$$

So the required set of values is

$$-a\sqrt{\frac{5}{2}} \leq x \leq a\sqrt{\frac{5}{2}}$$

10. A curve C_1 has equation

$$y = -2 + \frac{12}{x+3}$$

(a) Write down the equations of the asymptotes of curve C_1. **[2]**

(b) Sketch the graph of curve C_1. Indicate the values of the intercepts of the curve with the axes. **[3]**

(c) Hence, or otherwise, solve the inequality

$$-2 + \frac{12}{x+3} > 0$$

[2]

(d) Curve C_2 is a reflection of curve C_1 in the line $y = -x$. Find an equation for curve C_2 in the form $y = f(x)$. **[3]**

Solution

(a) For

$$y = -2 + \frac{12}{x+3}$$

the denominator is zero when $x + 3 = 0$, so the vertical asymptote is

$$x = -3$$

Also, as $x \to \pm\infty$,

$$\frac{12}{x+3} \to 0$$

so

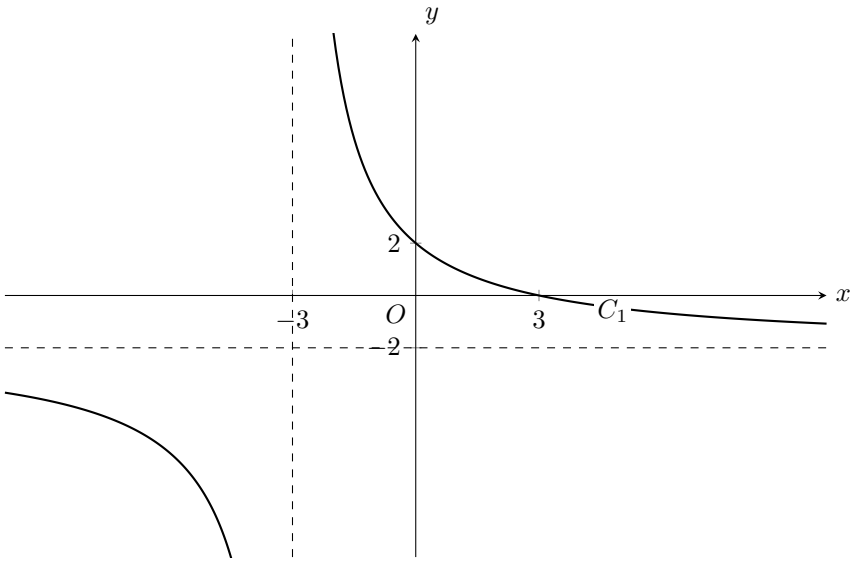
$$y \to -2$$

Hence the horizontal asymptote is

$$y = -2$$

Therefore the asymptotes are $x = -3$ and $y = -2$.

(b) The sketch should look like:



This is a rectangular hyperbola with centre $(-3, -2)$, with asymptotes $x = -3$ and $y = -2$. For $x > -3$, the branch lies above $y = -2$; for $x < -3$, the branch lies below $y = -2$.

To find the intercepts:

Setting $x = 0$,

$$y = -2 + \frac{12}{0+3} = -2 + 4 = 2$$

so the y-intercept is $(0, 2)$.

Setting $y = 0$,

$$\begin{aligned} -2 + \frac{12}{x+3} &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

so the x-intercept is $(3, 0)$.

Therefore the intercepts are $(3, 0)$ and $(0, 2)$.

(c) We need to solve

$$-2 + \frac{12}{x+3} > 0$$

From the sketch in part **(b)**, this means we want the part of the curve that lies *above* the x-axis.

Looking at the graph, curve C_1 is above the x-axis only between its vertical asymptote at $x = -3$ and its x-intercept at $x = 3$.

Hence

$$-3 < x < 3$$

(d) Reflection in the line $y = -x$ maps each point (x, y) to $(-y, -x)$.

