Questions

1. **(a)** Show that, for $r > 0$,

$$\ln\left(1 + \frac{2}{r(r+3)}\right) = \ln\left(\frac{r+1}{r}\right) - \ln\left(\frac{r+3}{r+2}\right)$$ [**1**]

(b) Hence, using the method of differences, show that for integer $n \geq 1$,

$$\sum_{r=1}^{n} \ln\left(1 + \frac{2}{r(r+3)}\right) = \ln\left(\frac{3(n+1)}{n+3}\right)$$ [**4**]

2. Let a be a positive constant.

(a) Use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(r+a)(r+a+1)(r+a+2)}$$

in terms of n and a. **[4]**

(b) Hence find the value of a for which

$$\sum_{r=1}^{\infty} \frac{1}{(r+a)(r+a+1)(r+a+2)} = \frac{1}{24}$$ **[3]**

3. **(a)** Show that, for $r \geq 2$,

$$\frac{4r}{\sqrt{r(r+2)}+\sqrt{r(r-2)}} = \sqrt{r(r+2)} - \sqrt{r(r-2)}$$ **[2]**

(b) Hence use the method of differences to determine

$$\sum_{r=2}^{n} \frac{4r}{\sqrt{r(r+2)}+\sqrt{r(r-2)}}$$

giving your answer in simplest form. **[3]**

(c) Hence show that

$$\sum_{r=3}^{7} \frac{4r}{\sqrt{r(r+2)}+\sqrt{r(r-2)}} = A\sqrt{2}+B\sqrt{3}+C\sqrt{7}$$

where A, B and C are integers to be determined. **[2]**

4. Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{12}{(2r-1)(2r+1)(2r+3)} = \frac{n(an+b)}{(2n+1)(2n+3)}$$

where a and b are integers to be determined. **[6]**

5. **(a)** Use the method of differences to prove that for $n \geq 1$

$$\sum_{r=1}^{n} \ln\left(1 + \frac{2}{r(r+3)}\right) = \ln\left(\frac{3(n+1)}{n+3}\right)$$ **[4]**

(b) Hence find the exact value of

$$\sum_{r=25}^{74} \ln\left(\left(1 + \frac{2}{r(r+3)}\right)^{28}\right)$$

Give your answer in the form $a \ln\left(\frac{b}{c}\right)$ where a, b and c are integers to be determined. **[3]**

6. Using partial fractions and a method of differences, prove that for positive integer n,

$$\sum_{r=1}^{n} \frac{6}{r(r+1)(r+2)} = \frac{n(an+b)}{2(n+1)(n+2)}$$

where a and b are constants to be found. **[6]**

7. Let $T_n = (n+1)^3 + (n+2)^3 + \cdots + (2n)^3$.

(a) By considering $(r+1)^4 - (r-1)^4$, use the method of differences to prove that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

[5]

(b) Show that

$$T_n = \frac{1}{4}n^2(an^2 + bn + c)$$

where a, b and c are integers to be determined. **[3]**

(c) State the value of

$$\lim_{n \to \infty} \frac{T_n}{n^4}$$

[1]

8. **(a)** By expressing $\frac{1}{r(r+2)}$ in partial fractions, show that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$ **[5]**

(b) Hence determine the value of

$$\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$$ **[2]**

9. Show that

$$\frac{1}{1\times 4}+\frac{1}{2\times 5}+\frac{1}{3\times 6}+\cdots+\frac{1}{n(n+3)}=\frac{11}{18}-\frac{an^2+bn+c}{3(n+1)(n+2)(n+3)}$$

where a, b and c are constants to be found. **[6]**

10. **(a)** Use standard results from the formula booklet to show that

$$\sum_{r=1}^{N} r(2r-1)(2r+1) = \frac{1}{2}N(N+1)(2N^2+2N-1)$$ **[3]**

(b) Express $\dfrac{6r+13}{(2r-1)(2r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{N} \frac{6r+13}{(2r-1)(2r+3)}\left(\frac{1}{2}\right)^{r+1}$$

in terms of N. **[4]**

(c) Deduce the value of

$$\sum_{r=1}^{\infty} \frac{6r+13}{(2r-1)(2r+3)}\left(\frac{1}{2}\right)^{r+1}$$ **[1]**

11. **(a)** Show that

$$(2n)^5 - (2n-2)^5 \equiv 10(2n-1)^4 + 20(2n-1)^2 + 2$$ **[2]**

(b) Hence, using the method of differences, show that for all positive integer values of n,

$$\sum_{r=1}^{n}(2r-1)^4 = \frac{1}{15}n(2n-1)(2n+1)(an^2+bn+c)$$

where a, b and c are integers to be determined. **[7]**

12. Let

$$S_n = \sum_{r=1}^{n} \frac{1}{(3r-2)(3r+4)}$$

where $n \geq 1$.

(a) Use the method of differences to show that

$$S_n = \frac{n(15n+17)}{8(3n+1)(3n+4)}$$ **[6]**

(b) Show that, for any number k greater than $\frac{24}{5}$, if the difference between $\frac{5}{24}$ and S_n is less than $\frac{1}{k}$, then

$$n > \frac{k - 15 + \sqrt{k^2+81}}{18}$$ **[6]**