Questions

1. The curve C_1 has equation

$$y = 2\cosh 2x - 2$$

and the curve C_2 has equation

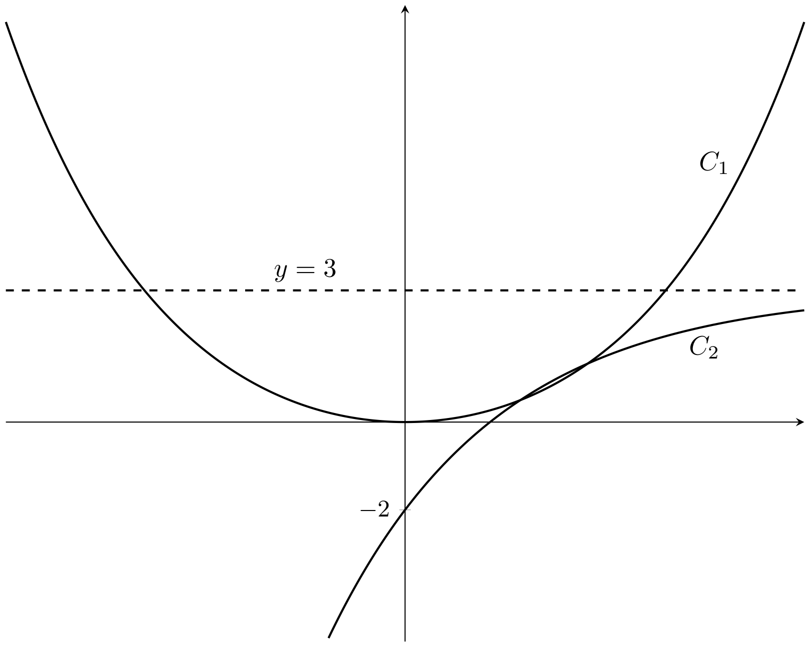
$$y = 3 - 5e^{-2x}$$

(a) Sketch the graphs of C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of the points where the curves meet the axes. **[4]**

(b) Solve the equation $2\cosh 2x - 2 = 3 - 5e^{-2x}$, giving your answers in the form $\frac{1}{2}\ln k$, where k is an integer. **[5]**

Solution

(a) The sketch should look like:



C_1 is the graph of $y = \cosh x$ scaled horizontally and vertically. Like $y = \cosh x$ it meets the coordinate axes at the origin $(0, 0)$ only, and has no asymptotes.

C_2 is the graph of $y = 3 - 5e^{-2x}$. As $x \to \infty$, $e^{-2x} \to 0$ so $y \to 3$. Thus C_2 has a horizontal asymptote at $y = 3$.

C_2 meets the y-axis when $x = 0$: $y = 3 - 5e^{-2(0)} = 3 - 5 = -2$. So the y-intercept is $(0, -2)$.

C_2 meets the x-axis when $3 - 5e^{-2x} = 0$. Solving this equation obtains $x = \frac{1}{2}\ln\frac{5}{3}$, so the x-intercept is

$$\left(\frac{1}{2}\ln\frac{5}{3}, 0\right)$$

(b) Solve

$$2\cosh 2x - 2 = 3 - 5e^{-2x}$$

Using

$$2\cosh 2x = e^{2x} + e^{-2x}$$

gives

$$e^{2x} + e^{-2x} - 2 = 3 - 5e^{-2x}$$

so

$$e^{2x} + e^{-2x} = 5 - 5e^{-2x}$$

Let

$$t = e^{2x}$$

Then $t > 0$ and $e^{-2x} = \dfrac{1}{t}$. Substituting,

$$t + \frac{1}{t} = 5 - \frac{5}{t}$$

Multiplying through by t,

$$t^2 + 1 = 5t - 5$$

$$t^2 - 5t + 6 = 0$$

$$(t-2)(t-3) = 0$$

Hence

$$t = 2 \quad \text{or} \quad t = 3$$

Since $t = e^{2x}$,

$$e^{2x} = 2 \quad \text{or} \quad e^{2x} = 3$$

Taking logarithms,

$$2x = \ln 2 \quad \text{or} \quad 2x = \ln 3$$

Therefore

$$x = \frac{1}{2}\ln 2 \quad \text{or} \quad x = \frac{1}{2}\ln 3$$

So the solutions are $x = \frac{1}{2}\ln 2$ and $x = \frac{1}{2}\ln 3$.

2. **(a)** Use the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials to show that

$$\tanh x \equiv \frac{e^{2x}-1}{e^{2x}+1}$$ **[2]**

(b) Hence find the value of x for which

$$\tanh(x - \ln 2) = \frac{3}{7}$$

giving your answer in the form $\frac{1}{2}\ln k$, where k is a rational number to be determined. **[5]**

Solution

(a) Using the definitions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

we have

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \end{aligned}$$

Now multiply numerator and denominator by e^x:

$$\begin{aligned} \tanh x &= \frac{(e^x - e^{-x})e^x}{(e^x + e^{-x})e^x} \\ &= \frac{e^{2x}-1}{e^{2x}+1} \end{aligned}$$

Hence

$$\tanh x \equiv \frac{e^{2x}-1}{e^{2x}+1}$$

(b) Let

$$y = x - \ln 2$$

Then the equation becomes

$$\tanh y = \frac{3}{7}$$

Using the result from part (a),

$$\frac{e^{2y}-1}{e^{2y}+1} = \frac{3}{7}$$

Cross-multiplying:

$$\begin{aligned} 7(e^{2y}-1) &= 3(e^{2y}+1) \\ 7e^{2y} - 7 &= 3e^{2y} + 3 \\ 4e^{2y} &= 10 \\ e^{2y} &= \frac{5}{2} \end{aligned}$$

Taking natural logarithms:

$$2y = \ln\left(\frac{5}{2}\right)$$

Since $y = x - \ln 2$,

$$2(x - \ln 2) = \ln\left(\frac{5}{2}\right)$$
$$2x - 2\ln 2 = \ln\left(\frac{5}{2}\right)$$

So

$$\begin{aligned} 2x &= \ln\left(\frac{5}{2}\right) + 2\ln 2 \\ &= \ln\left(\frac{5}{2}\right) + \ln 4 \\ &= \ln\left(\frac{5}{2} \times 4\right) \\ &= \ln 10 \end{aligned}$$

Therefore

$$x = \frac{1}{2}\ln 10$$

3. **(a)** Express $7\cosh x - 5\sinh x$ in the form $R\cosh(x-\alpha)$, where $R > 0$ and $\alpha > 0$. Give α to 3 decimal places. **[4]**

(b) Hence solve the equation $7\cosh x - 5\sinh x = 10$, giving all solutions correct to 2 decimal places. **[3]**

(c) Solve $7\cosh x - 5\sinh x = 10$ by instead using the exponential definitions of $\cosh x$ and $\sinh x$. **[4]**

Solution

(a) Using

$$\cosh(x-\alpha) = \cosh x\cosh\alpha - \sinh x\sinh\alpha$$

we have

$$R\cosh(x-\alpha) = R\cosh x\cosh\alpha - R\sinh x\sinh\alpha$$

Comparing this with

$$7\cosh x - 5\sinh x$$

gives

$$R\cosh\alpha = 7, \qquad R\sinh\alpha = 5$$

Now use $\cosh^2\alpha - \sinh^2\alpha = 1$:

$$\begin{aligned} R^2(\cosh^2\alpha - \sinh^2\alpha) &= 7^2 - 5^2 \\ R^2 &= 49 - 25 \\ R^2 &= 24 \end{aligned}$$

Since $R > 0$,

$$R = 2\sqrt{6}$$

Also,

$$\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha} = \frac{5/R}{7/R} = \frac{5}{7}$$

so

$$\begin{aligned} \alpha &= \operatorname{artanh}\left(\frac{5}{7}\right) \\ &= \frac{1}{2}\ln\left(\frac{1+\frac{5}{7}}{1-\frac{5}{7}}\right) \\ &= \frac{1}{2}\ln 6 \end{aligned}$$

Hence

$$\alpha \approx 0.896$$

Therefore

$$7\cosh x - 5\sinh x = 2\sqrt{6}\cosh\left(x - \tfrac{1}{2}\ln 6\right)$$

with

$$R = 2\sqrt{6}, \qquad \alpha = \frac{1}{2}\ln 6 \approx 0.896$$

(b) From part (a),

$$7\cosh x - 5\sinh x = 2\sqrt{6}\cosh\left(x - \tfrac{1}{2}\ln 6\right)$$

so the equation becomes

$$2\sqrt{6}\cosh\left(x - \tfrac{1}{2}\ln 6\right) = 10$$

Hence

$$\cosh\left(x - \tfrac{1}{2}\ln 6\right) = \frac{5}{\sqrt{6}}$$

Therefore

$$x - \tfrac{1}{2}\ln 6 = \pm \operatorname{arcosh}\left(\frac{5}{\sqrt{6}}\right)$$

so

$$x = \tfrac{1}{2}\ln 6 \pm \operatorname{arcosh}\left(\frac{5}{\sqrt{6}}\right)$$