So in the equation of C_1, replace x by $-y$ and y by $-x$:

$$-x = -2 + \frac{12}{-y+3}$$

that is,

$$-x = -2 + \frac{12}{3-y}$$

Now solve for y:

$$\begin{aligned} -x + 2 &= \frac{12}{3-y} \\ 3 - y &= \frac{12}{2-x} \end{aligned}$$

Since $2 - x = -(x - 2)$,

$$\frac{12}{2-x} = -\frac{12}{x-2}$$

so

$$\begin{aligned} 3 - y &= -\frac{12}{x-2} \\ y &= 3 + \frac{12}{x-2} \end{aligned}$$

Hence the equation of C_2 is

$$y = 3 + \frac{12}{x-2}$$

11. The curve C has equation

$$y = \frac{x^2 - 6x + 5}{x + 2}$$

(a) Find the equations of the asymptotes of C. **[3]**

(b) Find the exact coordinates of the stationary points on C. **[4]**

(c) Sketch C, stating the coordinates of any intersections with the axes. **[3]**

(d) Sketch the curve with equation

$$y = \left| \frac{x^2 - 6x + 5}{x + 2} \right|$$

and find in exact form the set of values of x for which

$$\left| \frac{x^2 - 6x + 5}{x + 2} \right| < 6$$ **[5]**

Solution

(a) Divide the numerator by the denominator:

$$x^2 - 6x + 5 = (x + 2)(x - 8) + 21$$

so

$$y = \frac{x^2 - 6x + 5}{x + 2} = x - 8 + \frac{21}{x + 2}$$

From this form:

- the denominator is zero when $x = -2$, so there is a vertical asymptote

$$x = -2$$

- as $x \to \pm\infty$, $\dfrac{21}{x + 2} \to 0$, so the curve approaches

$$y = x - 8$$

Hence the asymptotes are $x = -2$ and $y = x - 8$.

(b) Using

$$y = x - 8 + \frac{21}{x + 2}$$

differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 21(x + 2)^{-2} = 1 - \frac{21}{(x + 2)^2}$$

At a stationary point, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, so

$$\begin{aligned} 1 - \frac{21}{(x + 2)^2} &= 0 \\ \frac{21}{(x + 2)^2} &= 1 \\ (x + 2)^2 &= 21 \\ x + 2 &= \pm\sqrt{21} \end{aligned}$$

Hence

$$x = -2 \pm \sqrt{21}$$

Now find the corresponding y-coordinates from

$$y = x - 8 + \frac{21}{x+2}$$

If $x = -2 - \sqrt{21}$, then $x + 2 = -\sqrt{21}$, so

$$\begin{aligned} y &= (-2 - \sqrt{21}) - 8 + \frac{21}{-\sqrt{21}} \\ &= -10 - \sqrt{21} - \sqrt{21} \\ &= -10 - 2\sqrt{21} \end{aligned}$$

If $x = -2 + \sqrt{21}$, then $x + 2 = \sqrt{21}$, so

$$\begin{aligned} y &= (-2 + \sqrt{21}) - 8 + \frac{21}{\sqrt{21}} \\ &= -10 + \sqrt{21} + \sqrt{21} \\ &= -10 + 2\sqrt{21} \end{aligned}$$

So the stationary points are

$$(-2 - \sqrt{21},\ -10 - 2\sqrt{21}) \quad \text{and} \quad (-2 + \sqrt{21},\ -10 + 2\sqrt{21})$$

To identify them, differentiate again:

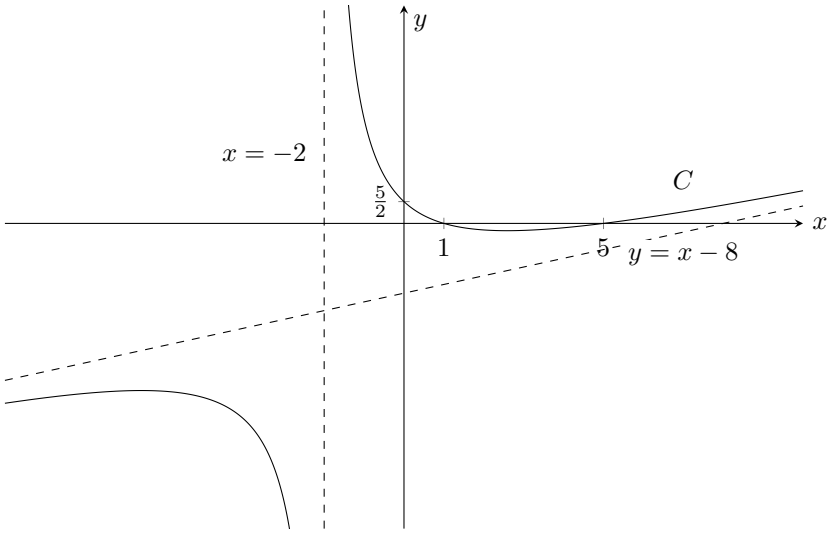
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{42}{(x+2)^3}$$

At $x = -2 - \sqrt{21}$, $(x+2)^3 < 0$, so $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$: this is a local maximum.

