Questions

1. The cubic equation

$$x^3 - 5x^2 + px - q = 0$$

where p and q are positive real constants, has roots α, β and γ.

Given that

$$(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 14$$

(a) show that $p = 6$ **[3]**

Given that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 3$$

(b) determine the value of q **[3]**

(c) Without solving the cubic equation, determine the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$ **[4]**

2. The quartic equation $x^4 + bx^3 + cx^2 + dx + 3 = 0$ has roots α, β, γ, δ. It is given that

$$\alpha + \beta + \gamma + \delta = -2, \qquad (\alpha+1)^2 + (\beta+1)^2 + (\gamma+1)^2 + (\delta+1)^2 = 6, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 2$$

(a) Find the values of b, c and d **[6]**

(b) Given also that

$$(\alpha+1)^3 + (\beta+1)^3 + (\gamma+1)^3 + (\delta+1)^3 = 20$$

find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$ **[2]**

3. The equation $x^3 - 4x^2 - x + 6 = 0$ has roots α, β and γ.

Determine the value of

$$\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$$

[4]

4. The quartic equation $3x^4 - x^3 - 10x^2 + 8x + 4 = 0$ has roots α, β, γ and δ.

Without solving the equation, find equations with integer coefficients whose roots are

(a) $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$ **[6]**

(b) $\frac{2}{\alpha - 1}$, $\frac{2}{\beta - 1}$, $\frac{2}{\gamma - 1}$ and $\frac{2}{\delta - 1}$ **[6]**

5. The cubic equation $x^3 + x^2 - 2x - 1 = 0$ has roots α, β and γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$ **[3]**

(b) Find the value of

$$\left(\frac{1}{\alpha+1}\right)^2 + \left(\frac{1}{\beta+1}\right)^2 + \left(\frac{1}{\gamma+1}\right)^2$$

[2]

(c) Find the value of

$$\left(\frac{1}{\alpha+1}\right)^3 + \left(\frac{1}{\beta+1}\right)^3 + \left(\frac{1}{\gamma+1}\right)^3$$

[2]

6. The equation $x^3 - 14x^2 + 56x + c = 0$, where c is a constant, has three roots in geometric progression.

(a) Determine the roots of the equation **[6]**

(b) Find c **[1]**

7. The quadratic equation

$$x^2 - kx + 1 = 0$$

where k is an integer, has roots α and β.

(a) Write down, in terms of k where appropriate, the value of $\alpha + \beta$ and the value of $\alpha\beta$ **[2]**

(b) Determine, in simplest form in terms of k, the value of

$$(\alpha^2 + \beta) + (\beta^2 + \alpha)$$ **[3]**

(c) Determine a quadratic equation which has roots

$$\alpha^2 + \beta \text{ and } \beta^2 + \alpha$$

giving your answer in the form $px^2 + qx + r = 0$, where p, q and r are integer values in terms of k **[4]**

8. The roots of the equation

$$x^3 + 5x^2 - 4x - 24 = 0$$

are α, β and γ.

Without solving the equation,

(a) write down the value of each of

$$\alpha + \beta + \gamma \qquad \alpha\beta + \beta\gamma + \gamma\alpha \qquad \alpha\beta\gamma \qquad \textbf{[1]}$$

(b) Hence determine the value of

(i)

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} \qquad \textbf{[2]}$$

(ii)

$$(\alpha + 2)(\beta + 2)(\gamma + 2) \qquad \textbf{[2]}$$

(iii)

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 \qquad \textbf{[3]}$$

9. The roots of the equation $3x^3 + px^2 - 12x - 8 = 0$ are α, β and γ.

(a) Given that $\alpha + \beta + \gamma = 2$, write down the value of p **[1]**

(b) Write down values for $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$ **[1]**

(c) Hence find the value of $\left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{1}{\beta}\right)\left(1 + \frac{1}{\gamma}\right)$ **[3]**

10. The roots of the equation $x^3 + 2x^2 - 5x - 3 = 0$ are α, β and γ.

(a) Find $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ **[4]**

(b) Find an equation with integer coefficients whose roots are α^2, β^2 and γ^2 **[4]**

11. The graph of $y = x^3 + 3x^2 + px - 20$, where p is a constant, crosses the x-axis at three points whose x-coordinates are equally spaced.

Find the roots of

$$x^3 + 3x^2 + px - 20 = 0$$ **[6]**

12. The cubic equation $x^3 - 3x + 1 = 0$ has roots α, β and γ.

(a) Find a cubic equation whose roots are $\dfrac{1}{\alpha - 1}$, $\dfrac{1}{\beta - 1}$, $\dfrac{1}{\gamma - 1}$. **[3]**

(b) Hence find the value of $\dfrac{1}{(\alpha - 1)^2} + \dfrac{1}{(\beta - 1)^2} + \dfrac{1}{(\gamma - 1)^2}$. **[2]**

(c) Find also the value of $\dfrac{1}{(\alpha - 1)^5} + \dfrac{1}{(\beta - 1)^5} + \dfrac{1}{(\gamma - 1)^5}$. **[3]**

13. The quartic equation $x^4 - 2x^2 + 2x + 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^2, β^2, γ^2, δ^2 and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. **[5]**

(b) Find the value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. **[3]**

(c) Find the value of $\alpha^8 + \beta^8 + \gamma^8 + \delta^8$. **[2]**

14. The equation $z^4 - 2z^3 - z^2 - 2z + 1 = 0$ has roots α, β, γ and δ.

(a) Show that a quartic equation whose roots are $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$, $\gamma + \frac{1}{\gamma}$ and $\delta + \frac{1}{\delta}$ is

$$w^4 - 4w^3 - 2w^2 + 12w + 9 = 0$$ **[3]**

(b) Hence determine the exact roots of the equation $z^4 - 2z^3 - z^2 - 2z + 1 = 0$. **[3]**

15. The quartic equation $x^4 - 2x^3 + 3x^2 - 4x + 1 = 0$ has roots α, β, γ, δ.

(a) Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$. **[2]**

(b) Find the value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} + \dfrac{1}{\delta^2}$. **[2]**

(c) Find the value of

$$\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 + \left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right)^2 + \left(\frac{1}{\gamma} + \frac{1}{\delta} + \frac{1}{\alpha}\right)^2 + \left(\frac{1}{\delta} + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2$$

[5]