Using

$$\operatorname{arcosh} u = \ln\left(u + \sqrt{u^2 - 1}\right)$$

gives

$$\begin{aligned}\operatorname{arcosh}\left(\frac{5}{\sqrt{6}}\right) &= \ln\left(\frac{5}{\sqrt{6}} + \sqrt{\frac{25}{6} - 1}\right)\\ &= \ln\left(\frac{5}{\sqrt{6}} + \sqrt{\frac{19}{6}}\right)\\ &= \ln\left(\frac{5+\sqrt{19}}{\sqrt{6}}\right)\end{aligned}$$

So

$$\begin{aligned}x &= \ln(\sqrt{6}) \pm \ln\left(\frac{5+\sqrt{19}}{\sqrt{6}}\right)\\ &= \ln(5+\sqrt{19}) \quad \text{or} \quad \ln\left(\frac{6}{5+\sqrt{19}}\right)\end{aligned}$$

Rationalising,

$$\frac{6}{5+\sqrt{19}} = 5 - \sqrt{19}$$

so

$$x = \ln(5+\sqrt{19}) \quad \text{or} \quad x = \ln(5-\sqrt{19})$$

Therefore the solutions are

$$x \approx 2.24 \quad \text{or} \quad x \approx -0.44$$

(c) Using the definitions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

the equation

$$7\cosh x - 5\sinh x = 10$$

becomes

$$\begin{aligned}7\left(\frac{e^x + e^{-x}}{2}\right) - 5\left(\frac{e^x - e^{-x}}{2}\right) &= 10\\ \frac{7e^x + 7e^{-x} - 5e^x + 5e^{-x}}{2} &= 10\\ \frac{2e^x + 12e^{-x}}{2} &= 10\\ e^x + 6e^{-x} &= 10\end{aligned}$$

Multiply through by e^x:

$$e^{2x} - 10e^x + 6 = 0$$

Let $y = e^x$, so $y > 0$. Then

$$y^2 - 10y + 6 = 0$$

Solving the quadratic,

$$\begin{aligned} y &= \frac{10 \pm \sqrt{100-24}}{2} \\ &= \frac{10 \pm \sqrt{76}}{2} \\ &= 5 \pm \sqrt{19} \end{aligned}$$

Since both values are positive,

$$e^x = 5 + \sqrt{19} \quad \text{or} \quad e^x = 5 - \sqrt{19}$$

Hence

$$x = \ln(5 + \sqrt{19}) \quad \text{or} \quad x = \ln(5 - \sqrt{19})$$

Therefore

$$x \approx 2.24 \quad \text{or} \quad x \approx -0.44$$

4. **(a)** Starting from the identity $\cosh 2x = 2\cosh^2 x - 1$, prove that

$$\operatorname{sech} 2x = \frac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}$$ **[2]**

(b) The function f is defined by

$$f(x) = \operatorname{sech} x \qquad (x > 0)$$

(i) State the range of f. **[1]**

(ii) Use part (a) and part (b)(i) to prove that $\operatorname{sech} 2x < \operatorname{sech} x$ if $x > 0$. **[3]**

Solution

(a) Starting with

$$\cosh 2x = 2\cosh^2 x - 1$$

take reciprocals:

$$\operatorname{sech} 2x = \frac{1}{\cosh 2x} = \frac{1}{2\cosh^2 x - 1}$$

Now use $\operatorname{sech} x = \dfrac{1}{\cosh x}$, so that

$$\cosh^2 x = \frac{1}{\operatorname{sech}^2 x}$$

Hence

$$\begin{aligned}\operatorname{sech} 2x &= \frac{1}{2\left(\dfrac{1}{\operatorname{sech}^2 x}\right) - 1} \\ &= \frac{1}{\dfrac{2 - \operatorname{sech}^2 x}{\operatorname{sech}^2 x}} \\ &= \frac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}\end{aligned}$$

So

$$\operatorname{sech} 2x = \frac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}$$

(b) (i) For $x > 0$, we have $\cosh x > 1$, so

$$0 < \operatorname{sech} x = \frac{1}{\cosh x} < 1$$

Also, as $x \to 0^+$, $\operatorname{sech} x \to 1$, and as $x \to \infty$, $\operatorname{sech} x \to 0$.
Therefore the range of f is

$$(0, 1)$$

(ii) Let

$$t = \operatorname{sech} x$$

From part (b)(i), since $x > 0$,

$$0 < t < 1$$

Using part (a),

$$\operatorname{sech} 2x = \frac{t^2}{2 - t^2}$$

Since $0 < t < 1$, we have $0 < t^2 < 1$, so

$$2 - t^2 > 1$$

Therefore

$$\frac{t^2}{2 - t^2} < t^2$$

and because $0 < t < 1$,

$$t^2 < t$$

Hence

$$\operatorname{sech} 2x = \frac{t^2}{2 - t^2} < t = \operatorname{sech} x$$

So, for $x > 0$,

$$\operatorname{sech} 2x < \operatorname{sech} x$$

5. By writing $\sinh x$ and $\cosh x$ in terms of exponentials, solve the equation

$$2\sinh x + 5\cosh x = 6$$

giving your solutions in exact logarithmic form. **[6]**

Solution

Write the hyperbolic functions in exponential form:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Substitute these into the equation $2\sinh x + 5\cosh x = 6$:

$$\begin{aligned} 2\left(\frac{e^x - e^{-x}}{2}\right) + 5\left(\frac{e^x + e^{-x}}{2}\right) &= 6 \\ e^x - e^{-x} + \frac{5}{2}e^x + \frac{5}{2}e^{-x} &= 6 \\ \frac{7}{2}e^x + \frac{3}{2}e^{-x} &= 6 \end{aligned}$$

Multiply through by $2e^x$:

$$\begin{aligned} 2e^x\left(\frac{7}{2}e^x + \frac{3}{2}e^{-x}\right) &= 2e^x \cdot 6 \\ 7e^{2x} + 3 &= 12e^x \end{aligned}$$

So

$$7e^{2x} - 12e^x + 3 = 0$$

Let

$$y = e^x$$

Then $y > 0$, and the equation becomes

$$7y^2 - 12y + 3 = 0$$

Solve this quadratic:

$$\begin{aligned} y &= \frac{12 \pm \sqrt{(-12)^2 - 4 \cdot 7 \cdot 3}}{2 \cdot 7} \\ &= \frac{12 \pm \sqrt{144 - 84}}{14} \\ &= \frac{12 \pm \sqrt{60}}{14} \\ &= \frac{12 \pm 2\sqrt{15}}{14} \\ &= \frac{6 \pm \sqrt{15}}{7} \end{aligned}$$

Hence

$$e^x = \frac{6 + \sqrt{15}}{7} \quad \text{or} \quad e^x = \frac{6 - \sqrt{15}}{7}$$

Both values are positive, so both are valid. Taking natural logarithms gives

$$x = \ln\left(\frac{6 + \sqrt{15}}{7}\right) \quad \text{or} \quad x = \ln\left(\frac{6 - \sqrt{15}}{7}\right)$$

Therefore the exact solutions are

$$x = \ln\left(\frac{6 + \sqrt{15}}{7}\right) \quad \text{and} \quad x = \ln\left(\frac{6 - \sqrt{15}}{7}\right)$$

6. **(a)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that

$$\cosh 2x = 1 + 2\sinh^2 x$$

[2]

(b) Hence solve the equation

$$2\cosh 2x - 7\sinh x = 4$$

giving all your answers in logarithmic form. **[5]**

Solution

(a) Using

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

we have

$$\begin{aligned} \cosh 2x &= \frac{e^{2x} + e^{-2x}}{2} \\ &= 1 + \frac{e^{2x} - 2 + e^{-2x}}{2} \\ &= 1 + \frac{(e^x - e^{-x})^2}{2} \\ &= 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= 1 + 2\sinh^2 x \end{aligned}$$

So

$$\cosh 2x = 1 + 2\sinh^2 x$$

(b) Using the result from part (a),

$$\begin{aligned} 2\cosh 2x - 7\sinh x &= 4 \\ 2(1 + 2\sinh^2 x) - 7\sinh x &= 4 \end{aligned}$$