At $x = -2 + \sqrt{21}$, $(x+2)^3 > 0$, so $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$: this is a local minimum.

Hence the stationary points are a local maximum at $(-2 - \sqrt{21}, -10 - 2\sqrt{21})$ and a local minimum at $(-2 + \sqrt{21}, -10 + 2\sqrt{21})$.

(c) The sketch should look like:



For the intersections with the axes:

For the x-axis, set $y = 0$:

$$\frac{x^2 - 6x + 5}{x + 2} = 0$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$

So the curve meets the x-axis at

$$(1, 0) \quad \text{and} \quad (5, 0)$$

For the y-axis, set $x = 0$:

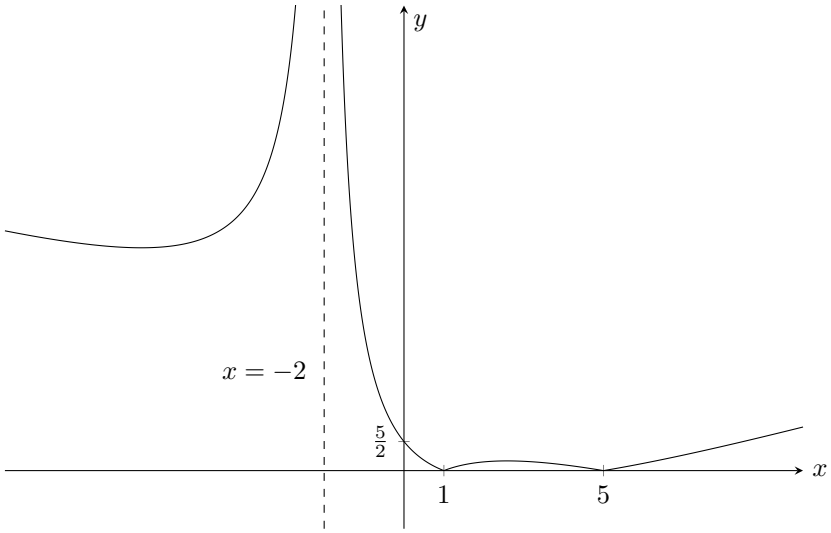
$$y = \frac{0^2 - 6(0) + 5}{0 + 2} = \frac{5}{2}$$

so the curve meets the y-axis at

$$\left(0, \frac{5}{2}\right)$$

The left-hand branch lies below the x-axis for $x < -2$, and the right-hand branch falls from $+\infty$, crosses at $(1, 0)$, reaches the minimum found in part (b), then rises through $(5, 0)$.

(d) The sketch should look like:



For $y = \left|\dfrac{x^2 - 6x + 5}{x + 2}\right|$, every part of the graph of C below the x-axis is reflected in the x-axis. So:

- the whole left-hand branch is reflected above the x-axis
- the part of the right-hand branch between $x = 1$ and $x = 5$ is reflected above the x-axis
- the points $(1, 0)$ and $(5, 0)$ stay on the curve
- $x = -2$ remains a vertical asymptote

Now solve

$$\left|\frac{x^2 - 6x + 5}{x + 2}\right| < 6$$

Since both sides are non-negative, we may square:

$$\frac{(x^2 - 6x + 5)^2}{(x + 2)^2} < 36$$

For $x \neq -2$, $(x + 2)^2 > 0$, so multiplying through gives

$$(x^2 - 6x + 5)^2 < 36(x + 2)^2$$

Rearranging as a difference of two squares:

$$\begin{aligned}(x^2-6x+5)^2-36(x+2)^2&<0\\\big((x^2-6x+5)-6(x+2)\big)\big((x^2-6x+5)+6(x+2)\big)&<0\\(x^2-12x-7)(x^2+17)&<0\end{aligned}$$

But $x^2+17>0$ for all real x, so we need

$$x^2-12x-7<0$$

Solve $x^2-12x-7=0$:

$$\begin{aligned}x&=\frac{12\pm\sqrt{(-12)^2-4(1)(-7)}}{2}\\&=\frac{12\pm\sqrt{144+28}}{2}\\&=\frac{12\pm\sqrt{172}}{2}\\&=6\pm\sqrt{43}\end{aligned}$$

Since the quadratic opens upwards, it is negative between its roots. Therefore

$$6-\sqrt{43}<x<6+\sqrt{43}$$

So the required set of values is $6-\sqrt{43}<x<6+\sqrt{43}$.

