Questions

1. The curve C_1 has equation

$$y = 2\cosh 2x - 2$$

and the curve C_2 has equation

$$y = 3 - 5e^{-2x}$$

(a) Sketch the graphs of C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of the points where the curves meet the axes. **[4]**

(b) Solve the equation $2\cosh 2x - 2 = 3 - 5e^{-2x}$, giving your answers in the form $\frac{1}{2}\ln k$, where k is an integer. **[5]**

2. **(a)** Use the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials to show that

$$\tanh x \equiv \frac{e^{2x}-1}{e^{2x}+1}$$

[2]

(b) Hence find the value of x for which

$$\tanh(x - \ln 2) = \frac{3}{7}$$

giving your answer in the form $\frac{1}{2}\ln k$, where k is a rational number to be determined. **[5]**

3. **(a)** Express $7\cosh x - 5\sinh x$ in the form $R\cosh(x - \alpha)$, where $R > 0$ and $\alpha > 0$.
Give α to 3 decimal places. **[4]**

(b) Hence solve the equation $7\cosh x - 5\sinh x = 10$, giving all solutions correct to 2 decimal places. **[3]**

(c) Solve $7\cosh x - 5\sinh x = 10$ by instead using the exponential definitions of $\cosh x$ and $\sinh x$. **[4]**

4. **(a)** Starting from the identity $\cosh 2x = 2\cosh^2 x - 1$, prove that

$$\operatorname{sech} 2x = \frac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}$$ **[2]**

(b) The function f is defined by

$$f(x) = \operatorname{sech} x \qquad (x > 0)$$

(i) State the range of f. **[1]**

(ii) Use part (a) and part (b)(i) to prove that $\operatorname{sech} 2x < \operatorname{sech} x$ if $x > 0$. **[3]**

5. By writing $\sinh x$ and $\cosh x$ in terms of exponentials, solve the equation

$$2\sinh x + 5\cosh x = 6$$

giving your solutions in exact logarithmic form. **[6]**

6. **(a)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that

$$\cosh 2x = 1 + 2\sinh^2 x$$

[2]

(b) Hence solve the equation

$$2\cosh 2x - 7\sinh x = 4$$

giving all your answers in logarithmic form. **[5]**

7. **(a)** Prove that, for $x \geq 1$,

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$ **[5]**

(b) Prove that the graphs of

$$y = \operatorname{arsinh} x \quad \text{and} \quad y = \operatorname{arcosh}(x + 2)$$

do not intersect. **[3]**

8. The function $f(x)$ is defined by $f(x) = \ln(\cosh x - 2\sinh x)$.

(a) Given that k lies in the domain of this function, explain why $k < \frac{1}{2}\ln 3$. **[2]**

(b) (i) Find $f'(x)$. **[2]**

(ii) Show that

$$f''(x) = \frac{a}{(\cosh x - 2\sinh x)^2}$$

where a is an integer to be determined. **[3]**

(c) Hence find a quadratic approximation to $f(x)$ for small values of x. **[3]**

(d) Find the percentage error in this approximation when $x = -0.1$. **[2]**

9. **(a)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh x + \sinh x = e^x \quad \text{and} \quad \cosh x - \sinh x = e^{-x}$$ **[3]**

(b) Solve the equation

$$4\cosh x - 3\sinh x = 5$$

giving your answers as exact logarithms. **[5]**

10. Using definitions of the hyperbolic functions in terms of exponentials, prove that, for all real numbers x and y,

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

[5]

11. **(a)** Using the definition of $\tanh x$ in terms of exponentials, prove that

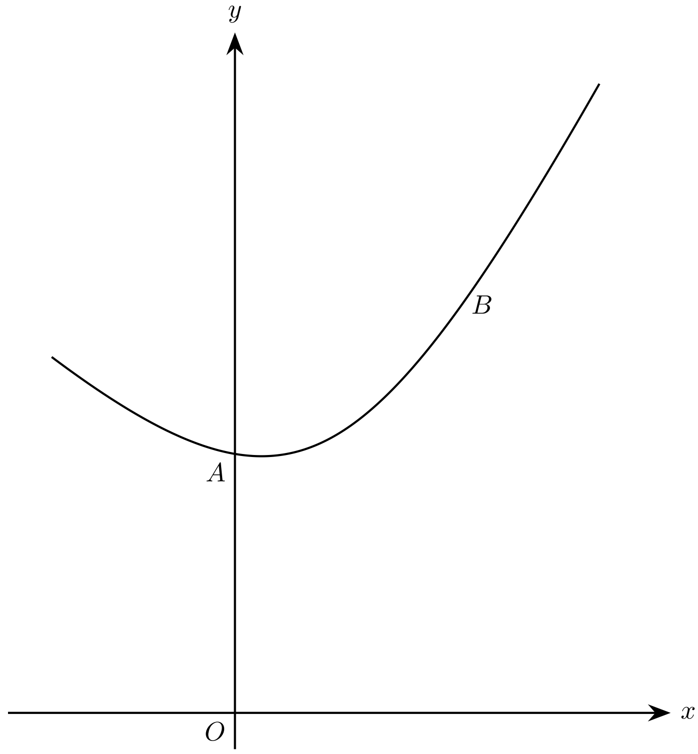
$$\tanh 3x \equiv \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