Let $s = \sinh x$. Then

$$\begin{aligned} 2 + 4s^2 - 7s &= 4 \\ 4s^2 - 7s - 2 &= 0 \\ (4s + 1)(s - 2) &= 0 \end{aligned}$$

So

$$\sinh x = 2 \qquad \text{or} \qquad \sinh x = -\frac{1}{4}$$

First, if $\sinh x = 2$, then

$$\begin{aligned} \frac{e^x - e^{-x}}{2} &= 2 \\ e^x - e^{-x} &= 4 \end{aligned}$$

Multiplying by e^x,

$$\begin{aligned} e^{2x} - 1 &= 4e^x \\ e^{2x} - 4e^x - 1 &= 0 \end{aligned}$$

Let $y = e^x$, where $y > 0$. Then

$$y^2 - 4y - 1 = 0$$

so

$$y = \frac{4 \pm \sqrt{16 + 4}}{2} = 2 \pm \sqrt{5}$$

Since $y > 0$, we take $y = 2 + \sqrt{5}$. Hence

$$x = \ln(2 + \sqrt{5})$$

Second, if $\sinh x = -\frac{1}{4}$, then

$$\begin{aligned} \frac{e^x - e^{-x}}{2} &= -\frac{1}{4} \\ e^x - e^{-x} &= -\frac{1}{2} \end{aligned}$$

Multiplying by $2e^x$,

$$\begin{aligned} 2e^{2x} - 2 &= -e^x \\ 2e^{2x} + e^x - 2 &= 0 \end{aligned}$$

Let $y = e^x$, where $y > 0$. Then

$$2y^2 + y - 2 = 0$$

so

$$y = \frac{-1 \pm \sqrt{1 + 16}}{4} = \frac{-1 \pm \sqrt{17}}{4}$$

Since $y > 0$, we take

$$y = \frac{\sqrt{17} - 1}{4}$$

Hence

$$x = \ln\left(\frac{\sqrt{17} - 1}{4}\right)$$

Therefore the solutions are

$$x = \ln(2 + \sqrt{5}) \qquad \text{or} \qquad x = \ln\left(\frac{\sqrt{17} - 1}{4}\right)$$

7. **(a)** Prove that, for $x \geq 1$,

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$ **[5]**

(b) Prove that the graphs of

$$y = \operatorname{arsinh} x \quad \text{and} \quad y = \operatorname{arcosh}(x + 2)$$

do not intersect. **[3]**

Solution

(a) Let

$$y = \operatorname{arcosh} x$$

Then, by definition of the inverse hyperbolic cosine, $y \geq 0$ and

$$x = \cosh y = \frac{e^y + e^{-y}}{2}$$

Now let

$$t = e^y$$

Since $e^y > 0$, we have $t > 0$, and also $e^{-y} = \frac{1}{t}$. So

$$2x = t + \frac{1}{t}$$

Multiplying by t,

$$2xt = t^2 + 1$$

so

$$t^2 - 2xt + 1 = 0$$

This is a quadratic in t, so

$$\begin{aligned} t &= \frac{2x \pm \sqrt{(2x)^2 - 4}}{2} \\ &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= x \pm \sqrt{x^2 - 1} \end{aligned}$$

Hence

$$e^y = x \pm \sqrt{x^2 - 1}$$

We now choose the correct sign. Since $y \geq 0$,

$$e^y \geq 1$$

Also,

$$\begin{aligned} x - \sqrt{x^2 - 1} &= \frac{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})}{x + \sqrt{x^2 - 1}} \\ &= \frac{x^2 - (x^2 - 1)}{x + \sqrt{x^2 - 1}} \\ &= \frac{1}{x + \sqrt{x^2 - 1}} \end{aligned}$$

For $x \geq 1$, the denominator is at least 1, so

$$x - \sqrt{x^2 - 1} \leq 1$$

Therefore the appropriate root is

$$e^y = x + \sqrt{x^2 - 1}$$

Taking natural logarithms,

$$y = \ln\left(x + \sqrt{x^2 - 1}\right)$$

Since $y = \operatorname{arcosh} x$, we have proved that

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad (x \geq 1)$$

(b) Suppose, for a contradiction, that the graphs intersect at some point (x, y).
Then

$$y = \operatorname{arsinh} x \qquad \text{and} \qquad y = \operatorname{arcosh}(x+2)$$

So, using the inverse definitions,

$$x = \sinh y \qquad \text{and} \qquad x + 2 = \cosh y$$

Subtracting the first equation from the second,

$$2 = \cosh y - \sinh y$$

Now

$$\begin{aligned} \cosh y - \sinh y &= \frac{e^y + e^{-y}}{2} - \frac{e^y - e^{-y}}{2} \\ &= e^{-y} \end{aligned}$$

So

$$2 = e^{-y}$$

But $y = \operatorname{arcosh}(x+2)$, and arcosh always gives $y \geq 0$. Hence

$$0 < e^{-y} \leq 1$$

which cannot equal 2.
This is a contradiction, so the graphs do not intersect.
Therefore, the graphs of $y = \operatorname{arsinh} x$ and $y = \operatorname{arcosh}(x+2)$ have no point of intersection.

8. The function $f(x)$ is defined by $f(x) = \ln(\cosh x - 2\sinh x)$.

(a) Given that k lies in the domain of this function, explain why $k < \frac{1}{2}\ln 3$. **[2]**

(b) (i) Find $f'(x)$. **[2]**

(ii) Show that

$$f''(x) = \frac{a}{(\cosh x - 2\sinh x)^2}$$

where a is an integer to be determined. **[3]**

(c) Hence find a quadratic approximation to $f(x)$ for small values of x. **[3]**

(d) Find the percentage error in this approximation when $x = -0.1$. **[2]**

Solution

(a) For $f(k)$ to exist, the argument of the logarithm must be positive:

$$\cosh k - 2\sinh k > 0$$

Using

$$\cosh k = \frac{e^k + e^{-k}}{2}, \qquad \sinh k = \frac{e^k - e^{-k}}{2}$$

gives

$$\begin{aligned}\cosh k - 2\sinh k &= \frac{e^k + e^{-k}}{2} - 2 \cdot \frac{e^k - e^{-k}}{2} \\ &= \frac{e^k + e^{-k} - 2e^k + 2e^{-k}}{2} \\ &= \frac{-e^k + 3e^{-k}}{2} \\ &= \frac{3 - e^{2k}}{2e^k}\end{aligned}$$

Since $2e^k > 0$, we need

$$3 - e^{2k} > 0$$

So

$$e^{2k} < 3$$

Taking natural logarithms,

$$2k < \ln 3$$

Hence

$$k < \frac{1}{2}\ln 3$$

Therefore the required explanation is that k must satisfy $k < \frac{1}{2}\ln 3$.

(b) (i) Differentiate using the chain rule:

$$f(x) = \ln(\cosh x - 2\sinh x)$$

So

$$f'(x) = \frac{1}{\cosh x - 2\sinh x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x - 2\sinh x)$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x - 2\sinh x) = \sinh x - 2\cosh x$$

Hence

$$f'(x) = \frac{\sinh x - 2\cosh x}{\cosh x - 2\sinh x}$$

So

$$f'(x) = \frac{\sinh x - 2\cosh x}{\cosh x - 2\sinh x}$$

(ii) Let

$$g = \cosh x - 2\sinh x$$

Then

$$f'(x) = \frac{g'}{g}$$

so

$$f''(x) = \frac{g''g - (g')^2}{g^2}$$

Now

$$g' = \sinh x - 2\cosh x, \qquad g'' = \cosh x - 2\sinh x = g$$

Therefore

$$f''(x) = \frac{g^2 - (g')^2}{g^2}$$

That is,

$$f''(x) = \frac{(\cosh x - 2\sinh x)^2 - (\sinh x - 2\cosh x)^2}{(\cosh x - 2\sinh x)^2}$$

Let $c = \cosh x$ and $s = \sinh x$. Then the numerator is

$$\begin{aligned}(c-2s)^2 - (s-2c)^2 &= (c^2 - 4cs + 4s^2) - (s^2 - 4cs + 4c^2)\\ &= c^2 + 4s^2 - s^2 - 4c^2\\ &= 3s^2 - 3c^2\\ &= -3(c^2 - s^2)\end{aligned}$$

Using the identity

$$\cosh^2 x - \sinh^2 x = 1$$

we have

$$c^2 - s^2 = 1$$

So the numerator is

$$-3$$

Hence

$$f''(x) = \frac{-3}{(\cosh x - 2\sinh x)^2}$$

Thus it has the form

$$f''(x) = \frac{a}{(\cosh x - 2\sinh x)^2}$$

with

$$a = -3$$

(c) A quadratic approximation for small x is the Maclaurin expansion up to the x^2 term:

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$$

First,

$$f(0) = \ln(\cosh 0 - 2\sinh 0) = \ln(1-0) = \ln 1 = 0$$

Next, from part (b)(i),

$$f'(0) = \frac{\sinh 0 - 2\cosh 0}{\cosh 0 - 2\sinh 0} = \frac{0-2}{1-0} = -2$$

From part (b)(ii),

$$f''(0) = \frac{-3}{(\cosh 0 - 2\sinh 0)^2} = \frac{-3}{1^2} = -3$$

Substituting into the Maclaurin formula,

$$f(x) \approx 0 + (-2)x + \frac{1}{2}(-3)x^2$$

So the quadratic approximation is

$$f(x) \approx -2x - \frac{3}{2}x^2$$

(d) Using the quadratic approximation,

$$Q(x) = -2x - \frac{3}{2}x^2$$

so

$$Q(-0.1) = -2(-0.1) - \frac{3}{2}(0.1)^2 = 0.2 - 0.015 = 0.185$$

The exact value is

$$f(-0.1) = \ln(\cosh(-0.1) - 2\sinh(-0.1))$$

Since cosh is even and sinh is odd,

$$f(-0.1) = \ln(\cosh 0.1 + 2\sinh 0.1) \approx 0.186760$$

Hence the percentage error is

$$\frac{|0.185 - 0.186760|}{0.186760} \times 100$$

which gives

$$0.942\%$$

Therefore the percentage error is

$$0.942\%$$

9. **(a)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh x + \sinh x = e^x \quad \text{and} \quad \cosh x - \sinh x = e^{-x}$$ **[3]**

(b) Solve the equation

$$4\cosh x - 3\sinh x = 5$$

giving your answers as exact logarithms. **[5]**

Solution

(a) Starting from the definitions,

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Then

$$\begin{aligned}\cosh x + \sinh x &= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \\ &= \frac{e^x + e^{-x} + e^x - e^{-x}}{2} \\ &= \frac{2e^x}{2} \\ &= e^x\end{aligned}$$

Also,

$$\begin{aligned}\cosh x - \sinh x &= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \\ &= \frac{e^x + e^{-x} - e^x + e^{-x}}{2} \\ &= \frac{2e^{-x}}{2} \\ &= e^{-x}\end{aligned}$$

Hence,

$$\cosh x + \sinh x = e^x \qquad \text{and} \qquad \cosh x - \sinh x = e^{-x}$$

(b) Using

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

substitute into

$$4\cosh x - 3\sinh x = 5$$

$$\begin{aligned}4\left(\frac{e^x + e^{-x}}{2}\right) - 3\left(\frac{e^x - e^{-x}}{2}\right) &= 5 \\ 2(e^x + e^{-x}) - \frac{3}{2}(e^x - e^{-x}) &= 5\end{aligned}$$

It is neater to combine over 2:

$$\begin{aligned}\frac{4(e^x + e^{-x}) - 3(e^x - e^{-x})}{2} &= 5 \\ \frac{4e^x + 4e^{-x} - 3e^x + 3e^{-x}}{2} &= 5 \\ \frac{e^x + 7e^{-x}}{2} &= 5\end{aligned}$$

So

$$e^x + 7e^{-x} = 10$$

Let $y = e^x$. Since $e^x > 0$, we have $y > 0$, and $e^{-x} = \frac{1}{y}$. Then

$$\begin{aligned} y + \frac{7}{y} &= 10 \\ y^2 + 7 &= 10y \\ y^2 - 10y + 7 &= 0 \end{aligned}$$

Solving the quadratic,

$$\begin{aligned} y &= \frac{10 \pm \sqrt{(-10)^2 - 4(1)(7)}}{2} \\ &= \frac{10 \pm \sqrt{100 - 28}}{2} \\ &= \frac{10 \pm \sqrt{72}}{2} \\ &= \frac{10 \pm 6\sqrt{2}}{2} \\ &= 5 \pm 3\sqrt{2} \end{aligned}$$

Both values are positive, so both are valid for $y = e^x$. Therefore

$$e^x = 5 + 3\sqrt{2} \quad \text{or} \quad e^x = 5 - 3\sqrt{2}$$

Taking logarithms,

$$x = \ln(5 + 3\sqrt{2}) \quad \text{or} \quad x = \ln(5 - 3\sqrt{2})$$

Hence the exact solutions are

$$x = \ln(5 + 3\sqrt{2}) \quad \text{or} \quad x = \ln(5 - 3\sqrt{2})$$

10. Using definitions of the hyperbolic functions in terms of exponentials, prove that, for all real numbers x and y,

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$ **[5]**

Solution

Using the exponential definition,

$$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

Multiplying top and bottom by e^t gives

$$\tanh t = \frac{e^{2t} - 1}{e^{2t} + 1}$$

Now let

$$a = e^{2x}, \qquad b = e^{2y}$$

so that $a > 0$ and $b > 0$. Then

$$\tanh x = \frac{a-1}{a+1}, \qquad \tanh y = \frac{b-1}{b+1}$$

and also

$$\tanh(x+y) = \frac{e^{2(x+y)} - 1}{e^{2(x+y)} + 1} = \frac{ab-1}{ab+1}$$

Now simplify the required right-hand side:

$$\begin{aligned}\frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} &= \frac{\frac{a-1}{a+1} + \frac{b-1}{b+1}}{1 + \frac{(a-1)(b-1)}{(a+1)(b+1)}} \\ &= \frac{\frac{(a-1)(b+1)+(b-1)(a+1)}{(a+1)(b+1)}}{\frac{(a+1)(b+1)+(a-1)(b-1)}{(a+1)(b+1)}} \\ &= \frac{(a-1)(b+1) + (b-1)(a+1)}{(a+1)(b+1) + (a-1)(b-1)}\end{aligned}$$

Expanding the numerator:

$$\begin{aligned}(a-1)(b+1) + (b-1)(a+1) &= (ab + a - b - 1) + (ab + b - a - 1) \\ &= 2ab - 2\end{aligned}$$

Expanding the denominator:

$$\begin{aligned}(a+1)(b+1) + (a-1)(b-1) &= (ab + a + b + 1) + (ab - a - b + 1) \\ &= 2ab + 2\end{aligned}$$

So

$$\begin{aligned}\frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} &= \frac{2ab-2}{2ab+2} \\ &= \frac{ab-1}{ab+1} \\ &= \tanh(x+y)\end{aligned}$$

Hence, for all real numbers x and y,

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

11. **(a)** Using the definition of $\tanh x$ in terms of exponentials, prove that

$$\tanh 3x \equiv \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$ **[3]**

(b) Hence solve the equation

$$\tanh 3x = 2\tanh x$$

giving your answers as simplified natural logarithms where appropriate. **[4]**

Solution

(a) Using

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

let

$$y = e^{2x}$$

Then

$$\tanh x = \frac{y-1}{y+1} \qquad \text{and} \qquad \tanh 3x = \frac{e^{6x} - 1}{e^{6x} + 1} = \frac{y^3 - 1}{y^3 + 1}$$

Now put

$$t = \tanh x = \frac{y-1}{y+1}$$

Then

$$\begin{aligned}
\frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x} &= \frac{3t + t^3}{1 + 3t^2} \\
&= \frac{3\left(\frac{y-1}{y+1}\right) + \left(\frac{y-1}{y+1}\right)^3}{1 + 3\left(\frac{y-1}{y+1}\right)^2} \\
&= \frac{3(y-1)(y+1)^2 + (y-1)^3}{(y+1)\big((y+1)^2 + 3(y-1)^2\big)} \\
&= \frac{(y-1)\big(3(y+1)^2 + (y-1)^2\big)}{(y+1)\big((y+1)^2 + 3(y-1)^2\big)}
\end{aligned}$$

Expand the brackets:

$$\begin{aligned}
3(y+1)^2 + (y-1)^2 &= 3(y^2 + 2y + 1) + (y^2 - 2y + 1) \\
&= 4y^2 + 4y + 4 \\
&= 4(y^2 + y + 1)
\end{aligned}$$

and

$$\begin{aligned}
(y+1)^2 + 3(y-1)^2 &= (y^2 + 2y + 1) + 3(y^2 - 2y + 1) \\
&= 4y^2 - 4y + 4 \\
&= 4(y^2 - y + 1)
\end{aligned}$$