12. A curve has equation $y = \dfrac{x-1}{x^2-x-6}$.

(a) (i) Write down the equations of the three asymptotes of the curve. **[3]**

(ii) Sketch the curve, showing the coordinates of any points of intersection with the coordinate axes. **[4]**

(b) Hence, or otherwise, solve the inequality

$$\frac{x-1}{x^2-x-6} > \frac{1}{2} \qquad \textbf{[3]}$$

Solution

(a) **(i)** Factorising the denominator,

$$x^2 - x - 6 = (x-3)(x+2)$$

So the vertical asymptotes are

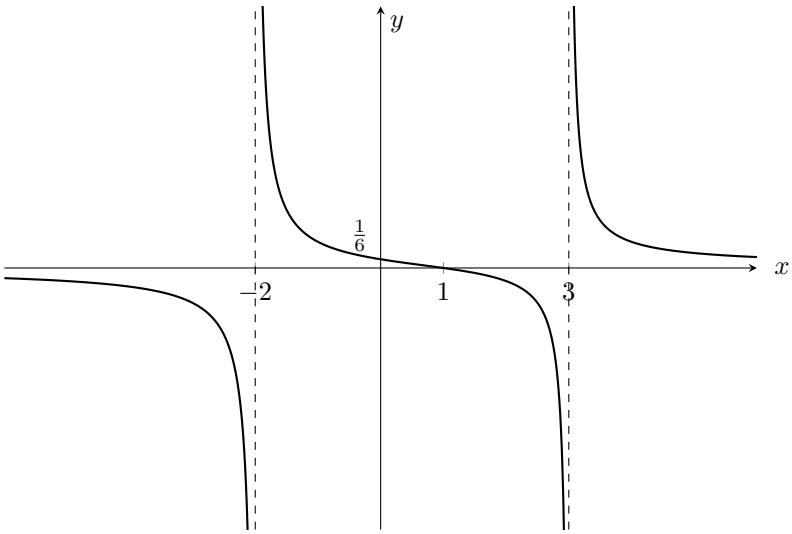
$$x = 3 \quad \text{and} \quad x = -2$$

Since the degree of the denominator is greater than the degree of the numerator, the horizontal asymptote is

$$y = 0$$

Hence the three asymptotes are $x = -2$, $x = 3$ and $y = 0$.

(ii) The sketch should look like:



The intersections with the axes are found as follows.

For the x-axis, $y = 0$, so

$$x - 1 = 0$$

giving $x = 1$. Hence the curve crosses the x-axis at $(1, 0)$.

For the y-axis, $x = 0$, so

$$y = \frac{0-1}{0^2-0-6} = \frac{-1}{-6} = \frac{1}{6}$$

Hence the curve crosses the y-axis at $\left(0, \frac{1}{6}\right)$.

Also, as $x \to \pm\infty$, $y \to 0$, so the curve approaches the asymptote $y = 0$. As $x \to -2^-$, $y \to -\infty$ and as $x \to -2^+$, $y \to +\infty$. As $x \to 3^-$, $y \to -\infty$ and as $x \to 3^+$, $y \to +\infty$. This gives the three branches shown in the sketch.