[3]

(b) Hence solve the equation

$$\tanh 3x = 2\tanh x$$

giving your answers as simplified natural logarithms where appropriate. **[4]**



12. The diagram shows the curve with parametric equations

$$x = 1 + 2\sinh t, \quad y = 3\cosh t + \sinh t$$

(a) The curve crosses the positive y-axis at A.

(i) Determine the value of the parameter t at A, giving your answer in logarithmic form. **[4]**

(ii) Find the y-coordinate of A, giving your answer correct to 3 significant figures. **[2]**

(b) The point B has parameter $t = \ln 2$.

Determine the equation of the tangent to the curve at B. **[6]**

13. **(a)** It is given that, for $|t| < 1$,

$$q = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$$

Starting from the exponential definition of the tanh function, show that

$$\tanh q = t$$

[4]

(b) Solve the equation

$$6\,\mathrm{sech}^2 x = 5 - \tanh x$$

Give your answers in logarithmic form. **[4]**

14. **(a)** Sketch the graph of $y = \sinh x$. **[2]**

(b) Given that $u = \sinh x$, use the definition of $\sinh x$ in terms of e^x and e^{-x} to show that

$$x = \ln\left(u + \sqrt{u^2 + 1}\right)$$ **[6]**

(c) (i) Show that the equation

$$9\cosh^2 x - 24\sinh x = -7$$

can be written as

$$9\sinh^2 x - 24\sinh x + 16 = 0$$ **[2]**

(ii) Hence show that the equation has only one real solution for x.

Find this solution in the form $\ln a$, where a is an integer. **[5]**

15. **(a)** Sketch the graph of $y = \operatorname{cosech} x$ and state the equations of its asymptotes. **[3]**

(b) Use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that, for $x \neq 0$,

$$\coth^2 x - \operatorname{cosech}^2 x = 1$$

[3]

(c) Solve the equation $\operatorname{cosech}^2 x = 5 - \coth x$, giving your answers in terms of natural logarithms. **[5]**

16. The hyperbola H has equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

(a) Use calculus to show that the equation of the tangent to H at the point $(a \sinh \theta,\ b \cosh \theta)$ may be written in the form

$$ay \cosh \theta - bx \sinh \theta = ab$$ **[4]**

The line ℓ_1 is the tangent to H at the point $(a \sinh \theta,\ b \cosh \theta)$, where $\theta > 0$.
Given that ℓ_1 meets the asymptote $y = \frac{b}{a}x$ at the point P

(b) find, in terms of a, b and θ, the coordinates of P. **[2]**

The line ℓ_2 is the tangent to H at the point $(0,\ b)$.
Given that ℓ_1 and ℓ_2 meet at the point Q

(c) find, in terms of a, b and θ, the coordinates of Q. **[2]**

(d) Show that, as θ varies, the locus of the mid-point of PQ has equation

$$2ay^2 - 2bxy = ab^2$$ **[6]**

17. **(a)** Show, by means of a sketch, that the curves with equations

$$y = \tanh x$$

and

$$y = \frac{1}{2}\big(1 + \operatorname{sech} x\big)$$

have exactly one point of intersection. **[4]**

(b) Find the x-coordinate of this point of intersection, giving your answer in the form $\ln k$, where k is an integer to be determined. **[4]**

18. (i) Sketch the graph of $y = \cosh x$ for $x \geq 0$ and state the value of the gradient when $x = 0$. On the same axes, sketch the graph of $y = \operatorname{arcosh} x$ for $x \geq 1$. Label each curve and give the equation of the line of symmetry. **[4]**

(ii) Find

$$\int_0^k \cosh x \, \mathrm{d}x$$

where $k > 0$. **[2]**

(iii) Deduce, or otherwise show, that

$$\int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k - \sinh k$$ **[4]**