So

$$\begin{aligned}
\frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x} &= \frac{(y-1)4(y^2 + y + 1)}{(y+1)4(y^2 - y + 1)} \\
&= \frac{(y-1)(y^2 + y + 1)}{(y+1)(y^2 - y + 1)} \\
&= \frac{y^3 - 1}{y^3 + 1} \\
&= \tanh 3x
\end{aligned}$$

Hence

$$\tanh 3x \equiv \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

(b) Let

$$t = \tanh x$$

Using part (a),

$$\tanh 3x = 2\tanh x \quad \Rightarrow \quad \frac{3t + t^3}{1 + 3t^2} = 2t$$

Since $1 + 3t^2 > 0$, we can multiply through:

$$\begin{aligned} 3t + t^3 &= 2t(1 + 3t^2) \\ 3t + t^3 &= 2t + 6t^3 \\ t - 5t^3 &= 0 \\ t(1 - 5t^2) &= 0 \end{aligned}$$

So

$$t = 0 \qquad \text{or} \qquad t^2 = \frac{1}{5}$$

Hence

$$\tanh x = 0 \qquad \text{or} \qquad \tanh x = \pm\frac{1}{\sqrt{5}}$$

Therefore

$$x = 0$$

or

$$x = \operatorname{artanh}\left(\pm\frac{1}{\sqrt{5}}\right) = \pm\operatorname{artanh}\left(\frac{1}{\sqrt{5}}\right)$$

Using

$$\operatorname{artanh} u = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$

we get

$$\begin{aligned} x &= \pm\frac{1}{2}\ln\left(\frac{1 + \frac{1}{\sqrt{5}}}{1 - \frac{1}{\sqrt{5}}}\right) \\ &= \pm\frac{1}{2}\ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) \\ &= \pm\frac{1}{2}\ln\left(\frac{(\sqrt{5}+1)^2}{5-1}\right) \\ &= \pm\frac{1}{2}\ln\left(\frac{6+2\sqrt{5}}{4}\right) \\ &= \pm\frac{1}{2}\ln\left(\frac{3+\sqrt{5}}{2}\right) \end{aligned}$$

Now

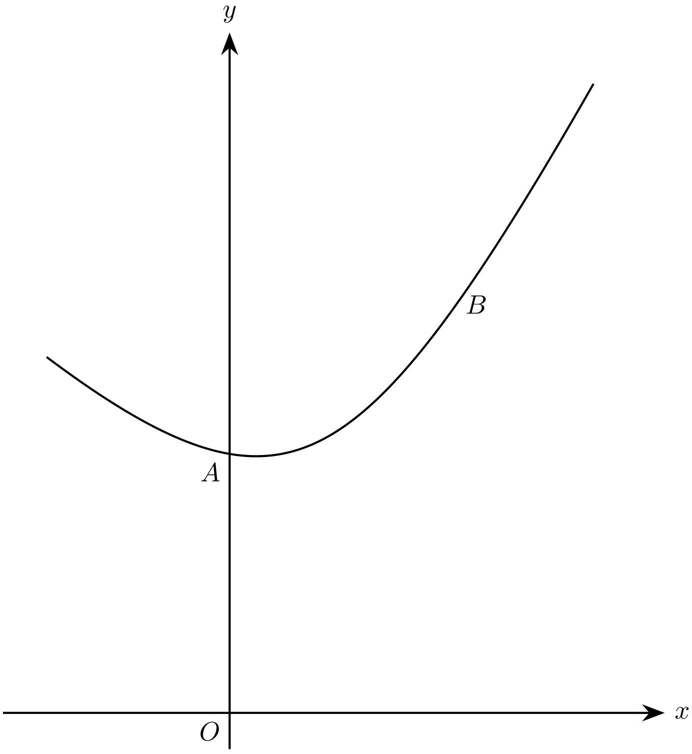
$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$$

so

$$\frac{1}{2}\ln\left(\frac{3+\sqrt{5}}{2}\right) = \ln\left(\frac{1+\sqrt{5}}{2}\right)$$

Hence the solutions are

$$x = 0,\ \pm\ln\left(\frac{1+\sqrt{5}}{2}\right)$$



12. The diagram shows the curve with parametric equations

$$x = 1 + 2\sinh t, \quad y = 3\cosh t + \sinh t$$

(a) The curve crosses the positive y-axis at A.

(i) Determine the value of the parameter t at A, giving your answer in logarithmic form. **[4]**

(ii) Find the y-coordinate of A, giving your answer correct to 3 significant figures. **[2]**

(b) The point B has parameter $t = \ln 2$.

Determine the equation of the tangent to the curve at B. **[6]**

Solution

(a) (i) At the positive y-axis, $x = 0$. So

$$1 + 2\sinh t = 0$$

Hence

$$\sinh t = -\frac{1}{2}$$

Using

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

gives

$$\begin{aligned} \frac{e^t - e^{-t}}{2} &= -\frac{1}{2} \\ e^t - e^{-t} &= -1 \\ e^{2t} - 1 &= -e^t \\ e^{2t} + e^t - 1 &= 0 \end{aligned}$$

Let $u = e^t$, so $u > 0$. Then

$$u^2 + u - 1 = 0$$

Solving,

$$u = \frac{-1 \pm \sqrt{5}}{2}$$

Since $u > 0$,

$$u = \frac{\sqrt{5} - 1}{2}$$

Therefore

$$e^t = \frac{\sqrt{5} - 1}{2}$$

so

$$t = \ln\left(\frac{\sqrt{5} - 1}{2}\right)$$

(ii) From part (i), $\sinh t = -\frac{1}{2}$. Using the identity

$$\cosh^2 t - \sinh^2 t = 1$$

we get

$$\cosh^2 t - \left(-\frac{1}{2}\right)^2 = 1$$
$$\cosh^2 t = 1 + \frac{1}{4} = \frac{5}{4}$$

Since $\cosh t > 0$,

$$\cosh t = \frac{\sqrt{5}}{2}$$

Now

$$\begin{aligned} y &= 3\cosh t + \sinh t \\ &= 3\left(\frac{\sqrt{5}}{2}\right) - \frac{1}{2} \\ &= \frac{3\sqrt{5} - 1}{2} \end{aligned}$$

Hence

$$y = \frac{3\sqrt{5} - 1}{2} \approx 2.85$$

So the y-coordinate of A is 2.85 correct to 3 significant figures.

(b) For parametric equations,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

Differentiate with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\cosh t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3\sinh t + \cosh t$$

At $t = \ln 2$,

$$\begin{aligned} \sinh(\ln 2) &= \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4} \\ \cosh(\ln 2) &= \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4} \end{aligned}$$

So the coordinates of B are

$$x = 1 + 2\left(\frac{3}{4}\right) = \frac{5}{2}$$
$$y = 3\left(\frac{5}{4}\right) + \frac{3}{4} = \frac{15}{4} + \frac{3}{4} = \frac{9}{2}$$

The gradient of the tangent is

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3\sinh t + \cosh t}{2\cosh t} \\ &= \frac{3\left(\frac{3}{4}\right) + \frac{5}{4}}{2\left(\frac{5}{4}\right)} \\ &= \frac{\frac{7}{2}}{\frac{5}{2}} = \frac{7}{5}\end{aligned}$$

Using the point-gradient form through $\left(\frac{5}{2}, \frac{9}{2}\right)$,

$$y - \frac{9}{2} = \frac{7}{5}\left(x - \frac{5}{2}\right)$$

Simplifying,

$$\begin{aligned}y - \frac{9}{2} &= \frac{7}{5}x - \frac{7}{2} \\ y &= \frac{7}{5}x - \frac{7}{2} + \frac{9}{2} \\ y &= \frac{7}{5}x + 1\end{aligned}$$

So the equation of the tangent at B is

$$y = \frac{7}{5}x + 1$$

13. **(a)** It is given that, for $|t| < 1$,

$$q = \frac{1}{2} \ln \left(\frac{1+t}{1-t} \right)$$

Starting from the exponential definition of the tanh function, show that

$$\tanh q = t$$

[4]

(b) Solve the equation

$$6 \operatorname{sech}^2 x = 5 - \tanh x$$

Give your answers in logarithmic form. **[4]**

Solution

(a) Starting from the exponential definition,

$$\tanh q = \frac{e^q - e^{-q}}{e^q + e^{-q}}$$

Multiply top and bottom by e^q:

$$\tanh q = \frac{e^{2q} - 1}{e^{2q} + 1}$$

Given

$$q = \frac{1}{2} \ln \left(\frac{1+t}{1-t} \right)$$

we have

$$2q = \ln \left(\frac{1+t}{1-t} \right)$$

so, exponentiating,

$$e^{2q} = \frac{1+t}{1-t}$$

Substitute this into the formula for $\tanh q$:

$$\begin{aligned} \tanh q &= \frac{\frac{1+t}{1-t} - 1}{\frac{1+t}{1-t} + 1} \\ &= \frac{\frac{1+t-(1-t)}{1-t}}{\frac{1+t+(1-t)}{1-t}} \\ &= \frac{\frac{2t}{1-t}}{\frac{2}{1-t}} \\ &= t \end{aligned}$$