(b) To solve

$$\frac{x-1}{x^2-x-6} > \frac{1}{2}$$

we compare the curve from part **(a)(ii)** with the line

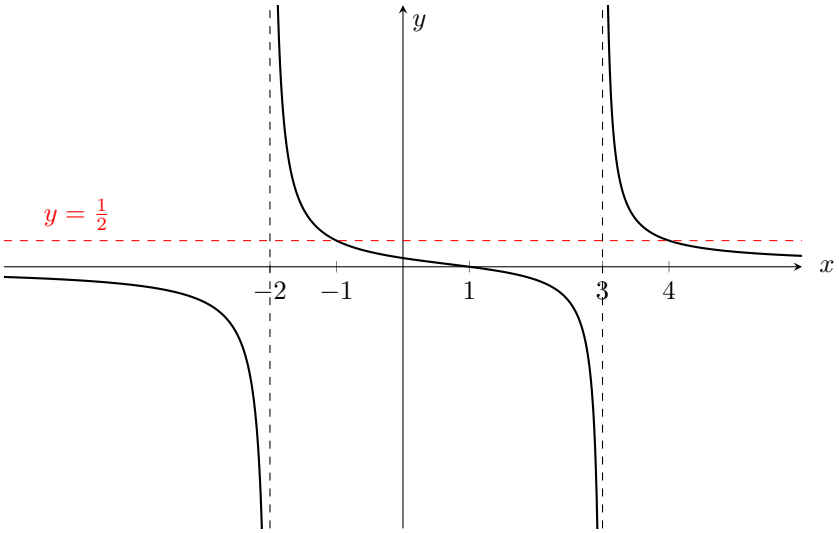
$$y = \frac{1}{2}$$

First find where the curve meets this line:

$$\begin{aligned} \frac{x-1}{x^2-x-6} &= \frac{1}{2} \\ 2(x-1) &= x^2-x-6 \\ x^2-3x-4 &= 0 \\ (x-4)(x+1) &= 0 \end{aligned}$$

So the graphs intersect when

$$x = -1 \quad \text{or} \quad x = 4$$



From the graph, the curve lies above the line $y = \frac{1}{2}$ for

$$-2 < x < -1 \quad \text{and} \quad 3 < x < 4$$

Hence the solution is

$$x \in (-2,-1) \cup (3,4)$$

13. The curve C has equation

$$y = \frac{(x-1)(x+3)}{x^2-2x+3}$$

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. **[3]**

(b) Find the coordinates of the stationary points on C. **[4]**

(c) Sketch C, stating the coordinates of the intersections with the axes. **[3]**

(d) Sketch the curve with equation

$$y = \left|\frac{(x-1)(x+3)}{x^2-2x+3}\right|$$

and state the set of values of k for which

$$\left|\frac{(x-1)(x+3)}{x^2-2x+3}\right| = k$$

has 4 distinct real solutions. **[2]**

Solution

(a) Write the denominator as

$$x^2-2x+3 = (x-1)^2+2$$

Since $(x-1)^2+2 > 0$ for all real x, the denominator is never 0.

So C has no vertical asymptotes.

The numerator and denominator are both quadratic, with the same leading coefficient 1, so as $x \to \pm\infty$,

$$y \to 1$$

Hence the horizontal asymptote is

$$y = 1$$

(b) First expand the numerator:

$$y = \frac{(x-1)(x+3)}{x^2-2x+3} = \frac{x^2+2x-3}{x^2-2x+3}$$

Differentiate using the quotient rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2x+2)(x^2-2x+3)-(x^2+2x-3)(2x-2)}{(x^2-2x+3)^2} \\ &= \frac{(2x^3-2x^2+2x+6)-(2x^3+2x^2-10x+6)}{(x^2-2x+3)^2} \\ &= \frac{-4x^2+12x}{(x^2-2x+3)^2} \\ &= \frac{4x(3-x)}{(x^2-2x+3)^2}\end{aligned}$$

For stationary points, set $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Since $(x^2-2x+3)^2 > 0$ for all x, this gives

$$4x(3-x) = 0$$

So

$$x = 0 \quad \text{or} \quad x = 3$$

Now find the corresponding y-coordinates:

$$y(0) = \frac{(-1)(3)}{3} = -1$$

$$y(3) = \frac{(2)(6)}{6} = 2$$

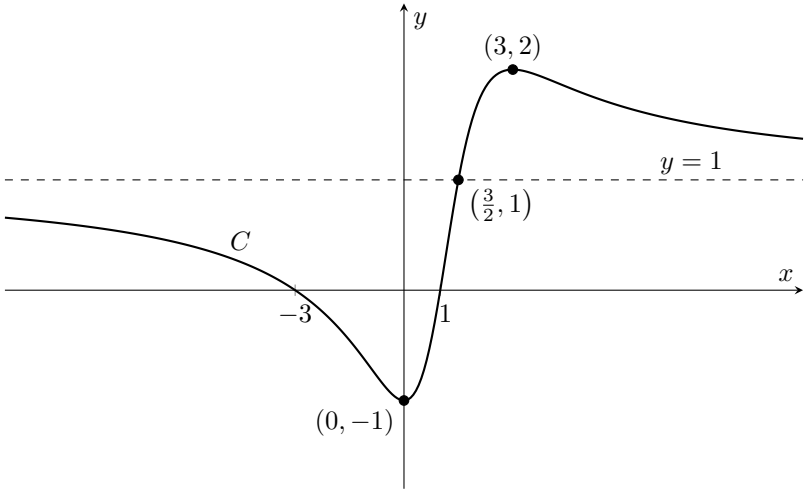
So the stationary points are

$$(0, -1) \quad \text{and} \quad (3, 2)$$

Also, since the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is the sign of $x(3-x)$, it is negative for $x < 0$, positive for $0 < x < 3$, and negative for $x > 3$.

Therefore $(0, -1)$ is a minimum point and $(3, 2)$ is a maximum point.

(c) The sketch should look like:



Intersections with the x-axis occur when $y = 0$, so the numerator must be 0:

$$(x-1)(x+3) = 0$$

Hence

$$x = 1 \quad \text{or} \quad x = -3$$

So the x-intercepts are

$$(1, 0) \quad \text{and} \quad (-3, 0)$$

The intersection with the y-axis is found by putting $x = 0$:

$$y = \frac{(-1)(3)}{3} = -1$$

So the y-intercept is

$$(0, -1)$$

Using parts (a) and (b), the curve is continuous for all x, has horizontal asymptote $y = 1$, minimum point $(0, -1)$ and maximum point $(3, 2)$.

It also crosses the horizontal asymptote when $y = 1$:

$$\frac{x^2 + 2x - 3}{x^2 - 2x + 3} = 1$$

$$x^2 + 2x - 3 = x^2 - 2x + 3$$

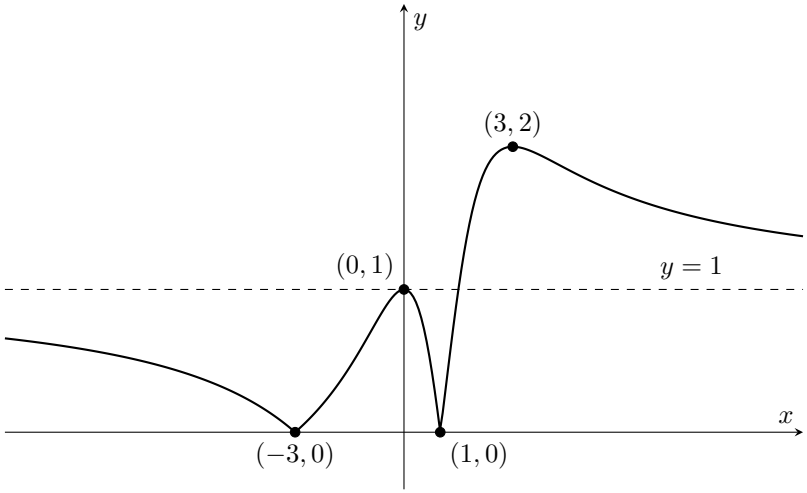
$$4x = 6$$

$$x = \frac{3}{2}$$

So it crosses the asymptote at

$$\left(\frac{3}{2}, 1\right)$$

(d) The sketch should look like:



To solve

$$\left|\frac{(x-1)(x+3)}{x^2-2x+3}\right| = k,$$

draw the horizontal line $y = k$.

For there to be 4 distinct real solutions, the line must meet the curve four times: once on the left-hand branch, twice on the middle arch, and once on the right-hand branch.

From the sketch, this happens exactly when the line lies above the x-axis and below the point $(0, 1)$, that is,

$$0 < k < 1$$

Hence the required set of values is

$$\boxed{0 < k < 1}$$

14. The curve C has equation

$$y = \frac{-x^2 + 4x + 5}{x + 2}$$

(a) Find the equations of the asymptotes of C. **[3]**

(b) Find the coordinates of the stationary points of C. **[4]**

(c) Sketch C, stating the coordinates of the intersections with the axes. **[3]**

(d) Sketch the curve with equation

$$y = \left| \frac{-x^2 + 4x + 5}{x + 2} \right| \quad \textbf{[2]}$$

(e) Using your sketch, or otherwise, find the set of values of x for which

$$\left| \frac{-x^2 + 4x + 5}{x + 2} \right| < 2 \quad \textbf{[4]}$$

Solution

(a) Divide the numerator by $x + 2$:

$$-x^2 + 4x + 5 = (-x + 6)(x + 2) - 7$$

so

$$y = -x + 6 - \frac{7}{x + 2}$$

Hence the denominator is zero when $x = -2$, so there is a vertical asymptote

$$x = -2$$

Also, as $x \to \pm\infty$, the term $\dfrac{-7}{x + 2} \to 0$, so the oblique asymptote is

$$y = -x + 6$$

Therefore the asymptotes are $x = -2$ and $y = -x + 6$.

(b) Using

$$y = -x + 6 - \frac{7}{x + 2}$$

differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 + \frac{7}{(x + 2)^2}$$

At a stationary point,

$$-1 + \frac{7}{(x + 2)^2} = 0$$

so

$$\frac{7}{(x + 2)^2} = 1$$
$$(x + 2)^2 = 7$$
$$x + 2 = \pm\sqrt{7}$$
$$x = -2 \pm \sqrt{7}$$

Now find the corresponding y-values.

For $x = -2 - \sqrt{7}$:

$$\begin{aligned} y &= -(-2 - \sqrt{7}) + 6 - \frac{7}{-\sqrt{7}} \\ &= 2 + \sqrt{7} + 6 + \sqrt{7} \\ &= 8 + 2\sqrt{7} \end{aligned}$$

For $x = -2 + \sqrt{7}$:

$$\begin{aligned} y &= -(-2 + \sqrt{7}) + 6 - \frac{7}{\sqrt{7}} \\ &= 2 - \sqrt{7} + 6 - \sqrt{7} \\ &= 8 - 2\sqrt{7} \end{aligned}$$

To classify them, differentiate again:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{14}{(x+2)^3}$$

At $x = -2 - \sqrt{7}$, $(x+2) = -\sqrt{7}$, so $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ and this is a minimum.

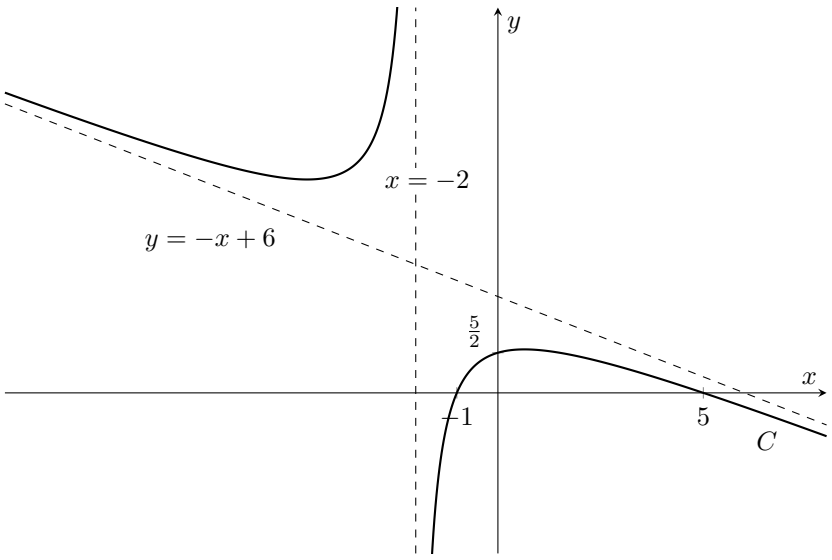
At $x = -2 + \sqrt{7}$, $(x+2) = \sqrt{7}$, so $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ and this is a maximum.

Therefore the stationary points are

$$(-2 - \sqrt{7},\ 8 + 2\sqrt{7}) \quad \text{and} \quad (-2 + \sqrt{7},\ 8 - 2\sqrt{7})$$

with the first a minimum and the second a maximum.