Hence

$$\tanh q = t$$

(b) Let

$$y = \tanh x$$

Using the identity

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

the equation

$$6 \operatorname{sech}^2 x = 5 - \tanh x$$

becomes

$$\begin{aligned} 6(1 - y^2) &= 5 - y \\ 6 - 6y^2 &= 5 - y \\ 6y^2 - y - 1 &= 0 \end{aligned}$$

Factorising,

$$6y^2 - y - 1 = (3y+1)(2y-1)$$

so

$$(3y+1)(2y-1) = 0$$

Hence

$$y = \frac{1}{2} \quad \text{or} \quad y = -\frac{1}{3}$$

Since $y = \tanh x$, we have

$$\tanh x = \frac{1}{2} \quad \text{or} \quad \tanh x = -\frac{1}{3}$$

From part (a), if $\tanh x = t$, then

$$x = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$$

For $t = \frac{1}{2}$:

$$\begin{aligned} x &= \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) \\ &= \frac{1}{2}\ln\left(\frac{3/2}{1/2}\right) \\ &= \frac{1}{2}\ln 3 \end{aligned}$$

For $t = -\frac{1}{3}$:

$$\begin{aligned} x &= \frac{1}{2}\ln\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right) \\ &= \frac{1}{2}\ln\left(\frac{2/3}{4/3}\right) \\ &= \frac{1}{2}\ln\left(\frac{1}{2}\right) \\ &= -\frac{1}{2}\ln 2 \end{aligned}$$

Therefore the solutions are

$$x = \frac{1}{2}\ln 3 \quad \text{or} \quad x = -\frac{1}{2}\ln 2$$

14. **(a)** Sketch the graph of $y = \sinh x$. **[2]**

(b) Given that $u = \sinh x$, use the definition of $\sinh x$ in terms of e^x and e^{-x} to show that

$$x = \ln\left(u + \sqrt{u^2 + 1}\right)$$

[6]

(c) (i) Show that the equation

$$9\cosh^2 x - 24\sinh x = -7$$

can be written as

$$9\sinh^2 x - 24\sinh x + 16 = 0$$

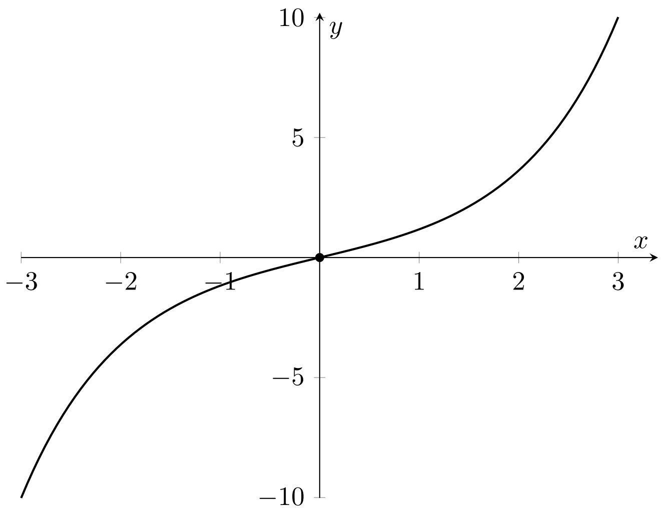
[2]

(ii) Hence show that the equation has only one real solution for x.

Find this solution in the form $\ln a$, where a is an integer. **[5]**

Solution

(a) The curve $y = \sinh x$ is an odd curve, so it is symmetric about the origin. It passes through $(0, 0)$, is strictly increasing for all x.



(b) Using the definition of $\sinh x$,

$$u = \sinh x = \frac{e^x - e^{-x}}{2}$$

Multiply through by 2:

$$2u = e^x - e^{-x}$$

Now multiply by e^x:

$$2ue^x = e^{2x} - 1$$

Let

$$t = e^x$$

so that $t > 0$. Then $e^{2x} = t^2$, and the equation becomes

$$2ut = t^2 - 1$$

Rearranging,

$$t^2 - 2ut - 1 = 0$$

This is a quadratic in t, so

$$t = \frac{2u \pm \sqrt{(-2u)^2 - 4(1)(-1)}}{2}$$

$$t = \frac{2u \pm \sqrt{4u^2 + 4}}{2}$$

$$t = \frac{2u \pm 2\sqrt{u^2 + 1}}{2}$$

$$t = u \pm \sqrt{u^2 + 1}$$

Since $t = e^x > 0$, we must choose the positive value. Also,

$$\sqrt{u^2 + 1} > |u|$$

so

$$u - \sqrt{u^2 + 1} < 0$$

and this root is not possible.
Therefore

$$e^x = u + \sqrt{u^2 + 1}$$

Taking natural logarithms,

$$x = \ln\left(u + \sqrt{u^2 + 1}\right)$$

So

$$x = \ln\left(u + \sqrt{u^2 + 1}\right)$$

(c) **(i)** Use the identity

$$\cosh^2 x - \sinh^2 x = 1$$

so

$$\cosh^2 x = 1 + \sinh^2 x$$

Substitute into the given equation:

$$9\cosh^2 x - 24\sinh x = -7$$

$$9(1 + \sinh^2 x) - 24\sinh x = -7$$

$$9 + 9\sinh^2 x - 24\sinh x = -7$$

$$9\sinh^2 x - 24\sinh x + 16 = 0$$

Hence

$$9\sinh^2 x - 24\sinh x + 16 = 0$$

(ii) Let

$$s = \sinh x$$

Then the equation from part (i) becomes

$$9s^2 - 24s + 16 = 0$$

Factorising,

$$9s^2 - 24s + 16 = (3s - 4)^2$$

So

$$(3s - 4)^2 = 0$$

Hence

$$3s - 4 = 0$$

and therefore

$$s = \frac{4}{3}$$

So

$$\sinh x = \frac{4}{3}$$

Since $\sinh x$ is strictly increasing on $\mathbb{R}$, it is one-to-one, so this gives only one real value of x. Using the result from part (b),

$$x = \ln\left(\frac{4}{3} + \sqrt{\left(\frac{4}{3}\right)^2 + 1}\right)$$

$$x = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right)$$

$$x = \ln\left(\frac{4}{3} + \sqrt{\frac{25}{9}}\right)$$

$$x = \ln\left(\frac{4}{3} + \frac{5}{3}\right)$$

$$x = \ln 3$$

So the equation has only one real solution, namely

$$x = \ln 3$$

and hence $a = 3$.

15. **(a)** Sketch the graph of $y = \operatorname{cosech} x$ and state the equations of its asymptotes. **[3]**

(b) Use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that, for $x \neq 0$,

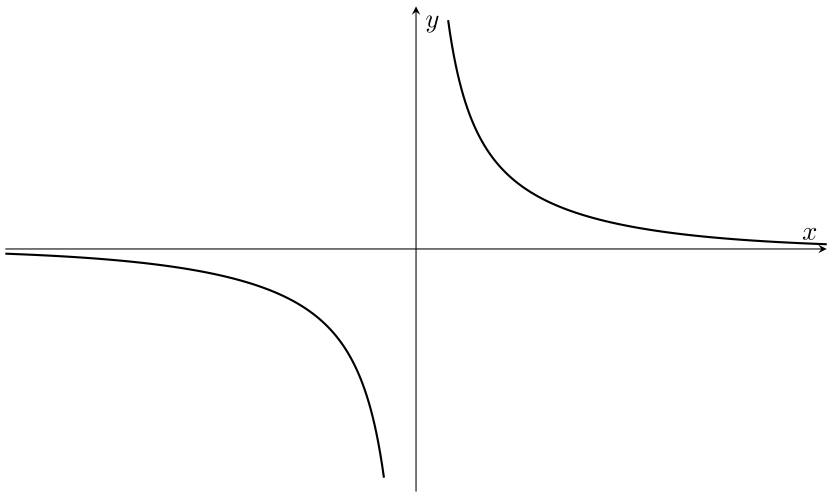
$$\coth^2 x - \operatorname{cosech}^2 x = 1$$

[3]

(c) Solve the equation $\operatorname{cosech}^2 x = 5 - \coth x$, giving your answers in terms of natural logarithms. **[5]**

Solution

(a) The sketch should look like:



Since

$$y = \operatorname{cosech} x = \frac{1}{\sinh x}$$

and $\sinh(-x) = -\sinh x$, we have

$$\operatorname{cosech}(-x) = \frac{1}{\sinh(-x)} = -\frac{1}{\sinh x} = -\operatorname{cosech} x$$

so the graph is odd.

Also,

$$x \to 0^+ \implies \sinh x \to 0^+ \implies \operatorname{cosech} x \to +\infty$$
$$x \to 0^- \implies \sinh x \to 0^- \implies \operatorname{cosech} x \to -\infty$$

and

$$x \to +\infty \implies \sinh x \to +\infty \implies \operatorname{cosech} x \to 0^+$$
$$x \to -\infty \implies \sinh x \to -\infty \implies \operatorname{cosech} x \to 0^-$$

So there is one branch in quadrant I and one in quadrant III.

Hence the asymptotes are $x = 0$ and $y = 0$.

(b) For $x \neq 0$,

$$\coth x = \frac{\cosh x}{\sinh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

so

$$\coth^2 x - \operatorname{cosech}^2 x = \frac{\cosh^2 x}{\sinh^2 x} - \frac{1}{\sinh^2 x} = \frac{\cosh^2 x - 1}{\sinh^2 x}$$

Now use the exponential definitions:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Then

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}\\ &= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4}\\ &= \frac{4}{4} = 1\end{aligned}$$

So

$$\cosh^2 x - 1 = \sinh^2 x$$

and therefore

$$\coth^2 x - \operatorname{cosech}^2 x = \frac{\sinh^2 x}{\sinh^2 x} = 1$$

Hence

$$\coth^2 x - \operatorname{cosech}^2 x = 1$$

(c) Using the identity from part (b),

$$\operatorname{cosech}^2 x = \coth^2 x - 1$$

so the equation becomes

$$\coth^2 x - 1 = 5 - \coth x$$

Rearranging,

$$\begin{aligned}\coth^2 x + \coth x - 6 &= 0\\ (\coth x - 2)(\coth x + 3) &= 0\end{aligned}$$

Hence

$$\coth x = 2 \quad \text{or} \quad \coth x = -3$$

Using

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}$$

First, if $\coth x = 2$, then

$$\frac{e^{2x} + 1}{e^{2x} - 1} = 2$$

so

$$\begin{aligned}e^{2x} + 1 &= 2e^{2x} - 2\\ e^{2x} &= 3\\ 2x &= \ln 3\\ x &= \frac{1}{2}\ln 3\end{aligned}$$

Second, if $\coth x = -3$, then

$$\frac{e^{2x} + 1}{e^{2x} - 1} = -3$$

so

$$\begin{aligned}e^{2x} + 1 &= -3e^{2x} + 3\\ 4e^{2x} &= 2\\ e^{2x} &= \frac{1}{2}\\ 2x &= \ln\left(\frac{1}{2}\right) = -\ln 2\\ x &= -\frac{1}{2}\ln 2\end{aligned}$$

Therefore the solutions are

$$x = \frac{1}{2}\ln 3 \quad \text{or} \quad x = -\frac{1}{2}\ln 2$$

16. The hyperbola H has equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

(a) Use calculus to show that the equation of the tangent to H at the point $(a \sinh\theta,\ b\cosh\theta)$ may be written in the form

$$ay\cosh\theta - bx\sinh\theta = ab$$ **[4]**

The line ℓ_1 is the tangent to H at the point $(a\sinh\theta,\ b\cosh\theta)$, where $\theta > 0$.
Given that ℓ_1 meets the asymptote $y = \frac{b}{a}x$ at the point P

(b) find, in terms of a, b and θ, the coordinates of P. **[2]**

The line ℓ_2 is the tangent to H at the point $(0,\ b)$.
Given that ℓ_1 and ℓ_2 meet at the point Q

(c) find, in terms of a, b and θ, the coordinates of Q. **[2]**

(d) Show that, as θ varies, the locus of the mid-point of PQ has equation

$$2ay^2 - 2bxy = ab^2$$ **[6]**

Solution

(a) Starting from

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

differentiate implicitly with respect to x:

$$\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2x}{a^2} = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2x}{a^2y}$$

At the point $(a\sinh\theta,\ b\cosh\theta)$, the gradient is

$$m = \frac{b^2(a\sinh\theta)}{a^2(b\cosh\theta)} = \frac{b\sinh\theta}{a\cosh\theta}$$

So the tangent is

$$y - b\cosh\theta = \frac{b\sinh\theta}{a\cosh\theta}(x - a\sinh\theta)$$

Multiply by $a\cosh\theta$:

$$ay\cosh\theta - ab\cosh^2\theta = bx\sinh\theta - ab\sinh^2\theta$$

Hence

$$ay\cosh\theta - bx\sinh\theta = ab\left(\cosh^2\theta - \sinh^2\theta\right) = ab$$

Therefore the tangent may be written as

$$ay\cosh\theta - bx\sinh\theta = ab$$

(b) Point P lies on the asymptote $y = \frac{b}{a}x$ and on the tangent

$$ay\cosh\theta - bx\sinh\theta = ab$$

Substitute $y = \frac{b}{a}x$:

$$\begin{aligned} a\left(\frac{b}{a}x\right)\cosh\theta - bx\sinh\theta &= ab \\ bx(\cosh\theta - \sinh\theta) &= ab \\ x &= \frac{a}{\cosh\theta - \sinh\theta} \end{aligned}$$

Now

$$\cosh\theta - \sinh\theta = \frac{e^{\theta} + e^{-\theta}}{2} - \frac{e^{\theta} - e^{-\theta}}{2} = e^{-\theta}$$

So

$$x = ae^{\theta}$$

and therefore

$$y = \frac{b}{a}x = be^{\theta}$$

Hence

$$P = (ae^{\theta},\ be^{\theta})$$

(c) For the point $(0, b)$, the gradient is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2x}{a^2y} = 0$$

so the tangent l_2 is

$$y = b$$

To find Q, substitute $y = b$ into l_1:

$$\begin{aligned} a(b)\cosh\theta - bx\sinh\theta &= ab \\ ab\cosh\theta - bx\sinh\theta &= ab \\ x\sinh\theta &= a(\cosh\theta - 1) \\ x &= a\frac{\cosh\theta - 1}{\sinh\theta} \end{aligned}$$

Hence

$$Q = \left(a\frac{\cosh\theta - 1}{\sinh\theta},\ b\right)$$

(d) Let the midpoint of PQ be $M(x, y)$.
From parts (b) and (c),

$$P = (ae^{\theta},\ be^{\theta}), \qquad Q = \left(a\frac{\cosh\theta - 1}{\sinh\theta},\ b\right)$$

So

$$\begin{aligned} x &= \frac{1}{2}\left(ae^{\theta} + a\frac{\cosh\theta - 1}{\sinh\theta}\right) = \frac{a}{2}\left(e^{\theta} + \frac{\cosh\theta - 1}{\sinh\theta}\right) \\ y &= \frac{1}{2}(be^{\theta} + b) = \frac{b}{2}(e^{\theta} + 1) \end{aligned}$$

Let

$$t = e^{\theta}$$

Then $t > 1$, and

$$\cosh\theta = \frac{t+t^{-1}}{2}, \qquad \sinh\theta = \frac{t-t^{-1}}{2}$$

Hence

$$\begin{aligned}\frac{\cosh\theta - 1}{\sinh\theta} &= \frac{\frac{t+t^{-1}}{2} - 1}{\frac{t-t^{-1}}{2}} \\ &= \frac{t+t^{-1}-2}{t-t^{-1}} \\ &= \frac{t^2+1-2t}{t^2-1} \\ &= \frac{(t-1)^2}{(t-1)(t+1)} \\ &= \frac{t-1}{t+1}\end{aligned}$$

Therefore

$$\begin{aligned}x &= \frac{a}{2}\left(t + \frac{t-1}{t+1}\right) \\ &= \frac{a}{2}\left(\frac{t(t+1)+t-1}{t+1}\right) \\ &= \frac{a(t^2+2t-1)}{2(t+1)}\end{aligned}$$

and

$$y = \frac{b}{2}(t+1)$$

Now calculate $2ay^2 - 2bxy$:

$$\begin{aligned}2ay^2 - 2bxy &= 2a\left(\frac{b}{2}(t+1)\right)^2 - 2b\left(\frac{a(t^2+2t-1)}{2(t+1)}\right)\left(\frac{b}{2}(t+1)\right) \\ &= \frac{ab^2}{2}(t+1)^2 - \frac{ab^2}{2}(t^2+2t-1) \\ &= \frac{ab^2}{2}\left(t^2+2t+1-t^2-2t+1\right) \\ &= \frac{ab^2}{2}(2) \\ &= ab^2\end{aligned}$$

So the midpoint $M(x, y)$ satisfies

$$2ay^2 - 2bxy = ab^2$$

Hence, as θ varies, the locus is

$$2ay^2 - 2bxy = ab^2$$

Since $\theta > 0$, we have $t = e^\theta > 1$, so

$$y = \frac{b}{2}(t+1) > b$$

so this is the relevant branch of the locus.