(c) Using the asymptotes and stationary points from earlier parts, the sketch should look like:



The intersections with the x-axis satisfy $y = 0$, so

$$\frac{-x^2 + 4x + 5}{x + 2} = 0$$

hence

$$-x^2 + 4x + 5 = 0$$

and

$$\begin{aligned} x^2 - 4x - 5 &= 0 \\ (x - 5)(x + 1) &= 0 \end{aligned}$$

so the x-intercepts are

$$(-1, 0) \quad \text{and} \quad (5, 0)$$

The intersection with the y-axis is found by putting $x = 0$:

$$y = \frac{5}{2}$$

so the y-intercept is

$$\left(0, \frac{5}{2}\right)$$

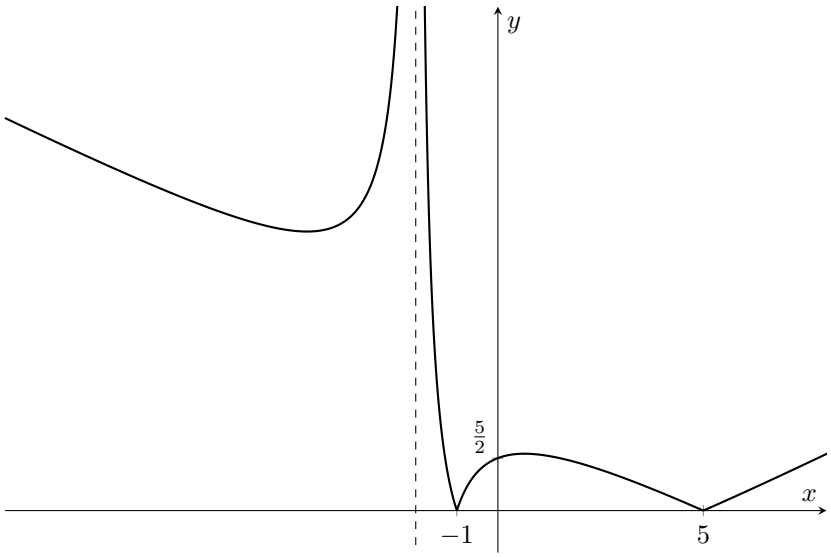
Also,

$$y - (-x + 6) = -\frac{7}{x + 2}$$

so for $x < -2$ the curve lies above the oblique asymptote, and for $x > -2$ it lies below it.

Therefore the axis intersections are $(-1, 0)$, $(5, 0)$ and $\left(0, \frac{5}{2}\right)$.

(d) The sketch should look like:



This graph is obtained by reflecting the parts of C below the x-axis in the x-axis.

From the sketch of C, those parts are for

$$-2 < x < -1 \qquad \text{and} \qquad x > 5$$

The zeros stay at $(-1, 0)$ and $(5, 0)$, the vertical asymptote remains $x = -2$, and for large positive x the reflected branch has asymptote

$$y = x - 6$$

(e) Using the sketch in part (d), we need the graph of

$$y = \left|\frac{-x^2 + 4x + 5}{x + 2}\right|$$

to lie below the line $y = 2$.

First find where it meets $y = 2$, so solve

$$\left|\frac{-x^2 + 4x + 5}{x + 2}\right| = 2$$

This means solving

$$\frac{-x^2 + 4x + 5}{x + 2} = 2 \qquad \text{and} \qquad \frac{-x^2 + 4x + 5}{x + 2} = -2$$

For $\dfrac{-x^2+4x+5}{x+2}=2$:

$$\begin{aligned} -x^2+4x+5 &= 2x+4 \\ -x^2+2x+1 &= 0 \\ x^2-2x-1 &= 0 \end{aligned}$$

so

$$x = 1 \pm \sqrt{2}$$

For $\dfrac{-x^2+4x+5}{x+2}=-2$:

$$\begin{aligned} -x^2+4x+5 &= -2x-4 \\ -x^2+6x+9 &= 0 \\ x^2-6x-9 &= 0 \end{aligned}$$

so

$$x = 3 \pm 3\sqrt{2}$$

In order, these values are

$$3-3\sqrt{2}, \quad 1-\sqrt{2}, \quad 1+\sqrt{2}, \quad 3+3\sqrt{2}$$

From the sketch, $\left|\dfrac{-x^2+4x+5}{x+2}\right| < 2$ between the first two intersection points, and between the last two intersection points:

$$3-3\sqrt{2} < x < 1-\sqrt{2} \qquad \text{or} \qquad 1+\sqrt{2} < x < 3+3\sqrt{2}$$

Therefore the solution set is

$$(3-3\sqrt{2},\ 1-\sqrt{2}) \ \cup \ (1+\sqrt{2},\ 3+3\sqrt{2})$$