17. (a) Show, by means of a sketch, that the curves with equations

$$y = \tanh x$$

and

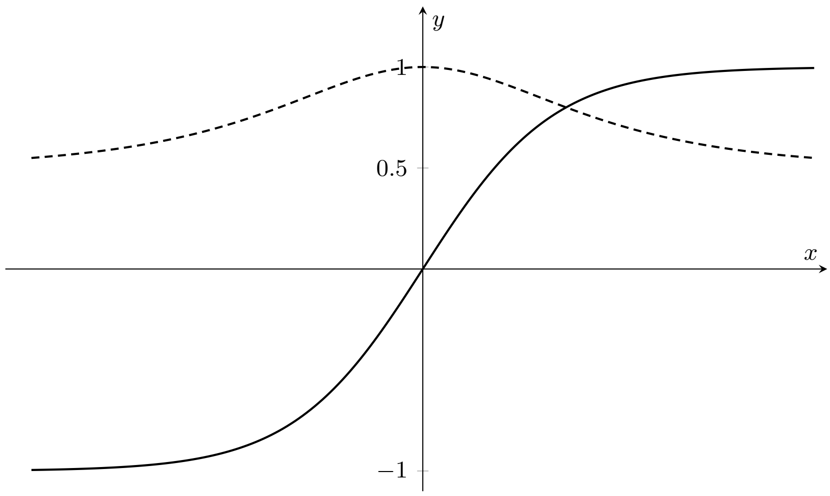
$$y = \frac{1}{2}(1 + \operatorname{sech} x)$$

have exactly one point of intersection. **[4]**

(b) Find the x-coordinate of this point of intersection, giving your answer in the form $\ln k$, where k is an integer to be determined. **[4]**

Solution

(a) The sketch should look like:



Where the solid curve is $y = \tanh x$ and the dashed curve is $y = \frac{1}{2}(1 + \operatorname{sech} x)$. So the curves have exactly one point of intersection.

Algebraic justification:

For $y = \tanh x$,

$$\tanh(-x) = -\tanh x$$

so the curve is odd. It passes through $(0, 0)$, is increasing, and has horizontal asymptotes $y = \pm 1$.

For $y = \frac{1}{2}(1 + \operatorname{sech} x)$,

$$\operatorname{sech}(-x) = \operatorname{sech} x$$

so this curve is even. Also

$$\frac{1}{2}(1 + \operatorname{sech} 0) = \frac{1}{2}(1 + 1) = 1$$

so it has a maximum at $(0, 1)$. As $x \to \pm\infty$,

$$\operatorname{sech} x \to 0 \quad \Rightarrow \quad \frac{1}{2}(1 + \operatorname{sech} x) \to \frac{1}{2}$$

Since $\cosh x$ increases for $x > 0$, $\operatorname{sech} x = \dfrac{1}{\cosh x}$ decreases for $x > 0$, so this curve decreases for $x > 0$.

Now compare the two graphs.

For $x < 0$,

$$\tanh x < 0, \qquad \frac{1}{2}(1 + \operatorname{sech} x) > 0$$

so there is no intersection for $x < 0$.

For $x > 0$, $y = \tanh x$ increases continuously from 0 to 1, while $y = \frac{1}{2}(1+\operatorname{sech} x)$ decreases continuously from 1 to $\frac{1}{2}$. Therefore the two curves cross exactly once for $x > 0$.

(b) At the point of intersection,

$$\tanh x = \frac{1}{2}(1 + \operatorname{sech} x)$$

Using

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \text{and} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

gives

$$\begin{aligned} \frac{\sinh x}{\cosh x} &= \frac{1}{2}\left(1 + \frac{1}{\cosh x}\right) \\ 2\sinh x &= \cosh x + 1 \end{aligned}$$

Now write $\sinh x$ and $\cosh x$ in exponential form:

$$\begin{aligned} 2\left(\frac{e^x - e^{-x}}{2}\right) &= \frac{e^x + e^{-x}}{2} + 1 \\ e^x - e^{-x} &= \frac{e^x + e^{-x}}{2} + 1 \\ 2e^x - 2e^{-x} &= e^x + e^{-x} + 2 \\ e^x - 3e^{-x} - 2 &= 0 \end{aligned}$$

Multiplying through by e^x,

$$e^{2x} - 2e^x - 3 = 0$$

Let $t = e^x$. Since $e^x > 0$, we have $t > 0$. Then

$$\begin{aligned} t^2 - 2t - 3 &= 0 \\ (t-3)(t+1) &= 0 \end{aligned}$$

So $t = 3$, and hence

$$e^x = 3$$

Therefore

$$x = \ln 3$$

So the x-coordinate of the point of intersection is $\ln 3$, and $k = 3$.

18. (i) Sketch the graph of $y = \cosh x$ for $x \geq 0$ and state the value of the gradient when $x = 0$. On the same axes, sketch the graph of $y = \operatorname{arcosh} x$ for $x \geq 1$. Label each curve and give the equation of the line of symmetry. **[4]**

(ii) Find

$$\int_0^k \cosh x \, \mathrm{d}x$$

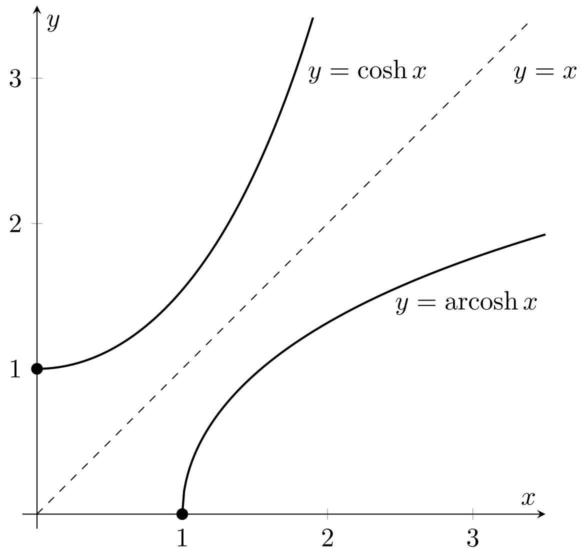
where $k > 0$. **[2]**

(iii) Deduce, or otherwise show, that

$$\int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k - \sinh k$$ **[4]**

Solution

(i) The sketch should look like:



because:

For $y = \cosh x$, when $x = 0$,

$$y = \cosh 0 = 1$$

so the curve starts at $(0, 1)$.

Its gradient is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh x$$

Hence at $x = 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh 0 = 0$$

so the tangent there is horizontal.

For $x \geq 0$, $y = \cosh x$ increases and is convex.

Now $y = \operatorname{arcosh} x$ is the inverse of $y = \cosh x$ for $x \geq 0$, so its graph is the reflection of $y = \cosh x$ in the line $y = x$. Therefore it starts at $(1, 0)$ and increases for $x \geq 1$.

So the line of symmetry is

$$y = x$$

Therefore the gradient at $x = 0$ is 0, and the line of symmetry is $y = x$.

(ii)

$$\int_0^k \cosh x \, \mathrm{d}x = [\sinh x]_0^k$$

So

$$\int_0^k \cosh x \, \mathrm{d}x = \sinh k - \sinh 0 = \sinh k$$

Hence

$$\int_0^k \cosh x \, \mathrm{d}x = \sinh k$$

(iii) Since $y = \operatorname{arcosh} x$ is the inverse of $y = \cosh x$, the two graphs are reflections of each other in the line $y = x$.

Therefore, the area under $y = \cosh x$ from $x = 0$ to $x = k$ and the area under $y = \operatorname{arcosh} x$ from $x = 1$ to $x = \cosh k$ together make the rectangle of width k and height $\cosh k$. So

$$\int_0^k \cosh x \, \mathrm{d}x + \int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k$$

From part (ii),

$$\int_0^k \cosh x \, \mathrm{d}x = \sinh k$$

Substituting gives

$$\sinh k + \int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k$$

Hence

$$\int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k - \sinh k$$

Therefore

$$\int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k - \sinh k$$